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On a subclass of analytic functions involving harmonic means

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Abstract

In the present paper, we consider a generalised subclass of analytic functions involving arithmetic, geometric and harmonic means. For this function class we obtain an inclusion result, Fekete-Szegö inequality and coefficient bounds for bi-univalent functions.

1 Introduction

Let $U_r = \{z \in \mathbb{C} : |z| < r\}$ $(r > 0)$ and let $U = U_1$ denote the unit disk.

Let $\mathcal{A}$ be the class of all analytic functions f in U of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{1}$$

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in U. It is known (see [6]) that if $f \in \mathcal{S}$, then $f(U)$ contains the disk $\left\{|w| < \frac{1}{4}\right\}$. Here $\frac{1}{4}$ is the best possible constant known as the Koebe constant for $\mathcal{S}$. Thus every univalent function f has an inverse f^{-1} defined on some disk containing the disk $\left\{|w| < \frac{1}{4}\right\}$ and satisfying:

$$\begin{aligned} &f^{-1}(f(z)) = z, \quad z \in U \ \ and \\ &f(f^{-1}(w)) = w, \quad |w| < r_0(f), r_0(f) \geq \tfrac{1}{4}, \end{aligned}$$

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where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots . \quad (2)$$

We denote by $\mathcal{S}^*$ the class of analytic functions which are starlike in U.

Let $f \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$. We define the following function:

$$F(z) = \left[f(z)^{1-\alpha}(zf'(z))^{\alpha}\right]^{1-\beta} \cdot \left[(1-\alpha)f(z) + \alpha zf'(z)\right]^{\beta}, \quad \alpha, \beta \in \mathbb{R}. \quad (3)$$

Remark 1. It is easy to observe that for specific values of β, the function $F(z)$ reduces to some generalised means. If $\beta = 0$ we obtain generalised geometric means, if $\beta = 1$ we obtain generalised arithmetic means and if $\beta = -1$ we obtain generalised harmonic means of functions $f(z)$ and $zf'(z)$.

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $H_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{R}$, if the function $F(z)$ defined by (3) is starlike, that is*

$$\Re\left\{(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] + \beta\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha zf'(z)}\right\} > 0, \quad z \in U. \quad (4)$$

In order to prove our main results we will need the following lemmas.

Lemma 1. *[9, p.24] Let $q \in Q$, with $q(0) = a$, and let $p(z) = a + a_nz^n + \cdots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q then there exist $z_0 = r_0e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ and $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$ and:*

1. $p(z_0) = q(\zeta_0)$,
2. $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$,
3. $\Re\dfrac{z_0p''(z_0)}{p'(z_0)} + 1 \geq m\Re\left[\dfrac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)} + 1\right]$.

Denote by $\mathcal{P}$ the class of analytic functions p normalized by $p(0) = 1$ and having positive real part in U.

Lemma 2. *[6] Let $p \in \mathcal{P}$ be of the form $p(z) = 1 + p_1z + p_2z^2 + \dots$, $z \in U$. Then the following estimates hold*

$$|p_n| \leq 2, \ n = 1, 2, \dots .$$

Lemma 3. *[4] If $p \in \mathcal{P}$ is of the form $p(z) = 1 + p_1z + p_2z^2 + \dots$, $z \in U$. Then*

$$\left|p_2 - vp_1^2\right| \leq \begin{cases} -4v + 2, & v \leq 0, \\ 2, & 0 \leq v \leq 1, \\ 4v - 2, & v \geq 1. \end{cases}$$

When $v<0$ or $v>1$, the equality holds if and only if $p_1(z)=\dfrac{1+z}{1-z}$ *or one of its rotations. If* $0<v<1$ *then the equality holds if and only if* $p_1(z)=\dfrac{1+z^2}{1-z^2}$ *or one of its rotations. If $v=0$, the equality holds if and only if*

$$p_1(z)=\left(\frac{1}{2}+\frac{1}{2}\lambda\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\frac{1-z}{1+z},\quad \lambda\in[0,1]$$

or one of its rotations. If $v=1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v=0$.

2 Inclusion result

In this section we show that the new class $H_{\alpha,\beta}$ is a subclass of the class of starlike functions.

Theorem 1. *Let $\alpha,\ \beta\in\mathbb{R}$ such that $\alpha\beta(1-\alpha)\geq 0$. Then*

$$H_{\alpha,\beta}\subset\mathcal{S}^*\subset\mathcal{S}.$$

Proof. Let f be in the class $H_{\alpha,\beta}$ and let $p(z)=\frac{zf'(z)}{f(z)}$. Then from (4) we obtain that $f\in H_{\alpha,\beta}$ if and only if

$$\Re\left\{\alpha(1-\beta)\frac{zp'(z)}{p(z)}+\alpha\beta\frac{zp'(z)}{1-\alpha+\alpha p(z)}\right\}>0. \tag{5}$$

Let

$$q(z)=\frac{1+z}{1-z}=1+q_1z+\cdots. \tag{6}$$

Then $\Delta=q(\mathbb{D})=\{w:\Re w>0\}$, $q(0)=1,E(q)=\{1\}$ and $q\in Q$. To prove that $f\in\mathcal{S}^*$ it is enough to show that

$$\Re\left\{\alpha(1-\beta)\frac{zp'(z)}{p(z)}+\alpha\beta\frac{zp'(z)}{1-\alpha+\alpha p(z)}\right\}>0\Rightarrow p(z)\prec q(z).$$

Suppose that $p(z)\not\prec q(z)$.Then, from Lemma 1,there exist a point $z_0\in U$ and a point $\zeta_0\in\partial U\setminus\{1\}$ such that $p(z_0)=q(\zeta_0)$ and $\Re p(z)>0$ for all $z\in U_{|z_0|}$. This implies that $\Re p(z_0)=0$, therefore we can choose $p(z_0)$ of the form $p(z_0):=ix$, where x is a real number. Due to symmetry, it is sufficient to consider only the case where $x>0$. We have

$$\zeta_0=q^{-1}(p(z_0))=\frac{p(z_0)-1}{p(z_0)+1},$$

then $z_0p'(z_0) = m\zeta_0 q'(\zeta_0) = -m(x^2+1) := y$, where $y < 0$.
Thus, we obtain:

$$\Re\left[\alpha(1-\beta)\frac{z_0p'(z_0)}{p(z_0)}\right] + \Re\left[\alpha\beta\frac{z_0p'(z_0)}{1-\alpha+\alpha p(z_0)}\right] =$$
$$= \Re\left[\alpha(1-\beta)\frac{y}{ix}\right] + \Re\left[\frac{\alpha\beta y}{1-\alpha+\alpha ix}\right] = 0 + \frac{y\left[\alpha\beta(1-\alpha)\right]}{|1-\alpha+\alpha ix|^2} \leq 0.$$

This contradicts the hypothesis of the theorem, therefore $p \prec q$ and the proof of Theorem 1 is complete. □

3 Fekete-Szegö problem

In 1933 M. Fekete and G. Szegö obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ for $f \in S$ and μ real number. For this reason, the determination of sharp upper bounds for the non-linear functional $|a_3 - \mu a_2^2|$ for any compact family $\mathbb{F}$ of functions $f \in \mathcal{A}$ is popularly known as the Fekete-Szegö problem for $\mathbb{F}$. For different subclasses of S, the Fekete-Szegö problem has been investigated by many authors (see [2], [4], [11]).

In this section we will solve the Fekete-Szegö problem for the class $H_{\alpha,\beta}$, where α and β are positive real numbers.

Theorem 2. *Let α, β, μ be positive real numbers. If the function f given by (1) belongs to the class $H_{\alpha,\beta}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{-4\mu}{(1+\alpha)^2} + \dfrac{2(\alpha-1)(1+\alpha\beta)+(\alpha+1)(5+\alpha)}{(1+2\alpha)(1+\alpha)^2} & , \quad \mu \leq \sigma_1, \\[2ex] \dfrac{1}{1+2\alpha} & , \quad \sigma_1 \leq \mu \leq \sigma_2, \\[2ex] \dfrac{4\mu}{(1+\alpha)^2} - \dfrac{2(\alpha-1)(1+\alpha\beta)+(\alpha+1)(3-\alpha)}{(1+2\alpha)(1+\alpha)^2} & , \quad \mu \geq \sigma_2. \end{cases}$$

where

$$\sigma_1 = \frac{1+3\alpha-\alpha\beta+\alpha^2\beta}{2(1+2\alpha)}, \qquad \sigma_2 = \frac{2+5\alpha+\alpha^2-\alpha\beta+\alpha^2\beta}{2(1+2\alpha)}$$

Proof. Let f be in the class H_α, β and let $p \in \mathcal{P}$. From (4) we obtain

$$\left\{(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] + \beta\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha zf'(z)}\right\} = p(z).$$

Since f has the Taylor series expansion (1) and $p(z) = 1+p_1z+p_2z^2+\ldots,\ z \in U$, we have

$$1+(1+\alpha)a_2z + \left[2(1+2\alpha)a_3 - (1+3\alpha-\alpha\beta+\alpha^2\beta)a_2^2\right]z^2 + \ldots = \qquad (7)$$
$$= 1+p_1z+p_2z^2+\ldots.$$

Therefore, equating the coefficients of z^2 and z^3 in (7), we obtain

$$a_2 = \frac{p_1}{1+\alpha}, \quad a_3 = \frac{1}{2(1+2\alpha}\left[p_2 + \frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)p_1^2}{(1+\alpha)^2}\right].$$

So, we have

$$a_3 - \mu a_2^2 = \frac{1}{2(1+2\alpha)}\left(p_2 - vp_1^2\right),$$

where

$$v = \frac{2(1+2\alpha)}{(1+\alpha)^2}\mu - \frac{1+3\alpha-\alpha\beta+\alpha^2\beta}{(1+\alpha)^2}. \tag{8}$$

Now, our result follows as an application of Lemma 3. □

4 Subclass of bi-univalent function

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let σ be the class of all functions $f \in \mathcal{S}$ such that the inverse function f^{-1} has an univalent analytic continuation to $\{|w| < 1\}$. The class σ, called the class of bi-univalent functions, was introduced by Levin [7] who showed that $|a_2| < 1.51$. Branan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$. On the other hand, Netanyahu [10] showed that $\max\limits_{f\in\sigma} |a_2| = \frac{4}{3}$. Several authors have studied similar problems in this direction (see [1] [5], [8], [12], [13]).

We notice that the class σ is not empty. For example, the following functions are members of σ:

$$z, \ \frac{z}{1-z}, \ -\log(1-z), \ \frac{1}{2}\log\frac{1+z}{1-z}.$$

However, the Koebe function is not a member of σ. Other examples of univalent functions that are not in the class σ are

$$z - \frac{z^2}{2}\ , \ \frac{z}{1-z^2}.$$

In the sequel we assume that φ is an analytic function with positive real part in the unit disk U, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and such that $\varphi(U)$ is symmetric with respect to the real axis. Assume also that:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots, \quad B_1 > 0. \tag{9}$$

Definition 2. *A function $f \in \mathcal{A}$ is said to be in the class $H_{\alpha,\beta}(\varphi)$, $\alpha \in [0,1]$, $\beta \geq 0$, if $f \in \sigma$ and satisfies the following conditions:*

$$(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] + \beta\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha zf'(z)} \prec \varphi(z),$$

and

$$(1-\beta)\left[(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1+\frac{wg''(w)}{g'(w)}\right)\right] + \beta\frac{wg'(w)+\alpha w^2 g''(w)}{(1-\alpha)g(w)+\alpha wg'(w)} \prec \varphi(w),$$

where g is the extension of f^{-1} to U.

Theorem 3. *If $f \in H_{\alpha,\beta}(\varphi)$ is in $\mathcal{A}$ then*

$$|a_2| \leq \frac{|\tau| B_1\sqrt{B_1}}{\sqrt{\left|[(1+\alpha)+\alpha\beta(1-\alpha)]B_1^2 - (B_2-B_1)(1+\alpha)^2\right|}}, \tag{10}$$

and

$$|a_3| \leq B_1\left[\frac{1}{1+\alpha} + \frac{\alpha\beta(1-\alpha)}{2(1+2\alpha)(1+\alpha)}\right] + \frac{|B_2-B_1|}{1+\alpha}. \tag{11}$$

Proof. Let $f \in H_{\alpha,\beta}(\varphi)$ and $g = f^{-1}$. Then there exist two analytic functions $u, v : U \to U$ with $u(0) = v(0) = 0$ such that:

$$\begin{aligned} &(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] + \beta\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha zf'(z)} = \varphi(u(z)) \text{ and} \\ &(1-\beta)\left[(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1+\frac{wg''(w)}{g'(w)}\right)\right] + \beta\frac{w'(w)+\alpha w^2 g''(w)}{(1-\alpha)g(w)+\alpha wg'(w)} = \varphi(v(w)). \end{aligned} \tag{12}$$

Define the functions p and q by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \ldots, \quad q(z) = \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \ldots$$

or equivalently,

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}z^2\right) + \ldots\right], \tag{13}$$

and

$$v(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}z^2\right) + \ldots\right]. \tag{14}$$

We observe that $p, q \in \mathcal{P}$ and, in view of Lemma 2, we have that $|p_n| \leq 2$ and $|q_n| \leq 2$, for $n \geq 1$.

Further, using (13) and (14) together with (9), it is evident that

$$\varphi(u(z)) = 1 + \frac{1}{2}B_1 p_1 z + \left([\frac{1}{2}B_1 \left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2\right] z^2 + \dots \tag{15}$$

and

$$\varphi(v(z)) = 1 + \frac{1}{2}B_1 q_1 z + \left[\frac{1}{2}B_1 \left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2\right] z^2 + \dots. \tag{16}$$

Therefore, in view of (12), (15) and (16) we have

$$(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] + \beta\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)}$$

$$= 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1 \left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2\right) z^2 + \dots, \tag{17}$$

and

$$(1-\beta)\left[(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right)\right] + \beta\frac{wg'(w) + \alpha w^2 g''(w)}{(1-\alpha)g(w) + \alpha w g'(w)}$$

$$= 1 + \frac{1}{2}B_1 q_1 w + \left(\frac{1}{2}B_1 \left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2\right) w^2 + \dots. \tag{18}$$

Since $f \in \sigma$ has the Taylor series expansion (1) and $g = f^{-1}$ the series expansion (2), we have

$$(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] + \beta\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)}$$

$$= 1 + (1+\alpha)a_2 z + \left[2(1+2\alpha)a_3 - (1+3\alpha-\alpha\beta+\alpha^2\beta)a_2^2\right] z^2 + \dots, \tag{19}$$

and

$$(1-\beta)\left[(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right)\right] + \beta\frac{wg'(w) + \alpha w^2 g''(w)}{(1-\alpha)g(w) + \alpha w g'(w)}$$

$$= 1 - (1+\alpha)a_2 w - \left[2(1+2\alpha)a_3 - (1+3\alpha-\alpha\beta+\alpha^2\beta)(a_3 - 2a_2^2)\right] w^2 + \dots. \tag{20}$$

Equating the coefficients in (17), (19) and (18), (20), we obtain

$$\begin{cases} (1+\alpha)a_2 = \frac{1}{2}B_1p_1, \\ 2(1+2\alpha)a_3 - (1+3\alpha-\alpha\beta+\alpha^2\beta)a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2, \\ -(1+\alpha)a_2 = \frac{1}{2}B_1q_1, \\ -2(1+2\alpha)(a_3-2a_2^2) - (1+3\alpha-\alpha\beta+\alpha^2\beta)a_2^2 = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2q_1^2. \end{cases} \tag{21}$$

From the first and the third equation of the system (21) it follows that

$$p_1 = -q_1, \tag{22}$$

and

$$a_2^2 = \left(\frac{B_1p_1\tau}{4(1+\gamma)}\right)^2. \tag{23}$$

Now, (22), (23) and the next two equations of the system (21) lead to

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{4[(1+\alpha)+\alpha\beta(1-\alpha)]B_1^2 - 4(B_2-B_1)(1+\alpha)^2}. \tag{24}$$

Thus, in view of Lemma 2, we obtain the desired estimation of $|a_2|$.

From the third and the fourth equation of (21), we obtain

$$a_3 = \frac{1}{2}B_1p_2\frac{3+5\alpha+\alpha\beta(1-\alpha)}{4(1+2\alpha)(1+\alpha)} + \frac{1+3\alpha-\alpha\beta(1-\alpha)}{4(1+2\alpha)(1+\alpha)} + \frac{1}{4}p_1^2(B_2-B_1)\frac{1}{1+\alpha},$$

which yields to the estimate given by (11) and so the proof of Theorem 3 is completed.

□

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