DOI: 10.1515/auom-2015-0017
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 23(1),2015, 247–266

An extragradient iterative scheme for common fixed point problems and variational inequality problems with applications

Adrian Petruşel, D.R. Sahu and Vidya Sagar

Abstract

In this paper, by combining a modified extragradient scheme with the viscosity approximation technique, an iterative scheme is developed for computing the common element of the set of fixed points of a sequence of asymptotically nonexpansive mappings and the set of solutions of the variational inequality problem for an α-inverse strongly monotone mapping. We prove a strong convergence theorem for the sequences generated by this scheme and give some applications of our convergence theorem.

1 Introduction

Let C be a nonempty subset of a real Hilbert space H with inner product $\langle ., . \rangle$ and norm $\|.\|$, respectively. A mapping $A : C \to H$ is called (see ([15]))
(i) monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \text{for all } u, v \in C;$$

(ii) η-strongly monotone if there exists a positive real number η such that

$$\langle Au - Av, u - v \rangle \geq \eta \|u - v\|^2, \quad \text{for all } u, v \in C;$$

Key Words: Modified extragradient method, viscosity approximation method, α-inverse strongly monotone mapping, sequence of asymptotically nonexpansive mappings, variational inequalities, AF point property.
2010 Mathematics Subject Classification: Primary 47H10; Secondary 54H25.
Received: 5 May, 2014.
Revised: 20 June, 2014.
Accepted: 27 June, 2014.

(iii) α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha \|Au - Av\|^2, \text{ for all } u, v \in C;$$

(iv) k-Lipschitzian if there exists $k > 0$ such that

$$\|Au - Av\| \leq k \|u - v\|, \quad \text{for all } u, v \in C;$$

(v) k-contraction if it is k-Lipschitzian with $k < 1$;
(vi) nonexpansive if

$$\|Au - Av\| \leq \|u - v\|, \quad \text{for all } u, v \in C.$$

Let C be a nonempty subset of a real Hilbert space H and $\{S_n\}$ a sequence of mappings from C into itself. Then the sequence $\{S_n\}_{n\in\mathbb{N}}$ is called a sequence of asymptotically nonexpansive mappings ([12]) on C if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|S_n u - S_n v\| \leq k_n \|u - v\|, \text{ for all } u, v \in C \text{ and } n \in \mathbb{N}.$$

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. A variational inequality problem is the problem of finding $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \text{for all } v \in C, \tag{1}$$

where A is a nonlinear mapping from C into H. The set of solutions of the variational inequality problem (1.1) is denoted by Ω. We denote by $F(S)$ the set of fixed points of mapping $S : C \to C$.

We give some examples of α-inverse strongly monotone mappings. Let H be a Hilbert space and C a nonempty closed convex subset of H. If T is a nonexpansive mapping from C into itself, then $A := I - T$ is $\frac{1}{2}$-inverse strongly monotone and $\Omega = F(T)$. Also, if A is η-strongly monotone and k-Lipschitz, then A is $\frac{\eta}{k^2}$-inverse strongly monotone. For the reverse implication, let us observe that there are examples of mappings which are inverse strongly monotone, but not strongly monotone. The metric projection P_C is one of these, see also [16]. Recall that mapping $T : C \to C$ is called λ-strictly pseudocontractive on C if there exists $\lambda \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda \|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in C.$$

Notice that if $T : C \to C$ is λ-strictly pseudocontractive, then the mapping $A := I - T$ is $\frac{1-\lambda}{2}$-inverse strongly monotone.

The variational inequality (1.1) was introduced by Stampacchia [14] in 1964. It has been shown that a large class of problems arising in engineering and applied sciences ([8] and the references therein) can be studied in the framework of the variational inequalities. It is known that the element $u \in C$ is a solution of the variational inequality problem (1.1) if and only if u satisfies the relation:

$$u = P_C(u - \lambda Au),$$

where $\lambda > 0$ is a constant and P_C is the metric projection mapping of H onto C.

It is obvious that fixed point problems and variational inequality problems are equivalent. This approach shows that a variational inequality can be regarded as a fixed point problem and, in this respect, the following iterative method could be important, in order to solve approximatively a variational inequality problem:
For a given $u_0 \in C$, compute u_{n+1} by the iterative scheme:

$$u_{n+1} = P_C(u_n - \lambda Au_n), \text{ for } n = 0, 1, 2, \ldots.$$

These ideas were the starting point of a large number of papers dealing with the problem of approximating the solution of a variational inequality problem, sometimes in connection to other related problems, such as the problem of finding and approximating the fixed points of a nonexpansive mapping.

Korpelevich [6] introduced an extragradient method and proved that the sequences generated by the extragradient method converge to the same point $z \in \Omega$.

Recently, Nadezhkina and Takahashi [7], Zeng and Yao [18] introduced new iterative schemes for finding an element of $F(S) \cap \Omega$ and obtained the weak and strong convergence theorems respectively. Chen, Zhang and Fan [2] introduced an iterative scheme by viscosity approximation method for finding a common element of the fixed point set of a nonexpansive operator and the solution set of a variational inequality problem and proved a strong convergence theorem.

More recently, Petruşel and Yao [10] introduced a modified extragradient scheme by viscosity approximation method and obtained a strong convergence result for an explicit scheme in a Hilbert space.

In this paper, inspired by Petruşel and Yao [10], we prove a strong convergence theorem for computation of the common element of the set of fixed points of a sequence of asymptotically nonexpansive mappings and the set of solutions of the variational inequality problem for an α-inverse strongly monotone mapping. Our results generalize the result of Petruşel and Yao [10] to the case of a sequence of asymptotically nonexpansive mappings and extend the results of Nadezhkina and Takahashi [7], Zeng and Yao [18] and Chen, Zhang and Fan [2]. Other related results are given in [17]-[4].

2 Preliminaries

Throughout this paper, H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|.\|$. We denote by I the identity operator of H. Also, we denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. The symbol $\mathbb{N}$ stands for the set of all natural numbers. Let C be a nonempty subset of H and $\mathcal{S} := \{S_n\}_{n\in\mathbb{N}}$ a sequence of self-mappings from C into itself. We denote by $F(\mathcal{S})$ the set of common fixed points of the sequence $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap\limits_{n=1}^{\infty} F(S_n)$.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C is characterized by the following properties (see [5, 1]):
(A) $P_C(x) \in C$, for all $x \in H$;
(B) $\langle x - P_C(x), P_C(x) - y\rangle \geq 0$, for all $x \in H, y \in C$;
(C) $\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$, for all $x \in H, y \in C$.

It is also known that H satisfies the Opial property (see [1, 9]), i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$, we have $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if the following implication holds: if for

$(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$, then $f \in Tx$.

Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and let $N_C(v)$ be the normal cone to C at $v \in C$, i.e.,

$$N_C(v) = \{w \in H : \langle v - y, w\rangle \geq 0, \text{ for all } \ y \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C(v), & \text{if} \quad v \in C, \\ \emptyset, & \text{if} \quad v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$, see ([11]).

We present now an important property of the α-inverse strongly monotone mappings.

Lemma 1. ([16]) *Let C be a nonempty subset of a real Hilbert space H. Let $\alpha > 0$ and $A : C \to H$ an α-inverse strongly monotone. Then, A is $\frac{1}{\alpha}$-Lipschitz continuous. Moreover, for all $u, v \in C$ and each $\lambda > 0$, we have*

$$\begin{aligned} \|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda \langle u - v, Au - Av\rangle \\ &\quad + \lambda^2 \|Au - Av\|^2 . \end{aligned}$$

As consequence, $(I - \lambda A)$ is a nonexpansive mapping from C into H if $\lambda \leq 2\alpha$.

Now we state an existence result for the solution of the variational inequality problem for inverse strongly monotone mappings.

Theorem 1. ([16]) *Let C be a closed convex bounded subset of a real Hilbert space H and let $A : C \to H$ be α-inverse strongly monotone. Then Ω is non-empty.*

In the proof of the main results, we need the following lemmas.

Lemma 2. *(Schu [13]) Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$, for all $n \in \mathbb{N}$ and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that*

$$\limsup_{n\to\infty} \|v_n\| \leq c, \ \limsup_{n\to\infty} \|w_n\| \leq c \ \text{and} \ \lim_{n\to\infty} \|\alpha_n v_n + (1 - \alpha_n)w_n\| = c$$

for some $c \geq 0$. *Then,* $\lim_{n\to\infty} \|v_n - w_n\| = 0$.

Lemma 3. *(Xu [17]) Let* $\{\alpha_n\}_{n=0}^{\infty}$ *be a sequence of non negative real numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\beta_n, \quad \text{for all } n \in \mathbb{N},$$

where $\{\gamma_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are sequences of real numbers which satisfy the conditions:*

(i) $\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*

(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 4. ([3]) *Assume that* S *is an asymptotically nonexpansive self-mapping of a nonempty closed convex subset* C *of a real Hilbert space* H*. Then* $I - S$ *is demiclosed, i.e., if* $\{x_n\}$ *is a sequence in* C *weakly converging to some* $x \in C$ *and the sequence* $\{(I-S)x_n\}$ *strongly converges to* 0*, then* $x \in F(S)$.

AF point property ([12]) Let C be a nonempty subset of a real Hilbert space H, and let $\mathcal{S} := \{S_n\}$ be a sequence of self-mappings on C. A sequence $\{x_n\}$ in C is said to have the *approximate fixed point property* (in short AF point property) for $\{S_n\}$ if $\lim_{n\to\infty} \|x_n - S_n x_n\| = 0$.

Condition $\mathcal{D}$ ([12]) Let C be a nonempty closed convex subset of a real Hilbert space H and $\mathcal{S} := \{S_n\}$ a sequence of self-mappings on C. A family $\{I - S_n\}$ is said to be *demi-closed at zero* if for every bounded sequence $\{x_n\}$ in C, the following condition holds:

$$(\mathcal{D}) \qquad \{x_n - S_n x_n\} \to 0 \Rightarrow w_w(x_n) \subset F(\mathcal{S}),$$

where $w_w(x_n)$ is the set of weak cluster points of the sequence $\{x_n\}$.

3 Main results

Theorem 2. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H*. Let* $A : C \to H$ *be an* α*-inverse strongly monotone mapping and* $\mathcal{S} := \{S_n\}$ *a sequence of asymptotically nonexpansive mappings from* C *into itself with sequence* $\{k_n\}$ *such that* $F(\mathcal{S}) \cap \Omega \neq \emptyset$. *Assume that* $\mathcal{S}$ *satisfies condition*

$(\mathcal{D})$ and $f : C \to C$ is a k-contraction. For arbitrary $x_1 \in C$, consider the sequences $\{x_n\}$ and $\{y_n\}$ generated by the following iterative process:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n P_C(x_n - \lambda_n A y_n), \text{ for all } n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are two sequences of positive numbers with $\{\alpha_n\} \subset (0, \overline{\alpha})$ and $\{\lambda_n\} \subset [a,b]$, with $0 < a < b < \alpha(1-\delta)$ $(for some\ \overline{\alpha}, \delta \in (0,1))$ satisfying the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(iv) $\lim_{n\to\infty} \dfrac{\|S_n t_n - S_{n+1} t_n\|}{\alpha_{n+1}} = 0$;

(v) $\lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0.$

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point p, such that p is the unique solution in $F(\mathcal{S}) \cap \Omega$ of the following variational inequality:

$$\langle f(p) - p, y - p \rangle \leq 0 \ for\ all\ y \in F(\mathcal{S}) \cap \Omega. \tag{3}$$

Proof. Denote $t_n := P_C(x_n - \lambda_n A y_n), \forall n \in \mathbb{N}$ and let $u \in F(\mathcal{S}) \cap \Omega$. Then $u = P_C(u - \lambda_n A u)$. We proceed in the following steps.

Step 1. *$\{x_n\}$ is bounded.*

Taking $x := x_n - \lambda_n A y_n$ and $y := u$ in relation (C), we have

$$\begin{aligned} \|t_n - u\|^2 &\leq \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - u\|^2 - 2\lambda_n \langle A y_n, x_n - u \rangle + \lambda_n^2 \|A y_n\|^2 - \|x_n - t_n\|^2 \\ &\quad + 2\lambda_n \langle A y_n, x_n - t_n \rangle - \lambda_n^2 \|A y_n\|^2 \\ &= \|x_n - u\|^2 + 2\lambda_n \langle A y_n, u - t_n \rangle - \|x_n - t_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 - 2\lambda_n \langle A y_n - A u, y_n - u \rangle \\ &\quad - 2\lambda_n \langle A u, y_n - u \rangle + 2\lambda_n \langle A y_n, y_n - t_n \rangle \\ &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + 2 \langle x_n - \lambda_n A y_n - y_n, t_n - y_n \rangle . \end{aligned}$$

From (B), we obtain

$$\begin{aligned}\langle x_n - \lambda_n A y_n - y_n, t_n - y_n \rangle &= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n \rangle \\ &\quad + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n \rangle \\ &= -\langle x_n - \lambda_n A x_n - y_n, y_n - t_n \rangle \\ &\quad + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n \rangle \\ &\leq \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n \rangle \\ &\leq \frac{\lambda_n}{\alpha} \|x_n - y_n\| \|t_n - y_n\|.\end{aligned}$$

Hence

$$\begin{aligned}\|t_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n \rangle \\ &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + \frac{{\lambda_n}^2}{\alpha^2} \|x_n - y_n\|^2 + \|t_n - y_n\|^2 \\ &= \|x_n - u\|^2 + (\frac{{\lambda_n}^2}{\alpha^2} - 1) \|x_n - y_n\|^2 \\ &\leq \|x_n - u\|^2.\end{aligned}$$

From (3.1), we have

$$\begin{aligned}\|x_{n+1} - u\| &= \|\alpha_n f(x_n) + (1-\alpha_n) S_n t_n - u\| \\ &\leq \alpha_n \|f(x_n) - u\| + (1-\alpha_n) \|S_n t_n - u\| \\ &\leq \alpha_n \|f(x_n) - f(u)\| + \alpha_n \|f(u) - u\| + (1-\alpha_n) \|S_n t_n - Su\| \\ &\leq \alpha_n k \|x_n - u\| + \alpha_n \|f(u) - u\| + k_n(1-\alpha_n) \|t_n - u\| \\ &\leq \alpha_n k \|x_n - u\| + \alpha_n \|f(u) - u\| + k_n(1-\alpha_n) \|x_n - u\| \\ &= [1 - \alpha_n(1-k)] \|x_n - u\| + \alpha_n \|f(u) - u\| \\ &\quad + (1-\alpha_n)(k_n - 1) \|x_n - u\| \\ &= [1 - \alpha_n(1-k)] \|x_n - u\| + (1-\alpha_n)(k_n - 1) \|x_n - u\| + \mu_n,\end{aligned}$$

where $\mu_n = \alpha_n \|f(u) - u\|$. Note $\lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$ and $\lim_{n\to\infty} \alpha_n = 0$, so there exist two constants $\beta \in (0,1)$ with $(\beta - k) \in (0,1)$ and $K_1 > 0$ such that $\frac{k_n - 1}{\alpha_n} \leq \frac{1-\beta}{1-\alpha_n}$ and $\frac{\mu_n}{\alpha_n} \leq K_1$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}\|x_{n+1} - u\| &\leq [1 - (\beta - k)\alpha_n] \|x_n - u\| + \alpha_n K_1 \\ &\leq \max\left\{ \|x_n - u\|, \frac{K_1}{\beta - k} \right\}.\end{aligned}$$

It follows that $\{x_n\}$ is bounded.

Step 2. $\|x_{n+1} - x_n\| \to 0$ *as* $n \to \infty$.

By Step 1, $\{x_n\}$ is bounded. So $\{f(x_n)\}$, $\{Ax_n\}$, $\{t_n\}$, $\{At_n\}$, $\{S_n t_n\}$ are bounded. Observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(\alpha_n - \alpha_{n-1})[f(x_{n-1}) - S_{n-1}t_{n-1}] \\
&\quad + (1-\alpha_n)(S_n t_n - S_{n-1}t_{n-1}) + \alpha_n[f(x_n) - f(x_{n-1})]\| \\
&\leq |\alpha_n - \alpha_{n-1}| \, \|f(x_{n-1}) - S_{n-1}t_{n-1}\| \\
&\quad + (1-\alpha_n) \|S_n t_n - S_{n-1}t_{n-1}\| + \alpha_n \|f(x_n) - f(x_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}| \, \|f(x_{n-1}) - S_{n-1}t_{n-1}\| \\
&\quad + (1-\alpha_n) \|S_n t_n - S_{n-1}t_{n-1}\| + \alpha_n k \|x_n - x_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}| \, \|f(x_{n-1}) - S_{n-1}t_{n-1}\| \\
&\quad + k_n(1-\alpha_n) \|t_n - t_{n-1}\| + \alpha_n k \|x_n - x_{n-1}\| + (1-\alpha_n)\epsilon_{n-1},
\end{aligned}$$

where $\epsilon_{n-1} = \|S_n t_{n-1} - S_{n-1}t_{n-1}\|$. Since $\lambda_n < \alpha(1-\delta) < 2\alpha$, from Proposition 1, we have

$$\begin{aligned}
\|t_{n+1} - t_n\| &= \|P_C(x_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(x_n - \lambda_n Ay_n)\| \\
&\leq \|x_{n+1} - \lambda_{n+1}Ay_{n+1} - x_n + \lambda_n Ay_n\| \\
&\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \, \|Ay_n\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq [k_n(1-\alpha_n) + k\alpha_n] \|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \, \|f(x_{n-1}) - S_{n-1}t_{n-1}\| \\
&\quad + k_n(1-\alpha_n) |\lambda_n - \lambda_{n-1}| \, \|Ay_{n-1}\| + (1-\alpha_n)\epsilon_{n-1} \\
&\leq [k_n(1-\alpha_n) + k\alpha_n] \|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| L + |\lambda_n - \lambda_{n-1}| M + (1-\alpha_n)\epsilon_{n-1} \\
&\leq [1 - (1-k)\alpha_n] \|x_n - x_{n-1}\| + (k_n - 1)N \\
&\quad + |\alpha_n - \alpha_{n-1}| L + |\lambda_n - \lambda_{n-1}| M + \epsilon_{n-1},
\end{aligned}$$

where $L = \sup_{n\in\mathbb{N}} \|f(x_n) - S_n t_n\|$, $M = \sup_{n\in\mathbb{N}} \|Ay_n\|$ and $N = \sup_{n\in\mathbb{N}} \|x_n - x_{n+1}\|$. Note $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$ and $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_{n+1}} = 0$. By Lemma 3, we obtain $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Step 3. $\|x_n - y_n\| \to 0$ *and* $\|t_n - y_n\| \to 0$ *as* $n \to \infty$.
From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)S_n t_n - u\|^2 \\
&= \|\alpha_n(f(x_n) - f(u)) + \alpha_n(f(u) - u) + (1-\alpha_n)(S_n t_n - u)\|^2 \\
&\leq \alpha_n[\|f(x_n) - f(u)\| + \|f(u) - u\|]^2 \\
&\quad + (1-\alpha_n)\|S_n t_n - u\|^2 \\
&\leq \alpha_n[k\|x_n - u\| + \|f(u) - u\|]^2 + k_n^2(1-\alpha_n)\|t_n - u\|^2 \\
&\leq k^2\alpha_n\|x_n - u\|^2 + \alpha_n[2k\|x_n - u\|\|f(u) - u\| \\
&\quad + \|f(u) - u\|^2] + k_n^2(1-\alpha_n)[\|x_n - u\|^2 \\
&\quad + (\frac{\lambda_n^2}{\alpha^2} - 1)\|x_n - y_n\|^2] \\
&\leq [k^2\alpha_n + k_n^2(1-\alpha_n)]\|x_n - u\|^2 \\
&\quad + \alpha_n[2k\|x_n - u\|\|f(u) - u\| + \|f(u) - u\|^2] \\
&\quad + k_n^2(1-\alpha_n)(\frac{\lambda_n^2}{\alpha^2} - 1)\|x_n - y_n\|^2 .
\end{aligned}$$

Hence

$$\begin{aligned}
(1-\overline{\alpha})(2\delta - \delta^2)\|x_n - y_n\|^2 &\leq k_n^2(1-\alpha_n)(1 - \frac{\lambda_n^2}{\alpha^2})\|x_n - y_n\|^2 \\
&\leq [k^2\alpha_n + k_n^2(1-\alpha_n)]\|x_n - u\|^2 - \|x_{n+1} - u\|^2 \\
&\quad + \alpha_n[2k\|x_n - u\|\|f(u) - u\| + \|f(u) - u\|^2] \\
&\leq [k^2\alpha_n + (1-\alpha_n)]\|x_n - u\|^2 - \|x_{n+1} - u\|^2 \\
&\quad + (1-\alpha_n)(k_n^2 - 1)\|x_n - u\|^2 \\
&\quad + \alpha_n[2k\|x_n - u\|\|f(u) - u\| + \|f(u) - u\|^2] \\
&\leq (\|x_n - u\|^2 - \|x_{n+1} - u\|^2) \\
&\quad + (k_n - 1)(k_n + 1)\|x_n - u\|^2 \\
&\quad + \alpha_n[2k\|x_n - u\|\|f(u) - u\| + \|f(u) - u\|^2] \\
&= [(\|x_n - u\| - \|x_{n+1} - u\|)(\|x_n - u\| + \|x_{n+1} - u\|)] \\
&\quad + (k_n - 1)\sup_{i\in\mathbb{N}}(k_i + 1)R^2 \\
&\quad + \alpha_n[2k\|x_n - u\|\|f(u) - u\| + \|f(u) - u\|^2] \\
&\leq 2\|x_n - x_{n+1}\|R + (k_n - 1)\sup_{i\in\mathbb{N}}(k_i + 1)R^2 \\
&\quad + \alpha_n[2Rk\|f(u) - u\| + \|f(u) - u\|^2],
\end{aligned}$$

where R is a positive constant such that $\|x_n - u\| \leq R$ for all $n \in \mathbb{N}$. Note $k_n \to 1, \|x_{n+1} - x_n\| \to 0$ and $\alpha_n \to 0$ as $n \to \infty$, we have, $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Observe that

$$\begin{aligned} \|y_n - t_n\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(x_n - \lambda_n A y_n)\| \\ &\leq \|(x_n - \lambda_n A x_n) - (x_n - \lambda_n A y_n)\| \\ &\leq \frac{\lambda_n}{\alpha} \|x_n - y_n\| \to 0 \text{ as } \ n \to \infty. \end{aligned}$$

Step 4. $\limsup_{n\to\infty} \langle f(p) - p, S_n t_n - p\rangle \leq 0$, *where* $p = P_{F(\mathcal{S})\cap\Omega} f(p)$.

For $u \in F(\mathcal{S}) \cap \Omega$, we have

$$\begin{aligned} \|S_n y_n - x_{n+1}\| &\leq \|S_n y_n - S_n t_n\| + \|S_n t_n - x_{n+1}\| \\ &\leq k_n \|y_n - t_n\| + \alpha_n \|S_n t_n - f(x_n)\| \\ &\leq k_n \|y_n - t_n\| + \alpha_n [\|S_n t_n - u\| + \|u - f(x_n)\|] \\ &\leq k_n \|y_n - t_n\| + \alpha_n [k_n \|t_n - u\| + \|u - f(x_n)\|] \\ &\leq k_n \|y_n - t_n\| + \alpha_n \left[k_n \max\left\{\|x_n - u\|, \frac{K_1}{\beta - k}\right\} + \|u - f(x_n)\|\right]. \end{aligned}$$

Hence $\|S_n y_n - x_{n+1}\| \to 0$ as $n \to \infty$. Note that $\|x_n - y_n\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned} \|S_n x_n - x_n\| &\leq \|S_n x_n - S_n y_n\| + \|S_n y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \\ &\leq k_n \|x_n - y_n\| + \|S_n y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \to 0 \text{ as } \ n \to \infty, \end{aligned}$$

and

$$\begin{aligned} \|S_n t_n - t_n\| &\leq \|S_n t_n - S_n x_n\| + \|S_n x_n - x_n\| + \|x_n - t_n\| \\ &\leq k_n \|t_n - x_n\| + \|S_n x_n - x_n\| + \|x_n - t_n\| \\ &\leq \|S_n x_n - x_n\| + (1 + k_n) \|x_n - t_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Now, let us choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty} \langle f(p) - p, S_n t_n - p\rangle = \lim_{i\to\infty} \langle f(p) - p, S_{n_i} t_{n_i} - p\rangle .$$

For the convenience we will denote this subsequence by $\{t_n\}$ too. As $\{t_n\}$ is bounded, we have that a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ converges weakly to some $z \in C$. Since $\|x_n - y_n\| \to 0$ and $\|y_n - t_n\| \to 0$ as $n \to \infty$, we have that

$\{x_{n_i}\}$ and $\{y_{n_i}\}$ converges weakly to $z \in C$. Also, since $\lim_{n\to\infty} \|S_n t_n - t_n\| = 0$, by condition $(\mathcal{D})$, we get that $z \in F(\mathcal{S})$. From the above arguments, we have

$$\limsup_{n\to\infty} \langle f(p) - p, S_n t_n - p\rangle = \lim_{i\to\infty} \langle f(p) - p, S_{n_i} t_{n_i} - p\rangle = \langle f(p) - p, z - p\rangle .$$

Notice now that, in order to prove that $\limsup_{n\to\infty} \langle f(p) - p, S_n t_n - p\rangle \leq 0$, it suffices to show that $z \in F(\mathcal{S}) \cap \Omega$.

Now, let us show that $z \in \Omega$. Let

$$Tv = \begin{cases} Av + N_C(v), & \text{if} \quad v \in C, \\ \emptyset, & \text{if} \quad v \notin C. \end{cases}$$

T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Av + N_C(v)$ and hence $w - Av \in N_C(v)$. Thus, we have $\langle v - u, w - Av\rangle \geq 0$, for all $u \in C$.

On the other hand, from $t_n := P_C(x_n - \lambda_n Ay_n)$ and $v \in C$, we have $\langle x_n - \lambda_n Ay_n - t_n, t_n - v\rangle \geq 0$, and hence $\left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + Ay_n \right\rangle \geq 0$. Therefore, from $w - Av \in N_C(v)$ and $t_{n_i} \in C$, We have

$$\begin{aligned} \langle v - t_{n_i}, w\rangle &\geq \langle v - t_{n_i}, Av\rangle \\ &\geq \langle v - t_{n_i}, Av\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \right\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle \\ &\quad - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle . \end{aligned}$$

Hence, letting $n_i \to \infty$ we obtain $\langle v - z, w\rangle \geq 0$. Thus, $z \in T^{-1}0$ together with the maximal monotonicity of T imply $z \in \Omega$.

Step 5. *$x_n \to p$ as $n \to \infty$, where $p = P_{F(\mathcal{S})\cap\Omega} f(p)$. i.e., p is the unique solution in $F(\mathcal{S}) \cap \Omega$ of the variational inequality*

$$\langle f(p) - p, y - p\rangle \leq 0, \ \ \textit{for all } y \in F(\mathcal{S}) \cap \Omega.$$

We have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(S_nt_n-p)\|^2 \\
&= \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\|S_nt_n-p\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n)-p, S_nt_n-p\rangle \\
&\le \alpha_n^2\|f(x_n)-p\|^2+k_n^2(1-\alpha_n)^2\|t_n-p\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle f(x_n)-f(p), S_nt_n-p\rangle \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(p)-p, S_nt_n-p\rangle \\
&\le \alpha_n^2\|f(x_n)-p\|^2+k_n^2(1-\alpha_n)^2\|x_n-p\|^2 \\
&\quad + 2kk_n\alpha_n(1-\alpha_n)\|x_n-p\|\,\|t_n-p\| \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(p)-p, S_nt_n-p\rangle \\
&\le \alpha_n^2\|f(x_n)-p\|^2+k_n^2(1-\alpha_n)^2\|x_n-p\|^2 \\
&\quad + 2kk_n\alpha_n(1-\alpha_n)\|x_n-p\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(p)-p, S_nt_n-p\rangle \\
&\le \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\|x_n-p\|^2 \\
&\quad + 2k\alpha_n(1-\alpha_n)\|x_n-p\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(p)-p, S_nt_n-p\rangle+(k_{n-1})\Gamma \\
&= [1-\alpha_n\{2-\alpha_n-2k(1-\alpha_n)\}]\|x_n-p\|^2+\alpha_n^2\|f(x_n)-p\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(p)-p, S_nt_n-p\rangle+(k_{n-1})\Gamma \\
&= (1-\gamma_n)\|x_n-p\|^2+\gamma_n\beta_n+(k_{n-1})\Gamma,
\end{aligned}
$$

where $\Gamma>0$ is some constant, $\gamma_n=\alpha_n(2-\alpha_n-2k(1-\alpha_n))$ and

$$\beta_n=\frac{\alpha_n\|f(x_n)-p\|^2+2(1-\alpha_n)\langle f(p)-p, S_nt_n-p\rangle}{2-\alpha_n-2k(1-\alpha_n)}.$$

Since $k_n\to 1$ and $\gamma_n\to 0$ as $n\to\infty$, $\sum_{n=1}^{\infty}\gamma_n=\infty$ and $\limsup_{n\to\infty}\beta_n\le 0$, by applying Lemma 3 and using Step 4, we obtain $x_n\to p$ as $n\to\infty$. Since $\|x_n-y_n\|\to 0$ and $\|y_n-t_n\|\to 0$ as $n\to\infty$, we also have $y_n\to p$ and $t_n\to p$ as $n\to\infty$. The proof is now complete. □

4 Applications

In this section, we present some applications of our main result.

Using Theorem 2, we state a strong convergence theorem for the common fixed point of a sequence of asymptotically nonexpansive mappings and a strictly pseudocontractive mapping.

Theorem 3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strictly pseudocontractive and $\mathcal{S} := \{S_n\}$ a sequence of asymptotically nonexpansive mappings from C into itself with sequence $\{k_n\}$ such that $F(\mathcal{S}) \cap F(T) \neq \emptyset$. Assume that $\mathcal{S}$ satisfies condition ($\mathcal{D}$) and $f : C \to C$ is a k- contraction. For arbitrary $x_1 \in C$, consider the sequences $\{x_n\}$ and $\{y_n\}$ generated by the following iterative process:*

$$\begin{cases} x_1 \in C, \\ y_n = (1-\lambda_n)x_n + \lambda_n T x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n(x_n - \lambda_n(y_n - Ty_n)), \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are two sequences of positive numbers with $\{\alpha_n\} \subset (0, \overline{\alpha})$ and $\{\lambda_n\} \subset [a, b]$, with $0 < a < b < \alpha(1-\delta)$ (for some $\overline{\alpha}, \delta \in (0,1)$) satisfying the conditions (i)- (v) of Theorem 2. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point p, such that p is the unique solution in $F(\mathcal{S}) \cap F(T)$ of the following variational inequality:

$$\langle f(p) - p, y - p \rangle \leq 0, \quad \text{for all } y \in F(\mathcal{S}) \cap F(T).$$

Proof. Put $A := I - T$ in Theorem 2. Then A is $\frac{1-\lambda}{2}$-inverse strongly monotone. We have that $F(T) = \Omega$, $P_C(x_n - \lambda_n A x_n) = x_n - \lambda_n A x_n = (1-\lambda_n)x_n + \lambda_n T x_n$ and $P_C(x_n - \lambda_n A y_n) = x_n - \lambda_n(y_n - Ty_n)$. So, by Theorem 2, we obtain the desired result. □

Theorem 3 extends the results of Theorem 3.1 of Petruşel and Yao [10] and Theorem 4.1 of Chen, Zhang and Fan [2].

The following theorem extends the results of Theorem 3.2 of Petruşel and Yao [10] and Theorem 4.1 of Zeng and Yao [18].

Theorem 4. *Let H be a real Hilbert space, $A : H \to H$ be an α-inverse strongly monotone mapping and $\mathcal{S} := \{S_n\}$ a sequence of asymptotically nonexpansive mappings from H into itself with sequence $\{k_n\}$ such that $F(\mathcal{S}) \cap A^{-1}(0) \neq \emptyset$. Assume that $\mathcal{S}$ satisfies condition ($\mathcal{D}$) and $f : C \to C$ is a k-contraction. For arbitrary $x_1 \in C$, consider the sequences $\{x_n\}$ and $\{y_n\}$ generated by the following iterative process:*

$$\begin{cases} x_1 \in C, \\ y_n = x_n - \lambda_n A x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n(x_n - \lambda_n A y_n), \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are two sequences of positive numbers with* $\{\alpha_n\} \subset (0, \overline{\alpha})$ *and* $\{\lambda_n\} \subset [a, b]$, *with* $0 < a < b < \alpha(1-\delta)$ *(for some* $\overline{\alpha}, \delta \in (0,1)$*) satisfying the conditions (i)- (v) of Theorem 2. Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* p*, such that* p *is the unique solution in* $F(\mathcal{S}) \cap A^{-1}(0)$ *of the following variational inequality:*

$$\langle f(p) - p, y - p \rangle \leq 0, \quad \textit{for all } y \in F(\mathcal{S}) \cap A^{-1}(0).$$

Proof. We have $A^{-1}(0) = \Omega$ and $P_H = I$. The conclusion follows from Theorem 2. □

Theorem 5 extends Theorem 3.3 of Petruşel and Yao [10] and Theorem 4.2 of Zeng and Yao [18].

Theorem 5. *Let* H *be a real Hilbert space and* $A : H \to H$ *be an* α*-inverse strongly monotone mapping. For each* $n \in \mathbb{N}$*, let* B_n *be a maximal monotone operator from* H *into* 2^H *with resolvent operator* $J_r^{B_n}$ *for some* $r > 0$ *such that* $F(\mathcal{J}) \cap A^{-1}(0) \neq \emptyset$*. Assume that* $\mathcal{J} = \{J_r^{B_n}\}$ *satisfies condition* $(\mathcal{D})$ *and* $f : C \to C$ *is a* k*-contraction. For arbitrary* $x_1 \in C$*, consider the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by the following iterative process:*

$$\begin{cases} x_1 \in H, \\ y_n = x_n - \lambda_n A x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_r^{B_n}(x_n - \lambda_n A y_n), \quad \textit{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are two sequences of positive numbers with* $\{\alpha_n\} \subset (0, \overline{\alpha})$ *and* $\{\lambda_n\} \subset [a, b]$, *with* $0 < a < b < \alpha(1-\delta)$ *(for some* $\overline{\alpha}, \delta \in (0,1)$*) satisfying the conditions (i)- (iii) of Theorem 2 and (iv)′:*

(iv)′ $\displaystyle \lim_{n\to\infty} \frac{\left\| J_r^{B_n} t_n - J_r^{B_{n+1}} t_n \right\|}{\alpha_{n+1}} = 0.$

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $p \in F(\mathcal{J}) \cap A^{-1}(0)$*, where* p *is the unique solution in* $F(\mathcal{J}) \cap A^{-1}(0)$ *of the following variational inequality:*

$$\langle f(p) - p, y - p \rangle \leq 0, \quad \textit{for all } y \in F(\mathcal{J}) \cap A^{-1}(0).$$

Proof. We have $A^{-1}(0) = \Omega$. The conclusion follows from Theorem 2, by putting $P_H = I$ and $J_r^{B_n} = S_n$. □

We now impose some condition on $\mathcal{S}$ to fulfill condition $(\mathcal{D})$ in Theorem 2.

Theorem 6. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse strongly monotone mapping and S an asymptotically nonexpansive mapping from C into itself with sequence $\{k_n\}$ such that $F(S) \cap \Omega \neq \emptyset$ and $f : C \to C$ a k-contraction. For arbitrary $x_1 \in C$, consider the sequences $\{x_n\}$ and $\{y_n\}$ generated by the following iterative process:*

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S^n P_C(x_n - \lambda_n A y_n), \quad \textit{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are two sequences of positive numbers with $\{\alpha_n\} \subset (0, \overline{\alpha})$ and $\{\lambda_n\} \subset [a, b]$, with $0 < a < b < \alpha(1-\delta)$ (for some $\overline{\alpha}, \delta \in (0,1)$) satisfying the conditions (i)- (v) of Theorem 2 with $S_n = S^n$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point p, such that p is the unique solution in $F(S) \cap \Omega$ of the following variational inequality:

$$\langle f(p) - p, y - p \rangle \leq 0, \quad \textit{for all } y \in F(S) \cap \Omega.$$

Proof. It is sufficient to show that $\{S^n : n \in \mathbb{N}\}$ holds condition $(\mathcal{D})$. Observe that

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^n t_n\| \\ &\quad + \left\|S^n t_n - S^{n+1} t_n\right\| + \left\|S^{n+1} t_n - Sx_n\right\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^n t_n\| + \left\|S^n t_n - S^{n+1} t_n\right\| \\ &\quad + k_1(\|S^n t_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \to 0 \text{ as } n \to \infty. \end{aligned}$$

One can see by Lemma 4, that condition $(\mathcal{D})$ holds. □

We now derive the main result of Petruşel and Yao ([10], Theorem 2.2) as Corollary.

Corollary 1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse strongly monotone mapping and $S : C \to C$ a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\lambda_n\}$ be two sequences of positive numbers with $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [a, b]$, with $0 < a < b < \alpha(1-\delta)$ (for some $\delta \in (0,1)$) satisfying the conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

For arbitrary $x_1 \in C$, consider the sequences $\{x_n\}$ and $\{y_n\}$ generated by the following iterative process:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \quad \textit{for all } n \in \mathbb{N}, \end{cases}$$

where $f : C \to C$ is a k-contraction.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point p, such that p is the unique solution in $F(S) \cap \Omega$ of the following variational inequality:

$$\langle f(p) - p, y - p \rangle \leq 0, \quad \textit{for all } y \in F(S) \cap \Omega.$$

Example. Let $H = C = \mathbb{R}$. Let $A, f : C \to C$ be mappings defined by $A(x) = \dfrac{x}{2}$ and $f(x) = \dfrac{3x}{16}, \forall x \in C$. Then A is 2-inverse strongly monotone mapping and f is a contraction mapping. Let $\mathcal{S} := \{S_n\}$ be a sequence of asymptotically nonexpansive mappings from C into C defined by $S_n(x) = (1 + \frac{1}{n})x, \forall x \in C$ and $n \in \mathbb{N}$. Clearly, $F(\mathcal{S}) = \{0\}$ and $F(\mathcal{S}) \cap \Omega = \{0\}$. Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ be two sequences of positive numbers defined by $\alpha_n = \dfrac{1}{n+1}$ and $\lambda_n = \dfrac{1}{2}$.

Then, the sequence $\{x_n\}$ generated by

$$\begin{cases} x_1 \in C, \\ y_n = x_n - \lambda_n A x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n(x_n - \lambda_n A y_n), \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

satisfying the inequality:

$$x_{n+1} \quad = \quad \left[\frac{3}{n+1} + 13\right] \frac{x_n}{16} \leq \frac{29}{32} x_n.$$

One can see easily that $\{x_n\}$ converges to $0 \in F(\mathcal{S}) \cap \Omega$.

Remark 1. *For the numerical simulation and the graphic representation of the above sequences, see Figure 1 and Figure 2 below.*

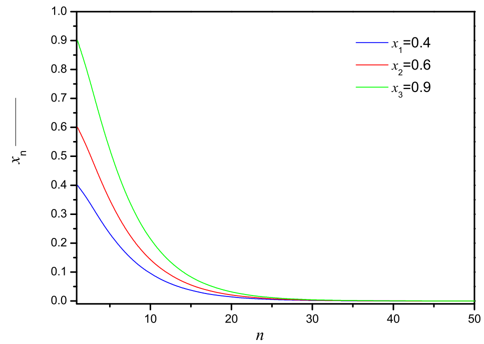

Figure 1: Convergence of the sequence $\{x_n\}$ for n iterations.

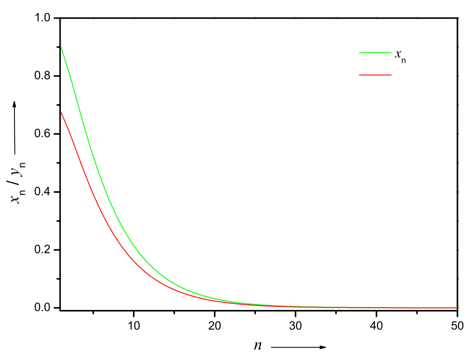

Figure 2: Convergence of the sequences $\{x_n\}$ and $\{y_n\}$ for n iterations.

References

[1] R.P. Agarwal, Donal O'Regan and D.R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Series: Topological Fixed Point Theory and Its Applications Springer, New York, 2009.

[2] J. Chen, L. Zhang and T. Fan, *Viscosity approximation methods for nonexpansive mappings and monotone mappings*, J. Math. Anal. Appl. **334** (2007), 1450-1461.

[3] Y.J. Cho, H. Zhou and G. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl., **47** (2004), 707-717.

[4] Q.-L. Dong and S. He, *A viscosity projection method for class $\mathcal{T}$ mappings*, Analele Ştiinţifice ale Univ. Ovidius Constanţa **21** (2013), 95-109.

[5] K. Goebel and W.A. Kirk, *Topics on Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[6] G.M. Korpelevich, *An extragradient method for finding saddle points and for other problems*, Ekonomika i Matematicheskie Metody **12** (1976), 747-756.

[7] N. Nadezhkina and W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl., **128** (2006), 191-201.

[8] M.A. Noor, *Modified projection method for pseudomonotone variational inequalities*, Appl. Math. Lett. **15** (2002), 315-320.

[9] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591-597.

[10] A. Petruşel and J.-C. Yao, *An extragradient iterative scheme by viscosity approximation methods for fixed point problems and variational inequality problems*, Cent. Eur. J. Math. **7** (2009), 335-347.

[11] R.T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75-88.

[12] D.R. Sahu, Zequing Liu, and Shin Min Kang, *Iterative approaches to common fixed points of asymptotically nonexpansive mappings*, Rocky Mountain J. Math. **39** (2009), 281-304.

[13] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **43** (1991), 153-159.

[14] G. Stampacchia, *Formes bilineaires coercivities sur les ensembles convexes*, C. R. Acad. Sci. Paris **258** (1964), 4413-4416.

[15] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

[16] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417-428.

[17] H.K. Xu and T.H. Kim, *Convergence of hybrid sleepest-descend methods for variational inequalities*, J. Optim. Theory Appl. **119** (2003), 185-201.

[18] L.C. Zeng and J.-C. Yao, *Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems*, Taiwan. J. Math., **10** (2006), 1293-1303.

Adrian PETRUŞEL,
Department of Mathematics,
Babeş-Bolyai University Cluj-Napoca,
Kogălniceanu Street, No. 1, 400084, Cluj-Napoca, Romania.
Email: petrusel@math.ubbcluj.ro

D.R. SAHU,
Department of Mathematics,
Banaras Hindu University,
Varanasi-221005, India.
Email: drsahudr@gmail.com

Vidya SAGAR,
Department of Mathematics,
Banaras Hindu University,
Varanasi-221005, India.
Email: luckvidya@yahoo.com