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Goldie-Rad-Supplemented Modules

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Abstract

In this paper we introduce β^{**} relation on the lattice of submodules of a module M. We say that submodules X, Y of M are β^{**} equivalent, $X\beta^{**}Y$, if and only if $\frac{X+Y}{X} \subseteq \frac{Rad(M)+X}{X}$ and $\frac{X+Y}{Y} \subseteq \frac{Rad(M)+Y}{Y}$. We show that the β^{**} relation is an equivalence relation. We also investigate some general properties of this relation. This relation is used to define and study classes of Goldie-Rad-supplemented and Rad-H-supplemented modules. We prove $M = A \oplus B$ is Goldie-Rad-supplemented if and only if A and B are Goldie-Rad-supplemented.

1 Introduction

Throughout this paper, R denotes an associative ring with an identity and modules are unital right R-modules. We use $N \leq M$ and $N \leq_{\oplus} M$ to signify that N is a submodule and a direct summand of M, respectively. $Rad(M)$ and $End(M)$ will denote the Jacobson radical of M and the ring of endomorphisms of M.

Let M be a module. A submodule K of M is called *small* in M (denoted by $K \ll M$) if $N + K \neq M$ for any proper submodule N of M. Lifting modules were studied by many authors (see [6] and [10]). A module M is called *lifting* if for every submodule N of M there exists a direct summand K of M such that $K \subseteq N$ and $N/K \ll M/K$. We call M, *($\oplus$-)supplemented* if for every submodule N of M, there is (a direct summand K of M) $K \leq M$, such that

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$M = N + K$ and $N \cap K \ll K$ (in this case K is a *($\oplus$-)supplement* of N in M). A module M is called *weakly supplemented* if for every submodule N of M, there exists a submodule L of M such that $M = N + L$ and $N \cap L \ll M$. H-supplemented modules were introduced in [10] as a generalization of lifting modules. According to [10] a module M is called *H-supplemented* if for every submodule A of M there exists a direct summand D of M such that $A+X = M$ if and only if $D + X = M$ for every submodule X of M. In [8], it is proved that M is H-supplemented if and only if for every submodule A of M there exists a direct summand D of M such that $\frac{A+D}{D} \ll \frac{M}{D}$ and $\frac{A+D}{A} \ll \frac{M}{A}$. For more information about H-supplemented modules we refer the reader to [8], [9] and [10].

Recall from [2] that a module M is said to have (P^*) *property or (P^*)-module* if for any submodule N of M there exists a direct summand D of M such that $D \subseteq N$ and $\frac{N}{D} \subseteq Rad(\frac{M}{D})$, equivalently, for every submodule N of M there exists a decomposition $M = K \oplus K'$ such that $K \subseteq N$ and $(N \cap K') \subseteq Rad(K')$.

Let $K, L \leq M$. We say K is a *(weak) Rad-supplement* of L in M, if $M = N+K$ and $(N \cap K \subseteq Rad(M))$ $N \cap K \subseteq Rad(K)$. A module M is called (*weakly*) *Rad-supplemented* if every submodule of M has a (weak) *Rad*-supplement.

Let M be a module. A submodule X of M is called *fully invariant*, if for every $f \in End(M)$, $f(X) \subseteq X$. A submodule N of M is *projection invariant*, if for every $e = e^2 \in End(M)$, $e(N) \subseteq N$.

In [3], the authors defined and studied the β^* relation and investigated some properties of this relation. Based on definition of β^* relation they introduced two new classes of modules namely Goldie*-lifting and Goldie*-supplemented. They showed that two concept of H-supplemented modules and Goldie*-lifting modules coincide. In this paper, motivated by [3], we change their definition of these two classes of modules.

Section 2 is devoted to introduce the β^{**} relation. We investigate some properties of this relation and prove that this relation is an equivalence relation.

In Section 3 we define Goldie-*Rad*-supplemented and *Rad*-H-supplemented modules. Motivated by [3] and based on the definition of β^{**} relation, we call a module M, *Goldie-Rad-supplemented (Rad-H-supplemented)* if for any submodule N of M, there exists a *Rad*-supplement submodule (a direct summand) D of M such that $N\beta^{**}D$. Clearly every (P^*)-module is *Rad*-H-supplemented and every *Rad*-H-supplemented module is Goldie-*Rad*-supplemented. Let $M = A \oplus B$ be a distributive module. Then M is Goldie-*Rad*-supplemented (*Rad*-H-supplemented) if and only if A and B are Goldie-*Rad*-supplemented (*Rad*-H-supplemented) (Theorem 3.9).

Also we obtain some conditions which under the factor module of a *Rad*-H-supplemented module will be *Rad*-H-supplemented.

Finally we obtain the relations between Goldie-*Rad*-supplemented modules

and Rad-H-supplemented modules with other types of supplemented modules. Let M be a projective module such that every Rad-supplement submodule of M is a direct summand. Then we show that the following statements are equivalent: (Theorem 3.23)
(1) M is Rad-supplemented;
(2) M is (P^*);
(3) M is amply Rad-supplemented;
(4) M is Rad-H-supplemented and $Rad(M)$ is QSL in M;
(5) M is Rad-$\oplus$-supplemented;
(6) M is Goldie-Rad-supplemented and $Rad(M)$ is QSL in M.

The texts by Mohamed and Müller [10] and Wisbauer [14] are the general references for notions of rings and modules not defined in this work.

2 The β^{**} Relation

The β^* relation is defined and studied in [3]. Let $X, Y \leq M$. The authors in [3], called X and Y are β^* *equivalent*, $X\beta^*Y$, provided $\frac{X+Y}{X} \ll \frac{M}{X}$ and $\frac{X+Y}{Y} \ll \frac{M}{Y}$.

Definition 2.1. Let M be a module and $X, Y \leq M$. We say X and Y are β^{**} equivalent, $X\beta^{**}Y$, if and only if $\frac{X+Y}{X} \subseteq \frac{Rad(M)+X}{X}$ and $\frac{X+Y}{Y} \subseteq \frac{Rad(M)+Y}{Y}$.

In this section we develop some basic properties of β^{**} relation on the set of submodules of M.

Lemma 2.2. *The β^{**} is an equivalence relation.*

Proof. The reflexive and symmetric properties are clear. For transitivity, assume $X\beta^{**}Y$ and $Y\beta^{**}Z$. So

$$\begin{array}{ccc} \frac{X+Y}{X} \subseteq \frac{Rad(M)+X}{X} & and & \frac{X+Y}{Y} \subseteq \frac{Rad(M)+Y}{Y} \\ \frac{Y+Z}{Y} \subseteq \frac{Rad(M)+Y}{Y} & and & \frac{Y+Z}{Z} \subseteq \frac{Rad(M)+Z}{Z}. \end{array}$$

So we have

$$\begin{array}{ccc} X+Y \subseteq Rad(M)+X & and & X+Y \subseteq Rad(M)+Y \\ Y+Z \subseteq Rad(M)+Y & and & Y+Z \subseteq Rad(M)+Z. \end{array}$$

It is easy to see that $X+Z \subseteq Rad(M)+X$ and $X+Z \subseteq Rad(M)+Z$. Thus, $X\beta^{**}Z$.

$\square$

It is clear that any submodule contained in $Rad(M)$ is β^{**} equivalent to zero submodule. Also, note that two submodules may be isomorphic but not β^{**} equivalent. For example, let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, $X = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. Then since $Rad(R_R) = X$, they are not β^{**} equivalent but they are R-isomorphic. Also in $M = \mathbb{Z}_{\mathbb{Z}}$, $m\mathbb{Z}\beta^{**}n\mathbb{Z}$ if and only if $m = n$ (see [3]).

Proposition 2.3. *Let $f : M \to N$ be an epimorphism. The following statements hold:*
(1) If $X, Y \leq M$ such that $X\beta^{**}Y$, then $f(X)\beta^{**}f(Y)$.
(2) If $X, Y \leq N$ such that $X\beta^{**}Y$, then $f^{-1}(X)\beta^{**}f^{-1}(Y)$.
(3) If $X \leq M$ such that $X \subseteq Rad(M)$, $K \leq N$ and $f(X)\beta^{**}K$, then $X\beta^{**}f^{-1}(K)$.

Proof. (1) Suppose that $X\beta^{**}Y$ for submodules X, Y of M. Then $X + Y \subseteq Rad(M) + X$ and $X + Y \subseteq Rad(M) + Y$. Therefore we have $f(X) + f(Y) \subseteq Rad(N) + f(X)$ and $f(X) + f(Y) \subseteq Rad(N) + f(Y)$. This implies that $f(X)\beta^{**}f(Y)$.

(2) Let $X\beta^{**}Y$ for submodules X, Y of N. Then $X + Y \subseteq Rad(N) + X$ and $X + Y \subseteq Rad(N) + Y$. Since f is an epimorphism $f^{-1}(X) + f^{-1}(Y) \subseteq Rad(M) + X$ and $f^{-1}(X) + f^{-1}(Y) \subseteq Rad(M) + Y$. It follows that $f^{-1}(X)\beta^{**}f^{-1}(Y)$.

(3) Assume that $f(X)\beta^{**}K$, $X \subseteq Rad(M)$ and $K \leq N$. Then, $f(X)+K \subseteq Rad(N)+f(X)$ and $f(X)+K \subseteq Rad(N)+K$. Since f is an epimorphism and $X \subseteq Rad(M)$, we get $f^{-1}(K) + X \subseteq Rad(M) + f^{-1}(K)$ and $f^{-1}(K) + X \subseteq Rad(M) + X$. Therefore, $X\beta^{**}f^{-1}(K)$. □

Proposition 2.4. *Let $X \leq M$ and K a maximal submodule of M.*
(1) If $C_1, C_2 \leq M$, $Rad(M) \subseteq C_2$ such that $C_1 + C_2 = M$, $C_2 \neq M$ and $X\beta^{**}C_1$. Then $X \nsubseteq C_2$.
(2) If $X\beta^{**}Y$ such that $X \subseteq K$, then $Y \subseteq K$.

Proof. (1) Assume that $X \subseteq C_2$. Since $Rad(M) \subseteq C_2$, we have $X + C_2 = M$. By assumption, $C_2 = M$, a contradiction.

(2) Assume that $Y \nsubseteq K$. Then $Y + K = M$. Since $X\beta^{**}Y$ and $Rad(M) \subseteq K$, we obtain $K + X = M$. But $X \subseteq K$ implies that $K = M$, a contradiction. □

Proposition 2.5. *Let $X_1, X_2, Y_1, Y_2 \leq M$ such that $X_1\beta^{**}Y_1$ and $X_2\beta^{**}Y_2$. Then $(X_1 + X_2)\beta^{**}(Y_1 + Y_2)$ and $(X_1 + Y_2)\beta^{**}(Y_1 + X_2)$.*

Proof. Suppose that $X_1\beta^{**}Y_1$ and $X_2\beta^{**}Y_2$. Then

$$\begin{array}{lll} X_1+Y_1\subseteq Rad(M)+X_1 & \textit{and} & X_1+Y_1\subseteq Rad(M)+Y_1 \\ X_2+Y_2\subseteq Rad(M)+X_2 & \textit{and} & X_2+Y_2\subseteq Rad(M)+Y_2. \end{array}$$

Hence by using above inequalities, we can easily see that $(X_1+X_2)\beta^{**}(Y_1+Y_2)$ and $(X_1+Y_2)\beta^{**}(Y_1+X_2)$. □

Corollary 2.6. *Let $X, Y \leq M$ and $K \subseteq Rad(M)$. Then $X\beta^{**}Y$ if and only if $X\beta^{**}(Y+K)$.*

Proof. ($\Rightarrow$) This implication follows from Proposition 2.5 and the fact that $0\beta^{**}K$.

($\Leftarrow$) Since $K \subseteq Rad(M)$, we have $Y\beta^{**}(Y+K)$. Now the implication follows from the transitivity of the β^{**} relation. □

Corollary 2.7. *Let $X, Y_1, \ldots, Y_n \leq M$. If $X\beta^{**}Y_i$ for $i = 1, \ldots, n$. Then $X\beta^{**}\sum_{i=1}^{n} Y_i$.*

3 Goldie-*Rad*-Supplemented Modules

In [3], the authors defined and study the β^* relation and investigated some properties of this relation. Based on definition of β^* relation they introduced two new classes of modules namely Goldie*-lifting and Goldie*-supplemented. A module M is called *Goldie*-lifting (Goldie*-supplemented)* (G^*-lifting (G^*-supplemented) for short) if for every submodule N of M there is a direct summand (supplement submodule) S of M such that $N\beta^*S$ (see [3]).

Next we introduce two new classes of modules.

Definition 3.1. Let M be a module.
(1) We say M is *Goldie-Rad-supplemented* if for every submodule N of M, there exists a *Rad*-supplement submodule S in M such that $N\beta^{**}S$.
(2) We say M is *Rad-H-supplemented* if for every submodule N of M, there exists a direct summand D of M such that $N\beta^{**}D$.

By the definitions every Goldie*-lifting module is Goldie*-supplemented. We give a general example of modules which are *Rad*-*H*-supplemented (Goldie-*Rad*-supplemented) but not Goldie*-supplemented(see Example 3.2). If M is a module with property that every *Rad*-supplement submodule is direct summand, then for M being Goldie-*Rad*-supplemented is equivalent to being *Rad*-*H*-supplemented.

We have the following implications:
(P^*)-module $\Rightarrow$ *Rad*-*H*-supplemented module $\Rightarrow$ Goldie-*Rad*-supplemented module.

The next example shows that *Rad*-H-supplemented modules (Goldie-*Rad*-supplemented) modules are a proper generalization of H-supplemented modules (Goldie*-supplemented modules).

Example 3.2. (1) A radical module M ($Rad(M) = M$) is *Rad*-H-supplemented and hence Goldie-*Rad*-supplemented. This yields that any non-supplemented module M with $Rad(M) = M$ is *Rad*-H-supplemented but not H-
supplemented. So all injective non-supplemented modules over a Dedekind domain (e.g. the quotient field of a non-local Dedekind domain (see [10, Proposition A.8])) are *Rad*-H-supplemented (hence Goldie-*Rad*-supplemented) but not Goldie*-supplemented (H-supplemented) by [3, Theorem 3.6]. In particular, $\mathbb{Q}_{\mathbb{Z}}$ is Goldie-*Rad*-supplemented but not Goldie*-supplemented.

(2) The $\mathbb{Z}$-module $\mathbb{Z}$ is neither *Rad*-H-supplemented nor Goldie-*Rad*-supplemented. In fact an (indecomposable) *Rad*-H-supplemented module with zero radical is (local) semisimple.

Proposition 3.3. *Let M be a H-supplemented module. Then M is Rad-H-supplemented. If $Rad(M) \ll M$, then the converse holds.*

Proof. Let $N \leq M$. By assumption, M has a decomposition $M = D \oplus D'$ such that $(N+D)/N \ll M/N$ and $(N+D)/D \ll M/D$. Then $M = D+D' = N + D'$ and $(N + D)/D \subseteq (Rad(M) + D)/D$. Let $\theta : (D + D')/D \to D'$, $\psi : D'/(N \cap D') \to (N + D')/N$ be natural isomorphisms and $f : D' \to D'/(N \cap D')$ be natural epimorphism. Set $h = \psi f \theta$. By a similar argument to [3, Proposition 2.5], $(N + D)/N = h((N + D)/D)$. Since $(N + D)/D \subseteq (Rad(M) + D)/D$, we have $(N + D)/N \subseteq (Rad(M) + N)/N$. Hence, M is *Rad*-H-supplemented. For the converse, when $Rad(M) \ll M$, it is easy to check that M is H-supplemented. □

Theorem 3.4. *([3, Theorem 3.8]) Let M be a Noetherian module such that each submodule is projection invariant. If M is Rad-H-supplemented, then M is a finite direct sum of local modules.*

Proposition 3.5. *Let R be a commutative local ring with maximal ideal m. If M is a finitely generated Rad-H-supplemented module, then $M \cong \frac{R}{I_1} \times \ldots \times \frac{R}{I_n}$ for some ideals $I_1, \ldots, I_n$ of R with $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subsetneq R$.*

Proof. It follows from [10, Proposition A.8] and Proposition 3.3. □

Proposition 3.6. *Let M be a module. Then M is Goldie-Rad-supplemented if and only if for every $X \leq M$ there exists a Rad-supplement submodule S of M such that $S + Rad(M) = X + Rad(M)$.*

Proof. Let M be Goldie-Rad-supplemented and $X \leq M$. Then, there is a Rad-supplement submodule S of M such that $X + S \subseteq Rad(M) + X$ and $X+S \subseteq Rad(M)+S$. Then $S+Rad(M) \subseteq X+Rad(M)$ and $X+Rad(M) \subseteq S + Rad(M)$. It follows that $S + Rad(M) = X + Rad(M)$. The converse is easy. $\square$

Proposition 3.7. *Let M be a module. If for every $X \leq M$, there is a Rad-supplement submodule S of M and a $H \subseteq Rad(M)$ such that $X = S + H$, then M is Goldie-Rad-supplemented.*

Proof. We prove that $X\beta^{**}S$. Since $X + S = S + H \subseteq Rad(M) + S + H = Rad(M)+X$ and $X+S = S+H+S \subseteq Rad(M)+S$, then $\frac{X+S}{X} \subseteq \frac{Rad(M)+X}{X}$ and $\frac{X+S}{S} \subseteq \frac{Rad(M)+S}{S}$ as required. $\square$

Proposition 3.8. *Let M be a Goldie-Rad-supplemented module. Then for each $X \leq M$ with $Rad(M) \subseteq X$, we have $X = S + H$ where S is a Rad-supplement in M and $H \subseteq Rad(M)$.*

Proof. Let $X \leq M$ such that $Rad(M) \subseteq X$. By assumption, there exists a Rad-supplement submodule S of M such that $X\beta^{**}S$. Then, $S \subseteq X$ and $X = Rad(M) + (S \cap X) = Rad(M) + S$. It completes the proof. $\square$

Let M be a module. Then M is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules K, L, N of M, $N+(K\cap L) = (N + K) \cap (N + L)$ or $N \cap (K + L) = (N \cap K) + (N \cap L)$

Theorem 3.9. *Let $M = A \oplus B$ be a distributive module. Then M is Goldie-Rad-supplemented (Rad-H-supplemented) if and only if A and B are Goldie-Rad-supplemented (Rad-H-supplemented).*

Proof. ($\Rightarrow$) Let $X \leq A$. Then there exist submodules S, L of M such that $S + L = M$ and $S \cap L \subseteq Rad(S)$ and $X\beta^{**}S$. We prove that $X\beta^{**}(A \cap S)$. Since $X\beta^{**}S$, we have $X+S \subseteq Rad(M)+X$ and $X+S \subseteq Rad(M)+S$. Since $X \subseteq A$, we get $X + (A \cap S) \subseteq Rad(A) + X$ and $X + (A \cap S) \subseteq (Rad(A) + A \cap S + B \cap S + Rad(B)) \cap A$. By modularity, $X + (A \cap S) \subseteq Rad(A) + X$ and $X + (A \cap S) \subseteq Rad(A) + (A \cap S)$. Thus $X\beta^{**}(A \cap S)$. By assumption, $(A\cap S)+(A\cap L) = A$ and $(A\cap S)\cap(A\cap L) = A\cap S\cap L \subseteq Rad(A\cap S)\oplus Rad(B\cap S)$. This implies that $A \cap S \cap L \subseteq Rad(A \cap S)$. So $(A \cap S)$ is a Rad-supplement of $(A \cap L)$ in A. Therefore A is Goldie-Rad-supplemented. Similarly, B is Goldie-Rad-supplemented.

($\Leftarrow$) Let $U \leq M$, $U_1 = A \cap U$ and $U_2 = B \cap U$. There exist $L_1, S_1 \leq A$ such that $U_1\beta^{**}S_1$, $L_1 + S_1 = A$ and $L_1 \cap S_1 \subseteq Rad(S_1)$. There also exist $L_2, S_2 \leq B$ such that $U_2\beta^{**}S_2$, $L_2 + S_2 = B$ and $L_2 \cap S_2 \subseteq Rad(S_2)$. By Proposition 2.5, $U\beta^{**}(S_1 + S_2)$. Moreover, $S_1 + S_2 + L_1 + L_2 = M$ and

$(S_1 + S_2) \cap (L_1 + L_2) = (S_1 \cap L_1) + (S_2 \cap L_2) \subseteq Rad(S_1) + Rad(S_2) \subseteq Rad(S_1 + S_2)$. This means that, $(S_1 + S_2)$ is a *Rad*-supplement submodule in M. Hence M is Goldie-*Rad*-supplemented. The proof for A and B being *Rad*-H-supplemented is similar. $\square$

Following example shows that a factor module of a *Rad*-H-supplemented module need not be *Rad*-H-supplemented in general.

A module M is called *finitely presented* if $M \cong F/K$ for some finitely generated free module F and finitely generated submodule K of M.

Example 3.10. Let R be a commutative local ring which is not a valuation ring and let $n \geq 2$. By [13, Theorem 2], there exists a finitely presented indecomposable module $M = R^{(n)}/K$ which cannot be generated by fewer than n elements. By [5, Corollary 1.6], $R^{(n)}$ is $\oplus$-supplemented and hence H-supplemented by [7, Proposition 2.1]. By Proposition 3.3, $R^{(n)}$ is *Rad*-H-supplemented. Since M is not cyclic, it is not $\oplus$-supplemented, and hence not H-supplemented. Since M is finitely generated, it is not *Rad*-H-supplemented by Proposition 3.3.

Let M be a module and N, A submodules of M such that $A \leq_{\oplus} M$. We say that A is an *Rad-H-supplement of N in M* if, there is a direct summand B of M such that $M = A \oplus B$ and $N\beta^{**}A$.

Proposition 3.11. *Let M_0 be a direct summand of a module M such that for every decomposition $M = N \oplus K$ of M, there exist submodules N' of N and K' of K such that $M = M_0 \oplus N' \oplus K'$. If M is Rad-H-supplemented, then M/M_0 is Rad-H-supplemented.*

Proof. Let $X/M_0 \leq M/M_0$. Since M is *Rad*-H-supplemented, there exists a decomposition $M = N \oplus K$ such that $X\beta^{**}N$. Then $(X+N)/N \subseteq (Rad(M)+N)/N$ and $(X+N)/X \subseteq (Rad(M)+X)/X$. By hypothesis, $M = M_0 \oplus N' \oplus K'$ for $N' \leq N$ and $K' \leq K$. Now it is easy to see that $(M_0 \oplus N')/M_0$ is a *Rad*-H-supplement of X/M_0 in M/M_0. $\square$

We call a module M *semilocal* provided that $M/Rad(M)$ is semisimple. Clearly *Rad*-supplemented modules are semilocal. We also show that every *Rad*-H-supplemented module is semilocal.

Lemma 3.12. *Let M be a Rad-H-supplemented module. Then $M/Rad(M)$ is semisimple.*

Proof. Let $N/Rad(M) \leq M/Rad(M)$. Since M is *Rad*-H-supplemented, there exists a direct summand D of M such that $N\beta^{**}D$. So $(N + D)/N \subseteq (Rad(M) + N)/N$ and $(N + D)/D \subseteq (Rad(M) + D)/D$. Since $D \leq_{\oplus} M$,

$M = D \oplus D'$ for some submodule D' of M. Then $M = D' + N$. It follows that $M/Rad(M) = N/Rad(M) + (D' + Rad(M))/Rad(M)$. Since $N \cap D' \subseteq Rad(D')$, $M/Rad(M) = N/Rad(M) \oplus (D' + Rad(M))/Rad(M)$. Hence $M/Rad(M)$ is semisimple. □

Proposition 3.13. *Let M be a module. Then the following are equivalent:*
(1) M is Rad-H-supplemented;
(2) M is semilocal and each direct summand of $M/Rad(M)$ lifts to a direct summand of M.

Proof. (1) $\Rightarrow$ (2) By Lemma 3.12, we only prove the last statement. Let $N/Rad(M) \leq M/Rad(M)$. Since M is *Rad*-H-supplemented, there exists $D \leq_{\oplus} M$ such that $N\beta^{**}D$, i.e. $(N + D)/N \subseteq (Rad(M) + N)/N$ and $(N + D)/D \subseteq (Rad(M) + D)/D$. Then $D \subseteq N$. Hence $N/Rad(M) = (D + Rad(M))/Rad(M)$. This means $N/Rad(M)$ lifts to D.

(2) $\Rightarrow$ (1) Let $N \leq M$. Then by assumption, $(N + Rad(M))/Rad(M) = \overline{N}$ is a direct summand of $M/Rad(M) = \overline{M}$. Hence by (2), $\overline{N} = \overline{L}$ such that $M = L \oplus K$. The rest is easy by taking L as a *Rad*-H-supplement of N in M. □

The next proposition introduces a module which is not G^*-supplemented (H-supplemented).

Proposition 3.14. *Let R be a commutative domain with only two maximal ideals. Then R is not a Goldie*-supplemented R-module.*

Proof. Let M_1 and M_2 be the maximal ideals of R. Note that R_R is not supplemented by [4, 27.21]. Also observe that if $Y \leq R_R$ then either $Y \leq M_1$ or $Y \leq M_2$, and that $Rad(R_R) = M_1 \cap M_2 \ll R_R$. Now Claim 1: Let $X \leq R_R$ such that X_R is not small in R_R. Then $X \leq M_i$ if and only if $X\beta^* M_i$ where $i \in \{1, 2\}$.
Proof of claim 1. Assume that $i = 1$. Since R_R is weakly supplemented from [4, 17.9], there exists $W \leq R_R$ such that $X + W = R$ and $X \cap W \ll R_R$. First assume $X \leq M_1$. Then $W \leq M_2$. By the modular law, $M_1 = X + (M_1 \cap W)$ and $M_1 \cap W \leq Rad(R) \ll R$. Let $K \leq R_R$ such that $X + M_1 + K = R_R$. Since $X \leq M_1$, $M_1 + K = R_R$. So $R_R = X + (M_1 \cap W) + K = X + K$. By [3, Theorem 2.3], $X\beta^* M_1$. Conversely assume, $X\beta^* M_1$. Suppose to the contrary that X is not a submodule of M_1. Then $X\beta^* M_2$. It follows that $M_1\beta^* M_2$. Then $R_R = M_1 + M_2 + M_1$. By [3, Lemma 2.2], $M_1 + M_1 = M_1 = R_R$, a contradiction. Thus $X \leq M_1$.
Claim 2. There exists no supplement $S \leq R_R$ such that $M_2\beta^* S$.
Proof of claim 2. Assume to the contrary that $M_2\beta^* S$ for some supplement $S \leq R$. By Claim 1, $S \leq M_2$. Hence there exists $V \leq R_R$ such that $V + S =$

R_R and $V \cap S \ll S$. Then $V \leq M_1$. From Claim 1, $V\beta^* M_1$. Since $X \leq M_1$, $X\beta^* M_1$, by claim 1. From [3, Lemma 2.2] , $X\beta^* V$, a contradiction. Thus Claim 2 is proved. It follows that R_R is not Goldie*-supplemented. □

Corollary 3.15. *Let $R = \{m/n \in \mathbb{Q} \mid p \nmid n, q \nmid n\}$ (see [11, p. 60, Exercise 3.67]) where p and q are distinct primes. Then R_R is not Goldie*-supplemented.*

Theorem 3.16. *Let $M = \oplus_{i \in I} H_i$ be a direct sum of Rad-H-supplemented modules H_i $(i \in I)$. Assume that each direct summand of $M/Rad(M)$ lifts to a direct summand of M. Then M is Rad-H-supplemented.*

Proof. Clearly $M/Rad(M)$ is semisimple by Lemma 3.12. Now M is *Rad-H*-supplemented by Proposition 3.13. □

The following example shows that any (finite) direct sum of *Rad-H*-supplemented modules need not be *Rad-H*-supplemented.

Example 3.17. Let R be a commutative local ring and M a finitely generated R-module. Assume $M \cong \bigoplus_{i=1}^{n} R/I_i$. Since every I_i is fully invariant in R, every R/I_i is H-supplemented by [9, Theorem 2.3] and hence *Rad-H*-supplemented by Proposition 3.3. By [10, Lemma A.4], M is *Rad-H*-supplemented if $I_1 \leq I_2 \leq \ldots \leq I_n$. If we don't have the condition $I_1 \leq I_2 \leq \ldots \leq I_n$, M is not *Rad-H*-supplemented by Proposition 3.3.

A module M is called *Rad-⊕-supplemented* if for every $A \leq M$, there exists a $B \leq_{\oplus} M$ such that $A + B = M$ and $A \cap B \subseteq Rad(B)$. Clearly every (P^*)-module is *Rad*-⊕-supplemented and every *Rad*-⊕-supplemented module is *Rad*-supplemented.

Now we investigate the relations between *Rad-H*-supplemented modules and the others. A module M is called *amply (Rad)-supplemented* if for any submodules K and V of M such that $M = K + V$, there is a submodule U of V such that $K + U = M$ and $(K \cap U \subseteq Rad(U))$ $K \cap U \ll U$. It is easy to show that every amply *Rad*-supplemented module is weakly *Rad*-supplemented.

Proposition 3.18. *Every amply Rad-supplemented module is Goldie-Rad-supplemented.*

Proof. Let M be amply *Rad*-supplemented and $X \leq M$. Let $X \subseteq Rad(M)$. Clearly $X\beta^{**}0$. So assume that $X \nsubseteq Rad(M)$. Since M is weakly *Rad*-supplemented, there exists a submodule L of M such that $X + L = M$ and $X \cap L \subseteq Rad(M)$. By assumption, there is a *Rad*-supplement S of L in X. So $M = S + L$ and $S \cap L \subseteq Rad(S)$. Since $S \subseteq X$, we have $X = S + (L \cap X) \subseteq Rad(M) + S$. It follows that $X\beta^{**}S$. Therefore, M is Goldie-*Rad*-supplemented. □

Example 3.19. *([3, Example 3.9])* (1) Let $R = \mathbb{Z}_8$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8$. By [10, p. 97], M is an H-supplemented R-module and hence *Rad*-H-supplemented R-module by Proposition 3.3. M is not lifting and since it is finitely generated, M is not P^*.

(2) Let R be a commutative local ring which has two incomparable ideals I and J. Let $M = R/I \oplus R/J$. By [10, Lemma A.4(1)], M is amply supplemented and hence amply *Rad*-supplemented. By Proposition 3.18, M is Goldie-*Rad*-supplemented but M is not H-supplemented by [10, Lemma A.4(3)]. Now by Proposition 3.3, M is not *Rad*-H-supplemented. Let F be a field and
$T = F[x]/ < x^4 >= \{a\overline{1} + b\overline{x} + c\overline{x}^2 + d\overline{x}^3 \mid a, b, c, d \in F, \overline{x} = x+ < x^4 >\}$. Let $R = \{a\overline{1} + c\overline{x}^2 + d\overline{x}^3 \in T\}$. Then R is a subring of T. Moreover, R is a commutative local Kasch ring. Then $F\overline{x}^2$ and $F\overline{x}^3$ are ideals of R and $F\overline{x}^2 \cap F\overline{x}^3 = 0$. Then $M = R/F\overline{x}^2 \oplus R/F\overline{x}^3$ is amply *Rad*-supplemented (Goldie-*Rad*-supplemented) but not *Rad*-H-supplemented.

Let M be any module. A submodule U of M is called *quasi strongly lifting* (QSL) in M if whenever $(A+U)/U$ is a direct summand of M/U, there exists a direct summand P of M such that $P \leq A$ and $P + U = A + U$ (see [1]).

Lemma 3.20. *Let M be any module. Then the following are equivalent:*
(1) M is (P^*)-module;
(2) M is Rad-H-supplemented and Rad(M) is QSL in M.

Proof. By Lemma 3.12 and [1, Lemma 3.5 and Proposition 3.6]. □

Lemma 3.21. *Let M be a projective module such that every Rad-supplement submodule of M is a direct summand of M. Then the following statements are equivalent:*
(1) M is Rad-supplemented;
(2) M is amply Rad-supplemented;
(3) M is (P^*);
(4) M is Rad-$\oplus$-supplemented.

Proof. (1) $\Leftrightarrow$ (2) By [12, Theorem 2.15].

(1) $\Rightarrow$ (3) In [1, Lemma 3.2] the assertion is proved for any preradical τ. Here we consider $\tau = Rad$.

(3) $\Rightarrow$ (1) and (1) $\Leftrightarrow$ (4) are clear by definitions and the assumption that every *Rad*-supplement submodule of M is a direct summand of M. □

We say that a module M is *strongly Rad-$\oplus$-supplemented* if M is *Rad*-$\oplus$-supplemented and every *Rad*-supplement submodule in M is a direct summand of M.

Proposition 3.22. *If M is Goldie-Rad-supplemented and strongly Rad-$\oplus$-supplemented, then M is Rad-H-supplemented.*

Proof. Let $N \leq M$. Then there exists a *Rad*-supplement submodule S in M such that $N\beta^{**}S$. By hypothesis, S is a direct summand of M. Hence M is *Rad*-H-supplemented. □

Now we have the following theorem:

Theorem 3.23. *Let M be a projective module such that every Rad-supplement submodule of M is a direct summand. Then the following are equivalent:*
(1) M is Rad-supplemented;
(2) M is (P^*);
(3) M is amply Rad-supplemented;
(4) M is Rad-H-supplemented and Rad(M) is QSL in M;
(5) M is Rad-$\oplus$-supplemented;
(6) M is Goldie-Rad-supplemented and Rad(M) is QSL in M.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$(5) are by Lemma 3.21.
(2) $\Leftrightarrow$ (4) It is by Lemma 3.20.
(4) $\Leftrightarrow$ (6) Follows from Proposition 3.22.
□

A module M is called *refinable* if whenever $M = A + B$ for submodules A, B, there is a direct summand C of M such that $C \subseteq A$ and $M = C + B$ (see [14]). By [1, Theorem 3.7], if M is refinable, then *Rad*(M) is QSL in M. Also by [1, Corollary 3.21], if R_R is lifting, then for every finitely generated projective R-module M, *Rad*(M) is QSL in M. Hence, we have following corollary:

Corollary 3.24. *Let M be a projective module such that every Rad-supplement submodule is direct summand. Then the following are equivalent in case M is refinable or R_R is lifting and M is finitely generated:*
(1) M is Rad-supplemented;
(2) M is (P^*);
(3) M is amply Rad-supplemented;
(4) M is Rad-H-supplemented;
(5) M is Rad-$\oplus$-supplemented;
(6) M is Goldie-Rad-supplemented.

Over a right perfect ring every right R-module is Goldie-*Rad*-supplemented. If R_R is *Rad*-H-supplemented, then R is a semiperfect ring. So if every module over a ring R is *Rad*-H-supplemented, then R is semiperfect. But there exists a semiperfect ring which has a module that is not *Rad*-H-supplemented.

Example 3.25. Let $R = F[[x, y]]$ be the ring of formal power series over a field F in the indeterminates x and y. Then R is a commutative noetherian local domain with maximal ideal $J = Rx + Ry$. Therefore the ring R is semiperfect. Since R is a domain, J_R is a uniform R-module. It follows that J_R is indecomposable. Now suppose that J_R is Rad-H-supplemented and $N \subsetneqq J$ such that $N \nsubseteq Rad(J_R)$. Then $N\beta^{**}0$ or $N\beta^{**}J$. Then $N \subseteq Rad(J_R)$ or $N = J_R$. It follows that J_R is not Rad-H-supplemented.

4 Open Problems

(1) By [8, Corollary 4.11], an H-supplemented module with (SIP) is a direct sum of hollow modules. When is every Goldie-Rad-supplemented module a direct sum of hollow modules?
(2) Determine when a Goldie-Rad-supplemented module is Rad-supplemented.
(3) When is an arbitrary direct sum of Goldie-Rad-supplemented modules, Goldie-Rad-supplemented?

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