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$(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals of Intra-Regular Abel Grassmann's-Groupoids

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Abstract

In this paper, we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals in an AG-groupoid. We characterize intra-regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals.

1 Introduction

The fuzzy set theory which was introduced by Zadeh in 1965 [15], plays an important role for solving real life problems involving ambiguities. Recently many theories like fuzzy set theory, theory of vague sets, theory of soft ideals, theory of intuitionistic fuzzy sets and theory of rough sets have been developed for handling such uncertainties. These theories can be used for latest development in almost every branch of science. Rosenfeld in 1971, introduced the concept of fuzzy set theory in groups [13]. Mordeson et al. [7] have discussed the applications of fuzzy set theory in fuzzy coding, fuzzy automata and finite state machines. The theory of soft sets (see [4, 5]) has many applications in different fields such as the smoothness of functions, game theory, operations research, Riemann integration etc.

Fuzzy set theory on semigroups has already been developed. In [8] Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence

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of a fuzzy point with a fuzzy set was defined in [10]. Bhakat and Das [1, 2] gave the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. In [12] regular semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals. In [11] semigroups were characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals.

An AG-groupoid is a non-associative algebraic structure lies in between a groupoid and a commutative semigroup. Although it is non-associative, some times it possesses some interesting properties of a commutative semigroup. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup. Now our aim is to find out some logical investigations for intra-regular AG-groupoids using the new generalized concept of fuzzy sets.

In this paper, we introduced some new types of fuzzy ideals namely $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy right ideals in AG-groupoids and develop some new results. We give some characterizations for intra-regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals.

2 Preliminaries

A groupoid $(S, .)$ is called an AG-groupoid (LA-semigroup in some articles [9]), if its elements satisfy left invertive law: $(ab)c = (cb)a$. In an AG-groupoid medial law [3], $(ab)(cd) = (ac)(bd)$, holds for all a, b, c, $d \in S$. It is also known that in an AG-groupoid with left identity, the paramedial law: $(ab)(cd) = (db)(ca)$, holds for all a, b, c, $d \in S$.

If an AG-groupoid contains a left identity, the following law holds

$$a(bc) = b(ac), \text{for all } a, b, c \in S. \tag{1}$$

Let S be an AG-groupoid. By an AG-subgroupoid of S we mean a nonempty subset A of S such that $A^2 \subseteq A$.

A left (right) ideal of S is a nonempty subset I of S such that $SI \subseteq I$ ($IS \subseteq I$). By a two-sided ideal or simply ideal, we mean a nonempty subset of S which is both a left and a right ideal of S.

A nonempty subset A of an AG-groupoid S is called semiprime of S if $a^2 \in A$ implies $a \in A$.

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \longrightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For any two fuzzy subsets f and g of S, $f \leq g$ means that, $f(x) \leq g(x)$ and $(f \cap g)(x) = f(x) \wedge g(x)$ for all x in S.

Let f and g be any fuzzy subsets of an AG-groupoid S. Then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c)\} & \text{if there exist } b,\, c \in S, \text{ such that } a = bc, \\ 0 & \text{otherwise.} \end{cases}$$

The following definitions can be found in [7].

A fuzzy subset f of an AG-groupoid S is called a fuzzy AG-subgroupoid of S if $f(xy) \geq f(x) \wedge f(y)$ for all $x,\, y \in S$.

A fuzzy subset f of an AG-groupoid S is called a fuzzy left (right) ideal of S if $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$ for all $x,\, y \in S$. A fuzzy subset f of an AG-groupoid S is called a fuzzy ideal of S if it is both a fuzzy left and fuzzy right ideal of S.

Let $\mathcal{F}(X)$ denote the collection of all fuzzy subsets of an AG-groupoid S with a left identity. Note that S can be considered as a fuzzy subset of S itself and we write $S = C_S$, that is, $S(x) = 1$ for all $x \in S$. Moreover $S \circ S = S$.

Definition 2.1. *A fuzzy subset f of X of the form*

$$f(y) = \begin{cases} r(\neq 0) \ if \ y = x, \\ 0 \ otherwise, \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r, where $r \in (0, 1]$.

3 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals of AG-groupoids

Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For any $B \subseteq A$, let $X^\delta_{\gamma B}$ be a fuzzy subset of X such that $X^\delta_{\gamma B}(x) \geq \delta$ for all $x \in B$ and $X^\delta_{\gamma B}(x) \leq \gamma$ otherwise. Clearly, $X^\delta_{\gamma B}$ is the characteristic function of B if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and a fuzzy subset f of X, we say that

(1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.

(2) $x_r q_\delta f$ if $f(x) + r > 2\delta$.

(3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

Now we introduce a new relation on $\mathcal{F}(X)$, denoted by "$\subseteq \vee q_{(\gamma,\delta)}$", as follows:

For any $f, g \in \mathcal{F}(X)$, by $f \subseteq \vee q_{(\gamma,\delta)} g$ we mean that $x_r \in_\gamma f$ implies $x_r \in_\gamma \vee q_\delta g$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover f and g are said to be (γ, δ)-equal, denoted by $f =_{(\gamma,\delta)} g$, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$.

The above definitions can be found in [14].

Lemma 3.1. *[14] Let f and g be fuzzy subsets of $\mathcal{F}(X)$. Then $f \subseteq \vee q_{(\gamma,\delta)} g$ if and only if* $\max\{g(x), \gamma\} \geq \min\{f(x), \delta\}$ *for all $x \in X$.*

Lemma 3.2. *[14] Let f, g and $h \in \mathcal{F}(X)$. If $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} h$, then $f \subseteq \vee q_{(\gamma,\delta)} h$.*

The relation "$=_{(\gamma,\delta)}$" is equivalence relation on $\mathcal{F}(X)$, see [14]. Moreover, $f =_{(\gamma,\delta)} g$ if and only if $\max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$ for all $x \in X$.

Lemma 3.3. *Let A, B be any nonempty subsets of an AG -groupoid S with a left identity. Then we have*

(1) *$A \subseteq B$ if and only if $X^{\delta}_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma B}$, where $r \in (\gamma, 1]$ and $\gamma, \delta \in [0, 1]$.*

(2) $X^{\delta}_{\gamma A} \cap X^{\delta}_{\gamma B} =_{(\gamma,\delta)} X^{\delta}_{\gamma(A \cap B)}$.

(3) $X^{\delta}_{\gamma A} \circ X^{\delta}_{\gamma B} =_{(\gamma,\delta)} X^{\delta}_{\gamma(AB)}$.

Lemma 3.4. *If S is an AG-groupoid with a left identity then $(ab)^2 = a^2 b^2 = b^2 a^2$ for all a and b in S.*

Proof. It follows by medial and paramedial laws.

Definition 3.1. *A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$, such that $x_t \in_\gamma f$, $y_s \in_\gamma f$ we have $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.*

Theorem 3.1. *Let f be a fuzzy subset of an AG groupoid S with a left identity. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG subgroupoid of S if and only if*

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \text{ where } \gamma, \delta \in [0, 1].$$

Proof. Let f be a fuzzy subset of an AG-groupoid S which is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subgroupoid of S. Assume that there exist $x, y \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}.$$

This implies that $f(xy) < t$, which further implies that $(xy)_{\min t} \overline{\in_\gamma \vee q_\delta} f$ and $\min\{f(x), f(y), \delta\} \geq t$. Therefore $\min\{f(x), f(y)\} \geq t \Rightarrow f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$, whence $x_t \in_\gamma f$, $y_s \in_\gamma f$. But $(xy)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$, which contradicts the definition. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \text{ for all } x, y \in S.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$, $y_s \in_\gamma f$ but $(xy)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$. Then $f(x) \geq t > \gamma$, $f(y) \geq s > \gamma$, $f(xy) < \min\{f(x), f(y), \delta\}$ and $f(xy) + \min\{t, s\} \leq 2\delta$. It follows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$, which is a contradiction. Hence $x_t \in_\gamma f$, $y_s \in_\gamma f$ which means that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y in S.

Definition 3.2. *A fuzzy subset f of an AG-groupoid S with a left identity is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (respt-right) ideal of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $y_t \in_\gamma f$ we have $(xy)_t \in_\gamma \vee q_\delta f$ (resp $x_t \in_\gamma f$ implies that $(xy)_t \in_\gamma \vee q_\delta f$).*

Theorem 3.2. *A fuzzy subset f of an AG-groupoid S with a left identity is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S if and only if for all $a, b \in S$,*

$$\max\{f(ab), \gamma\} \geq \min\{f(a), \delta\}.$$

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S. Suppose that there are $a, b \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f(ab), \gamma\} < t \leq \min\{f(a), \delta\}.$$

Then $\max\{f(ab), \gamma\} < t \leq \gamma$ implies that $(ab)_t \bar{\in}_\gamma f$ which further implies that $(ab)_t \overline{\in_\gamma \vee q_\delta} f$. From $\min\{f(a), \delta\} \geq t > \gamma$ it follows that $f(a) \geq t > \gamma$, which implies that $a_t \in_\gamma f$. But $(ab)_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Thus

$$\max\{f(ab), \gamma\} \geq \min\{f(a), \delta\}.$$

Conversely, assume that there exist $a, b \in S$ and $t, s \in (\gamma, 1]$ such that $a_s \in_\gamma f$ but $(ab)_t \overline{\in_\gamma \vee q_\delta} f$. Then $f(a) \geq t > \gamma$, $f(ab) < \min\{f(a), \delta\}$ and $f(ab) + t \leq 2\delta$. It follows that $f(ab) < \delta$ and so $\max\{f(ab), \gamma\} < \min\{f(a), \delta\}$, which is a contradiction. Hence $a_t \in_\gamma f$ which implies that $(ab)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ (respectively $a_t \in_\gamma f$ implies that $(ab)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$) for all a, b in S.

Example 3.1. *Consider the AG-groupoid defined by the following multiplication table on $S = \{1, 2, 3\}$:*

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.4 \text{ if } x = 1, \\ 0.41 \text{ if } x = 2, \\ 0.38 \text{ if } x = 3. \end{cases}$$

Then, we have

- *f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.22})$-fuzzy right ideal,*
- *f is not an $(\in, \in \vee q_{0.22})$-fuzzy right ideal, because*

$$f(2 \cdot 3) < \min\{f(2), \frac{1 - 0.22}{2} = 0.39\}.$$

- *f is not a fuzzy right ideal because $f(2 \cdot 3) < f(2)$.*

Example Let $S = \{1, 2, 3\}$ and the binary operation $\circ$ be defined on S as follows:

$\circ$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then $(S, \circ)$ is an AG-groupoid. Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.44 \text{ if } x = 1, \\ 0.6 \text{ if } x = 2, \\ 0.7 \text{ if } x = 3. \end{cases}$$

Then, we have

- f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy subgroupoid of S.
- f is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy right ideal of S.

Example Let $S = \{1, 2, 3\}$ and define the binary operation $\cdot$ on S as follows:

$\cdot$	1	2	3
1	1	1	1
2	3	3	3
3	1	1	1

$(S, \cdot)$ is an AG-groupoid. Let us define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.6 \text{ if } x = 1 \\ 0.5 \text{ if } x = 2 \\ 0.55 \text{ if } x = 3 \end{cases}$$

f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S.

Lemma 3.5. *R is a right ideal of an AG-groupoid S if and only if $X^\delta_{\gamma R}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S.*

Proof. (i) Let $x, y \in R$, it means that $xy \in R$. Then $X^\delta_{\gamma R}(xy) \geq \delta$, $X^\delta_{\gamma R}(x) \geq \delta$ and $X^\delta_{\gamma R}(y) \geq \delta$ but $\delta > \gamma$. Thus

$$\max\{X^\delta_{\gamma R}(xy), \gamma\} = X^\delta_{\gamma R}(xy) \text{ and } \min\{X^\delta_{\gamma R}(x), \delta\} = \delta.$$

Hence $\max\{X^\delta_{\gamma R}(xy), \gamma\} \geq \min\{X^\delta_{\gamma R}(x), \delta\}$.

(ii) Let $x \notin R$, $y \in R$

Case(a): If $xy \notin R$. Then $X^{\delta}_{\gamma R}(x) \leq \gamma$, $X^{\delta}_{\gamma R}(y) \geq \delta$ and $X^{\delta}_{\gamma R}(xy) \leq \gamma$. Therefore

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = \gamma \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = X^{\delta}_{\gamma R}(x).$$

Hence $\max\{X^{\delta}_{\gamma R}(xy), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(x), \delta\}$.

Case(b): If $xy \in R$. Then $X^{\delta}_{\gamma R}(xy) \geq \delta$, $X^{\delta}_{\gamma R}(x) \leq \gamma$ and $X^{\delta}_{\gamma R}(y) \geq \delta$. Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = X^{\delta}_{\gamma R}(xy) \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = X^{\delta}_{\gamma R}(x).$$

Hence $\max\{X^{\delta}_{\gamma R}(xy), \gamma\} > \min\{X^{\delta}_{\gamma R}(x), \delta\}$.

(iii) Let $x \in R, y \notin R$. Then $xy \in R$. Thus $X^{\delta}_{\gamma R}(xy) \geq \delta$, $X^{\delta}_{\gamma R}(y) \leq \gamma$ and $X^{\delta}_{\gamma R}(x) \geq \delta$. Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = X^{\delta}_{\gamma R}(xy) \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = \delta.$$

Hence $\max\{X^{\delta}_{\gamma R}(xy), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(x), \delta\}$.

(iv) Let $x, y \notin R$, then

Case (a) Assume that $xy \notin R$. Then by definition we get $X^{\delta}_{\gamma R}(xy) \leq \gamma$, $X^{\delta}_{\gamma R}(y) \leq \gamma$ and $X^{\delta}_{\gamma R}(x) \leq \gamma$. Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = \gamma \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = X^{\delta}_{\gamma R}(x).$$

Therefore $\max\{X^{\delta}_{\gamma R}(xy), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(x), \delta\}$.

Case (b) Assume that $xy \in R$. Then by definition we get $X^{\delta}_{\gamma R}(xy) \geq \gamma$, $X^{\delta}_{\gamma R}(y) \leq \gamma$ and $X^{\delta}_{\gamma R}(x) \leq \gamma$. Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = X^{\delta}_{\gamma R}(xy) \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = X^{\delta}_{\gamma R}(x).$$

Therefore $\max\{X^{\delta}_{\gamma R}(xy), \gamma\} > \min\{X^{\delta}_{\gamma R}(x), \delta\}$.

Conversely, let $rs \in RS$, where $r \in R$ and $s \in S$. By hypothesis $\max\{X^{\delta}_{\gamma R}(rs), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(r), \delta\}$. Since $r \in R$, thus $X^{\delta}_{\gamma R}(r) \geq \delta$ which implies that $\min\{X^{\delta}_{\gamma R}(r), \delta\} = \delta$. Thus

$$\max\{X^{\delta}_{\gamma R}(rs), \gamma\} \geq \delta.$$

This implies that $X^{\delta}_{\gamma R}(rs) \geq \delta$ which implies that $rs \in R$. Hence R is a right ideal of S.

Here we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideals.

Definition 3.3. *A fuzzy subset f of an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime if $x_t^2 \in_\gamma f$ implies that $x_t \in_\gamma \vee q_\delta f$ for all $x \in S$ and $t \in (\gamma, 1]$.*

Example Consider an AG-groupoid $S = \{1, 2, 3, 4, 5\}$ with the following multiplication table

.	1	2	3	4	5
1	4	5	1	2	3
2	3	4	5	1	2
3	2	3	4	5	1
4	1	2	3	4	5
5	5	1	2	3	4

Clearly $(S, .)$ is intra-regular because $1 = (3.1^2).2$, $2 = (1.2^2).5$, $3 = (5.3^2).2$, $4 = (2.4^2).1$, $5 = (3.5^2).1$. Define a fuzzy subset f on S as given:

$$f(x) = \begin{cases} 0.7 & if\ x = 1, \\ 0.6 & if\ x = 2, \\ 0.68 & if\ x = 3, \\ 0.63 & if\ x = 4, \\ 0.52 & if\ x = 5. \end{cases}$$

Then f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.5})$-fuzzy semiprime of S.

Theorem 3.3. *A fuzzy subset f of an AG-groupoid S is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime if and only if $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}$.*

Proof. Let f be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. Assume that there exists $a \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(a), \gamma\} < t \leq \min\{f(a^2), \delta\}.$$

Then $\max\{f(a), \gamma\} < t$. This implies that $f(a) < t > \gamma$. Now since $\delta \geq t$, so $f(a) + t < 2\delta$. Thus $a_t \overline{\in_\gamma \vee q_\delta} f$. Also since $\min\{f(a^2), \delta\} \geq t$, so $f(a^2) \geq t > \gamma$. This implies that $a_t^2 \in_\gamma f$. Thus by definition of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime $a_t \in_\gamma \vee q_\delta f$ which is a contradiction to $a_t \overline{\in_\gamma \vee q_\delta} f$. Hence

$$\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}, \text{ for all } a \in S.$$

Conversely assume that there exist $a \in S$ and $t \in (\gamma, 1]$ such that $a_t^2 \in_\gamma f$, then $f(a^2) \geq t > \gamma$, thus $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\} \geq \min\{t, \delta\}$. We consider two cases here,

Case(i): if $t \leq \delta$, then $f(a) \geq t > \gamma$, this implies that $a_t \in_\gamma f$.

Case(ii) : if $t > \delta$, then $f(a) + t > 2\delta$. Thus $a_t q_\delta f$.

Hence from (i) and (ii) we write $a_t \in_\gamma \vee q_\delta f$. Hence f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Theorem 3.4. *For a right ideal R of an AG-groupoid S with a left identity, the following conditions are equivalent:*
(i) R is semiprime.
(ii) $X^\delta_{\gamma R}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (ii)$ Let R be a semiprime ideal of an AG-groupoid S. Let a be an arbitrary element of S such that $a \in R$. Then $a^2 \in R$. Hence $X^\delta_{\gamma R}(a) \geq \delta$ and $X^\delta_{\gamma R}(a^2) \geq \delta$ which implies that $\max\{X^\delta_{\gamma R}(a), \gamma\} \geq \min\{X^\delta_{\gamma R}(a^2), \delta\}$.

Now let $a \notin R$. Since R is semiprime, we have $a^2 \notin R$. This implies that $X^\delta_{\gamma R}(a) \leq \gamma$ and $X^\delta_{\gamma R}(a^2) \leq \gamma$. Therefore, $\max\{X^\delta_{\gamma R}(a), \gamma\} \geq \min\{X^\delta_{\gamma R}(a^2), \delta\}$. Hence, $\max\{X^\delta_{\gamma R}(a), \gamma\} \geq \min\{X^\delta_{\gamma R}(a^2), \delta\}$ for all $a \in S$.

$(ii) \Rightarrow (i)$ Let $X^\delta_{\gamma R}$ be fuzzy semiprime. If $a^2 \in R$, for some a in S, then $X^\delta_{\gamma R}(a^2) \geq \delta$. Since $X^\delta_{\gamma R}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime, we have $\max\{X^\delta_{\gamma R}(a), \gamma\} \geq \min\{X^\delta_{\gamma R}(a^2), \delta\}$. Therefore $\max\{X^\delta_{\gamma R}(a), \gamma\} \geq \delta$. But $\delta > \gamma$, so $X^\delta_{\gamma R}(a) \geq \delta$. Thus $a \in R$. Hence R is semiprime.

4 Intra-Regular AG-groupoids

An element a of an AG-groupoid S is called **intra-regular** if there exist $x, y \in S$ such that $a = (xa^2)y$. S is called **intra-regular**, if every element of S is intra-regular.

Theorem 4.1. *Let S be an AG- groupoid with a left identity. Then the following conditions are equivalent:*
$(i)$$S$ is intra-regular.
(ii)For a right ideal R of an AG-groupoid S, $R \subseteq R^2$ and R is semiprime.
(iii)For an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f of S, $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$ and f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular AG-groupoid S with a left identity. Since S is intra-regular, for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using (1), medial law, paramedial law and left invertive law, we get

$$a = (xa^2)y = [(x(aa))y] = [(a(xa))y] = [(y(xa))a]$$

$$\begin{aligned} y(xa) &= [y(x((xa^2)y))] = [y((xa^2)(xy))] = [(xa^2)(xy^2)] = [(y^2x)(a^2x)] \\ &= [a^2((y^2x)x)] = [\{(a(y^2x)\}(ax)]. \end{aligned}$$

For any a in S there exist p and q in S such that $a = pq$. Then

$$\begin{aligned}
\max\{(f \circ f)(a), \gamma\} &= \max\left\{ \bigvee_{a=pq} \{\{f(p) \wedge f(q)\}, \gamma\} \right\} \\
&\geq \max\left\{\min\{\{f(a(y^2x)), f(ax)\}, \gamma\right\} \\
&\geq \max\{\min\{f(a(y^2x)), f(ax)\}, \gamma\} \\
&= \min\{\max\{f(a(y^2x)), \gamma\}, \max\{f(ax), \gamma\}\} \\
&\geq \min\left\{\min\{f(a), \delta\}, \min\{f(a), \delta\}\right\} \\
&= \min\{(f)(a), \delta\}.
\end{aligned}$$

Thus by Lemma 3.1, $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$. Next we show that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. Since $S = S^2$, for each y in S there exist y_1, y_2 in S such that $y = y_1y_2$. Thus using medial law, paramedial law and (1), we get

$$\begin{aligned}
a &= (xa^2)y = (xa^2)(y_1y_2) = (y_2y_1)(a^2x) \\
&= a^2[(y_2y_1)x] = a^2t, \text{ where } [(y_2y_1)x] = t.
\end{aligned}$$

Then

$$\begin{aligned}
\max\{f(a), \gamma\} &= \max\{f(a^2t), \gamma\} \\
&\geq \min\{f(a^2), \delta\}.
\end{aligned}$$

$(iii) \Rightarrow (ii)$ Let R be any right ideal of an AG-groupoid S. By (iii), $X^\delta_{\gamma R}$ is semiprime and by Theorem 3.4, R is semiprime. Now using Lemma 3.3 and (iii), we get

$$X^\delta_{\gamma R} = X^\delta_{\gamma R \cap R} =_{(\gamma,\delta)} X^\delta_{\gamma R} \cap X^\delta_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma R} \circ X^\delta_{\gamma R} =_{(\gamma,\delta)} X^\delta_{\gamma R^2}.$$

Hence by Lemma 3.3, we get $R \subseteq R^2$.

$(ii) \Rightarrow (i)$ Since Sa^2 is a right ideal containing a^2, using (ii) we get $a \in Sa^2 \subseteq (Sa^2)^2 = (Sa^2)(Sa^2) \subseteq (Sa^2)S$. Hence S is intra-regular.

Theorem 4.2. *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*

(i) S is intra-regular.

(ii) For any right ideal R and any subset A of S, $R \cap A \subseteq RA$ and R is a semiprime ideal.

(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy right ideal f and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset g, $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ g)$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of an intra regular AG-groupoid S. Since S is intra-regular, then for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using (1), medial law, paramedial law and left invertive law, we get

$$a = (xa^2)y = [(x(aa))y] = [(a(xa))y] = [y(xa)]a.$$

$$\begin{aligned} y(xa) &= [y\{x((xa^2)y)\}] = [y\{(xa^2)(xy)\}] = [(xa^2)(xy^2)] \\ &= [(y^2x)(a^2x)] = [a^2(y^2x^2)]. \end{aligned}$$

Thus $a = (a^2t)a$, where $(y^2x^2) = t$.

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{\bigvee_{a=bc}\{f(b) \wedge g(c)\}, \gamma\right\} \\ &\geq \max\{\min\{f(a^2t), g(a)\}, \gamma\} \\ &= \min\{\max\{f(a^2t), \gamma\}, \max\{g(a), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

By Lemma 3.1, $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$. The rest of proof is similar as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be an right ideal and A be a subset of S. By Lemma 3.3 and (iii), we get

$$X^\delta_{\gamma(R \cap A)} =_{(\gamma,\delta)} X^\delta_{\gamma R} \cap X^\delta_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma R} \circ X^\delta_{\gamma A} =_{(\gamma,\delta)} X^\delta_{\gamma RA}.$$

By Lemma 3.3, $R \cap A \subseteq RA$. The rest of the proof is similar as in Theorem 4.1.

$(ii) \Rightarrow (i)$ Sa^2 is a right ideal containing a^2. By (ii), it is semiprime. Therefore $a \in Sa^2 \cap Sa \subseteq (Sa^2)(Sa) \subseteq (Sa^2)S$. Hence S is intra-regular.

Theorem 4.3. *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*

(i) *S is intra regular.*

(ii) *For any right ideal R and any subset A of S, $R \cap A \subseteq AR$ and R is a semiprime ideal.*

(iii) *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset g, $f \cap g \subseteq \vee q_{(\gamma,\delta)}(g \circ f)$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.*

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of an intra-regular AG-groupoid S. Since S is intra-regular, then for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using left invertive law, we get

$$\begin{aligned} a &= (xa^2)y = (xa^2)(y_1y_2) = (y_2y_1)(a^2x) = a^2[(y_2y_1)x] = [x(y_2y_1)]a^2 \\ &= a[\{x(y_2y_1)\}a] = a[\{x(y_2y_1)\}\{(xa^2)y\}] = a[(xa^2)[\{x(y_2y_1)\}y]] \\ &= a[[y\{x(y_2y_1)\}](a^2x)] = a[a^2([y\{x(y_2y_1)\}]x)] = a[(x[y\{x(y_2y_1)\}])a^2] \\ &= a[(x[y\{x(y_2y_1)\}])(aa)] = a[a((x[y\{x(y_2y_1)\}])a)] \\ &= a(au), \text{ where } u = (x[y\{x(y_2y_1)\}])a. \end{aligned}$$

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{ \bigvee_{a=bc} \{g(b) \wedge f(c)\}, \gamma \right\} \\ &\geq \max\{\min\{g(a), f(au)\}, \gamma\} \\ &= \min[\max\{g(a), \gamma\}, \max\{f(au), \gamma\}] \\ &\geq \min[\min\{g(a), \delta\}, \min\{f(a), \delta\}] \\ &= \min[g(a), f(a), \delta\}] \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

By Lemma 3.1, we have$f \cap g \subseteq \vee q_{(\gamma,\delta)} g \circ f$. The rest of the proof is similar as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A be a subset of S. By lemma 3.3 and (iii), we get

$$X^\delta_{\gamma(R\cap A)} = X^\delta_{\gamma(A\cap R)} =_{(\gamma,\delta)} X^\delta_{\gamma A} \cap X^\delta_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma A} \circ X^\delta_{\gamma R} =_{(\gamma,\delta)} X^\delta_{\gamma AR}.$$

By Lemma 3.3, $R \cap A \subseteq AR$. The rest of the proof is similar as in Theorem 4.1.

$(ii) \Rightarrow (i)$ Sa^2 is a right ideal containing a^2. By (ii), it is semiprime. Therefore

$$\begin{aligned} a &\in Sa^2 \cap Sa \subseteq (Sa)(Sa^2) = (a^2S)(aS) = [(aa)(SS)](aS) \\ &= [(SS)(aa)](aS) \subseteq (Sa^2)S. \end{aligned}$$

Hence S is intra-regular.

Theorem 4.4. *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*

(i) S is intra-regular.

(ii) For any subset A and any right ideal R of S, $A \cap R \subseteq AR$ and R is a semiprime.

(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy subset f and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal g of S, $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ g)$ and g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular AG-groupoid S. Since S is intra-regular it follows that for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using medial law, paramedial law and (1), we get

$$\begin{aligned} a &= (xa^2)y = [(xa^2)(y_1y_2)] = [(y_2y_1)(a^2x)] \\ &= [a^2((y_2y_1)x)] = [(x(y_2y_1))(aa)] = [a\{(x(y_2y_1))a\}] \\ &= a(ta), \text{ where } x(y_2y_1) = t, \text{ and} \\ ta &= t[(xa^2)y] = (xa^2)(ty) = (yt)(a^2x) = a^2[(yt)x]. \end{aligned}$$

Thus $a = a(a^2v)$, where $(yt)x = v$ and $x(y_2y_1) = t$.

For any a in S there exist s and t in S such that $a = st$. Then

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{ \bigvee_{a=st} \{f(s) \wedge g(t)\}, \gamma \right\} \\ &\geq \max\{\min\{f(a), g(a^2v)\}, \gamma\} \\ &= \min\{\max\{f(a), \gamma\}, \max\{g(a^2v), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

Thus by Lemma 3.1, $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$. The rest of the proof is similar as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A be a subset of S. By Lemma 3.3 and (iii), we get

$$X^\delta_{\gamma(A\cap R)} =_{(\gamma,\delta)} X^\delta_{\gamma A} \cap X^\delta_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma A} \circ X^\delta_{\gamma R} =_{(\gamma,\delta)} X^\delta_{\gamma AR}.$$

By Lemma 3.3, $A \cap R \subseteq AR$. The rest of the proof is similar as in Theorem 4.1.

$(ii) \Rightarrow (i)$ Sa^2 is a right ideal containing a^2. By (ii), it is semiprime. Therefore

$$\begin{aligned} a &\in Sa \cap Sa^2 \subseteq (Sa)(Sa^2) = (a^2S)(aS) \subseteq (a^2S)S \\ &= [a^2(SS)]S = [(SS)a^2]S = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular.

Theorem 4.5. *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*

(i) S is intra-regular.

(ii) For any subsets A, B and for any right ideal R of S, $A \cap B \cap R \subseteq (AB)R$ and R is a semiprime ideal.

(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets f, g and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal h, $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$ and h is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S.

Proof. $(i) \Rightarrow (iii)$ Let f, g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and h be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular AG-groupoid S. Since S is intra-regular then for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using medial, paramedial laws and (1), we get

$$\begin{aligned}
a &= (xa^2)y = (y_2y_1)(a^2x) = a^2[(y_2y_1)x] = [x(y_2y_1)]a^2 \\
&= a[\{x(y_2y_1)\}a] = a(pa), \text{ where } x(y_2y_1) = p \text{ and} \\
pa &= p[(xa^2)y] = (xa^2)(py) = [(yp)(a^2x)] \\
&= a^2[(yp)x] = [x(yp)](aa) = a[\{x(yp)\}a] \\
&= [a(qa)], \text{ where } x(yp) = q, \text{ and} \\
qa &= q[(xa^2)y] = (xa^2)(qy) = (yq)(a^2x) = a^2[(yq)x].
\end{aligned}$$

Thus $a = a[a(a^2c)] = a[a^2(ac)] = a^2[a(ac)]$, where $(yq)x = c$ and $x(yp) = q$ and $(y_2y_1) = p$.

For any a in S there exist b and c in S such that $a = bc$. Then

$$\begin{aligned}
\max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{\bigvee_{a=bc}\{(f \circ g)(b) \wedge h(c)\}, \gamma\right\} \\
&\geq \max\{\min\{(f \circ g)(aa), h(a(ac))\}, \gamma\} \\
&= \max\left\{\bigvee_{aa=pq}\{\{f(p) \wedge g(q)\}, h(a(ac))\}, \gamma\right\} \\
&\geq \max[\min\{f(a), g(a), h(a(ac))\}, \gamma] \\
&= \min[\max\{f(a), \gamma\}, \max\{g(a), \gamma\}, \max\{h(a(ac)), \gamma] \\
&\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\
&= \min[\min\{f(a), g(a), h(a)\}, \delta] \\
&= \min\{(f \cap g \cap h)(a), \delta\}.
\end{aligned}$$

Thus by Lemma 3.1, $(f \cap g) \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. The rest of the proof is similar as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A, B be any subsets of S. Then by Lemma 3.3 and (iii), we get

$$X^\delta_{\gamma(A\cap B)\cap R} =_{(\gamma,\delta)} X^\delta_{\gamma A} \cap X^\delta_{\gamma B} \cap X^\delta_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} (X^\delta_{\gamma A} \circ X^\delta_{\gamma B}) \circ X^\delta_{\gamma R} =_{(\gamma,\delta)} X^\delta_{\gamma(AB)R}.$$

By Lemma 3.3, we get $(A \cap B) \cap R \subseteq (AB)R$. The rest of the proof is similar as in Theorem 4.1.

$(ii) \Rightarrow (i)$ Sa^2 is a right-ideal of an AG-groupoid S containing a^2. By (ii), it is semiprime. Thus (ii), we get

$$Sa \cap Sa \cap Sa^2 \subseteq [(Sa)(Sa)](Sa^2) = [(SS)(aa)](Sa^2) \subseteq (Sa^2)S.$$

Hence S is intra-regular.

Theorem 4.6. *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*

(i) *S is intra-regular.*

(ii) *For any subsets A, B and for any right ideal R of S, $A \cap R \cap B \subseteq (AR)B$ and R is a semiprime.*

(iii) *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets f, h and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal g, $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$ and g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S.*

Proof. $(i) \Rightarrow (iii)$ Let f, h be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular AG-groupoid S. Now since S is intra-regular it follows that for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using medial, paramedial laws and (1), we get

$$\begin{aligned} a &= [(x(aa))y] = [(a(xa))y] = [(y(xa))a], \\ y(xa) &= y[x((xa^2)y)] = y[(xa^2)(yx)] = [(xa^2)(xy^2)] = (y^2x)(a^2x) \\ &= a^2(y^2x^2) = (aa)(y^2x^2) = (x^2y^2)(aa) = a[(x^2y^2)a], \\ (x^2y^2)a &= (x^2y^2)[(xa^2)y] = (xa^2)[(x^2y^2)y] = [y(y^2x^2)](a^2x) = a^2[\{(y(y^2x^2)\}x] \end{aligned}$$

Thus $a = [a(a^2v)]a$, where $\{y(y^2x^2)\}x = v$.

For any a in S there exist p and q in S such that $a = pq$. Then

$$\begin{aligned}
\max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{ \bigvee_{a=pq} \{(f \circ g)(p) \wedge h(q)\}, \gamma \right\} \\
&\geq \max\{\min\{(f \circ g)(a(a^2v)), h(a)\}, \gamma\} \\
&= \max\left\{ \bigvee_{a(a^2v)=cd} \{\{f(c) \wedge g(d)\}, h(a)\}, \gamma \right\} \\
&\geq \max[\min\{f(a), g(a^2v), h(a)\}, \gamma] \\
&= \min[\max\{f(a), \gamma\}, \max\{g(a^2v), \gamma\}, \max\{h(a), \gamma\}] \\
&\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\
&= \min[\min\{f(a), g(a), h(a)\}, \delta] \\
&= \min\{(f \cap g \cap h)(a), \delta\}.
\end{aligned}$$

Thus by Lemma 3.1, $(f \cap g) \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. The rest of the proof is same as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A, B be any subsets of S. Then by Lemma 3.3 and (iii), we get

$$X^\delta_{\gamma(A \cap R) \cap B} =_{(\gamma,\delta)} X^\delta_{\gamma A} \cap X^\delta_{\gamma R} \cap X^\delta_{\gamma B} \subseteq \vee q_{(\gamma,\delta)}(X^\delta_{\gamma A} \circ X^\delta_{\gamma R}) \circ X^\delta_{\gamma B} =_{(\gamma,\delta)} X^\delta_{\gamma(AR)B}.$$

Hence by Lemma 3.3, we get $(A \cap R) \cap B \subseteq (AR)B$. The rest of the proof is same as in Theorem 4.1.

$(ii) \Rightarrow (i)$ Sa^2 is a right-ideal of an AG-groupoid S containing a^2. By (ii), it is semiprime. Thus (ii), we get

$$\begin{aligned}
a \in Sa \cap Sa^2 \cap Sa &\subseteq [(Sa)(Sa^2)](Sa) \subseteq [S(Sa^2)]S \\
&= [S(Sa^2)](SS) = (SS)[(Sa^2)S] \\
&= S[(Sa^2)S] = (Sa^2)(SS) = (Sa^2)S.
\end{aligned}$$

Hence S is intra-regular.

Theorem 4.7. *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*

(i) *S is intra-regular.*

(ii) *For any subsets A, B and for any right ideal R of S, $R \cap A \cap B \subseteq (RA)B$ and R is a semiprime.*

(iii) *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets g, h, $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S.*

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g, h be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets of an intra-regular AG-groupoid S. Now since S is intra-regular it follows that for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using medial, paramedial laws and (1), we get

$$\begin{aligned} a &= (x(aa))y = (a(xa))y = (y(xa))a, \\ y(xa) &= y[x\{(xa^2)y\}] = y[(xa^2)(yx)] = (xa^2)(xy^2) \\ &= (y^2x)(a^2x) = a^2[(y^2x)x] = [(x^2y^2)a]a \text{ and} \\ (x^2y^2)a &= (x^2y^2)[(xa^2)y] = (xa^2)[(x^2y^2)y] = [y(y^2x^2)](a^2x) \\ &= a^2[\{y(y^2x)\}x] = a^2v. \end{aligned}$$

Thus $a = [(a^2v)a]a$, where $[y(y^2x)]x = v$.

For any a in S there exist b and c in S such that $a = bc$. Then

$$\begin{aligned} \max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{ \bigvee_{a=bc} \{(f \circ g)(b) \wedge h(c)\}, \gamma \right\} \\ &\geq \max\{\min\{(f \circ g)((a^2v)a), h(a)\}, \gamma\} \\ &= \max\left\{ \bigvee_{(a^2v)a=pq} \{\{f(p) \wedge g(q)\}, h(a)\}, \gamma \right\} \\ &\geq \max[\min\{f(a^2v), g(a), h(a)\}, \gamma] \\ &= \min\left\{\max\{f(a^2v), \gamma\}, \max\{g(a), \gamma\}, \max\{h(a), \gamma\}\right\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}\} \\ &= \min\{\min\{f(a), g(a), h(a)\}, \delta\} \\ &= \min\{(f \cap g \cap h)(a), \delta\}. \end{aligned}$$

Thus by Lemma 3.1, $(f \cap g) \cap h \subseteq \vee q_{(\gamma,\delta)} (f \circ g) \circ h$. The rest of the proof is similar as in Theorem 4.1.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A, B be any subsets of S. Then by Lemma 3.3 and (iii), we get

$$X^\delta_{\gamma(R\cap A)\cap B} =_{(\gamma,\delta)} X^\delta_{\gamma R} \cap X^\delta_{\gamma A} \cap X^\delta_{\gamma B} \subseteq \vee q_{(\gamma,\delta)} (X^\delta_{\gamma R} \circ X^\delta_{\gamma A}) \circ X^\delta_{\gamma B} =_{(\gamma,\delta)} X^\delta_{\gamma (RA)B}.$$

Hence by Lemma 3.3, we get $(R \cap A) \cap B \subseteq (RA)B$. The rest of the proof is similar as in theorem 4.1.

$(ii) \Rightarrow (i)$ Sa^2 is a right-ideal of an AG-groupoid S containing a^2. By (ii), it is semiprime. Thus (ii), we get

$$\begin{aligned} a \in Sa^2 \cap Sa \cap Sa &\subseteq [(Sa^2)(Sa)](Sa) \subseteq [(Sa^2)S]S \\ &= (SS)(Sa^2) = (SS)[(SS)(aa)] = (SS)[(aa)(SS)] \\ &= (SS)(a^2S) = (Sa^2)(SS) = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular.

Conclusion

In this paper, we characterize intra-regular AG-groupoids with a left identity using the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals. This study can give a new direction for applications of fuzzy set theory in algebraic logic, non-classical logics, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

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