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A posteriori analysis of the spectral element discretization of heat equation

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Abstract

In this paper, we present a posteriori analysis of the discretization of the heat equation by spectral element method. We apply Euler's implicit scheme in time and spectral method in space. We propose two families of error indicators both of them are built from the residual of the equation and we prove that they satisfy some optimal estimates. We present some numerical results which are coherent with the theoretical ones.

1 Introduction

The a posteriori error analysis and mesh adaptivity methods have received considerable attention by mathematicians and engineers in the last two decades [2] However, the majority of works dealing with this theory are limited to finite element method see Verfürth [12] [13] [14] [15] and still insufficient in spectral methods. The spectral element method consists on approaching the solution of a partial differential equation by polynomial functions of high degree on each element of a decomposition. The main results consist on optimizing the discretization of the heat equation. The later relies on a spectral element method with respect to space variables and Euler's implicit scheme with respect to time. The parameter of discretization is a K-tuple formed by the maximum degrees N_k of polynomial on each element. However, like the $h-p$

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version of finite element see [1][9], it can also involve in this parameter a quantity h_k related to the diameter of elements.

In this paper we are interested in the a posteriori analysis of the spectral element discretization of heat equation. This work is an extension in spectral element method of some results obtained by Bergam *and al*. [3] in the case of finite element method. More precisely, we introduce two kinds of indicators, both of them of residual type. The first family, which is similar to that introduced in [8], is global with respect to the space variables but local with respect to the time discretization. Thus, at each time, the error indicator provides appropriate information for the choice of the next time step. The second family is local with respect to both the time and space variables, and the idea is that at each time is an efficient tool for the mesh adaptivity. These indicators are local quantities which can be computed explicitly as a function of the discrete solution and the data of the problem. They are said to be optimal if their Hilbertian sum is equivalent to the error such that the equivalence constants are independent of the parameter of discretization.

The outline of the paper is as follows :

• In Section 2, we present the linear heat equation and we describe the time semi-discret problem and its space discretization.

• Section 3 is devoted to the construction of error indicators for the heat equation and for the proof of upper and lower bounds based on time and space indicators.

• Section 4 deals with some numerical experiments which confirm the interest of the discretization.

2 Time and space discretisation of the heat equation

Let Ω be a connected and bounded open set in $\mathbb{R}$ and T be a fixed positive integer. We consider the one dimensional heat equation: Find u the solution of

$$\left\{\begin{array}{ll} \partial_t u - \dfrac{\partial^2 u}{\partial x^2} = f & \text{in } \Omega\times\ ]0,T[, \\ u = 0 & \text{on } \partial\Omega\times\ ]0,T[, \\ u_{|t=0} = u_0 & \text{in } \Omega. \end{array}\right. \tag{1}$$

The data f and the function u_0 are given. It is readily checked that the equation (1) admits the equivalent variational formulation

Find $u \in C^0(0,T;L^2(\Omega)) \ \cap\ L^2(0,T;H^1_0(\Omega))$ such that for all t in $]0,T[$,

$$(\partial_t u(t), v) + (\frac{\partial u}{\partial x}(t), \frac{\partial v}{\partial x}) = (f(t), v), \quad \forall v \in H^1_0(\Omega). \tag{2}$$

$$u_{|t=0} = u_0 \quad \text{in } \Omega, \tag{3}$$

It is well-known that, for all f in $L^2(0,T;H^{-1}(\Omega))$ and u_0 in $L^2(\Omega)$, problem (2)-(3) admits a unique solution.(see [10] [11]).

First by taking v equal to $u(t)$ in (2) and integrating on the interval $]0,T[$, we derive the following estimate, for all t in $[0,T]$,

$$\|u(t)\|^2_{L^2(\Omega)} + \int_0^t |u(s)|^2_{H^1(\Omega)} ds \leq \|u_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;H^{-1}(\Omega))}. \tag{4}$$

We introduce the norm

$$[[v]](t) = \Big(\|v(t)\|^2_{L^2(\Omega)} + \int_0^t |v(s)|^2_{H^1(\Omega)} ds\Big)^{\frac{1}{2}}. \tag{5}$$

then the (4) is written

$$[[v]](t) \leq \|u_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;H^{-1}(\Omega))}. \tag{6}$$

On the other hand if the function f belongs to $L^2(0,T;L^2(\Omega))$ and u_0 belongs to $H^1(\Omega)$, by replacing v in (2) by $\partial_t u(t)$, we obtain for all t in $[0,T]$,

$$|u(t)|^2_{H^1(\Omega)} + \int_0^t \|\partial_t u(s)\|^2_{L^2(\Omega)} ds \leq |u_0|^2_{H^1(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))}. \tag{7}$$

2.1 Time discretisation

We suppose that f is a continuous function on t with values in $H^{-1}(\Omega)$. We introduce a partition of the interval $[0,T]$ into subintervals $[t_{n-1},t_n], 1 \leq n \leq N$, such that $0 = t_0 < t_1 < .... < t_N = T$. We denote $\tau_n = t_n - t_{n-1}$, by τ the N-tuple $(\tau_1, ..., \tau_N)$ and by $|\tau| = \max_{1\leq n\leq N} \tau_n$. We also define the regularity parameter

$$\sigma_\tau = \max_{2\leq n\leq N} \frac{\tau_n}{\tau_{n-1}} \tag{8}$$

With each family $(v^n)_{0\leq n\leq N}$, we agree to associate the function v_τ on $[0,T]$ which is affine on each interval $[t_{n-1},t_n], 1 \leq n \leq N$, and equal to v^n at $t_n, 0 \leq n \leq N$.

For simplicity, we introduce the notation

$$f^n = f(t_n).$$

The semi-discrete problem issued from Euler's implicit scheme now writes

$$\begin{cases} \dfrac{u^n - u^{n-1}}{\tau_n} - \dfrac{\partial^2 u^n}{\partial x^2} = f^n & \text{in } \Omega, \quad 1 \le n \le N, \\ u^n = 0 & \text{on } \partial\Omega, \quad 1 \le n \le N, \\ u^0 = u_0 & \text{in } \Omega. \end{cases} \tag{9}$$

Equivalently, its admits the variational formulation Find

$$(u^n)_{0 \le n \le N} \in L^2(\Omega) \times (H_0^1(\Omega))^N$$

satisfying for $1 \le n \le N$,

$$(u^n, v) + \tau_n(\frac{\partial u^n}{\partial x}(t), \frac{\partial v}{\partial x}) = (u^{n-1}, v) + \tau_n(f^n, v), \quad \forall v \in H_0^1(\Omega), \tag{10}$$

$$u^0 = u_0 \quad \text{in}\, \Omega. \tag{11}$$

The existence and uniqueness of a solution $(u^n)_{0 \le n \le N}$ for any data $f \in C^0(0, T; H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$ is a simple consequence of the Lax-Milgram Lemma. Moreover, let us introduce the "local" norm on each v^n in $H_0^1(\Omega)$

$$[[v^n]] = \left(\|v^n\|^2_{L^2(\Omega)} + \tau_n |v^n|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}. \tag{12}$$

By Taking v equal to u^n in (10), we easily derive the estimate

$$[[u^n]] \le \|u^{n-1}\|^2_{L^2(\Omega)} + \tau_n \|f^n\|^2_{H^{-1}(\Omega)}. \tag{13}$$

The global norm is now defined on whole sequences $(u^m)_{0 \le m \le n}$ by

$$[[(v^m)]]_n = (\|v^n\|^2_{L^2(\Omega)} + \sum_{m=1}^{n} \tau_n |v^m|^2_{H^1(\Omega)})^{\frac{1}{2}}. \tag{14}$$

By summing up estimate (13) on n, we derive the semi-discrete analogue of (4)

$$[[(u^m)]]_n \le (\|u_0\|^2_{L^2(\Omega)} + \sum_{m=1}^{n} \tau_m \|f^m\|^2_{H^{-1}(\Omega)})^{\frac{1}{2}}. \tag{15}$$

It can be observed that the norm $[[(u^m)]]_n$ involved in this estimate is not equal to the norm $[[(u_\tau)]](t_n)$. However, when u_0 is supposed to be in $H^1(\Omega)$, there are equivalent, as proven in [3] which is of great use in what follows

Lemma 2.1. *For any family* $(v^n)_{0\le n\le N} \in H^1(\Omega)^{N+1}$*, we have*

$$\frac{1}{4}\ [[(v^m)]]_n^2 \le [[v_\tau]]^2(t_n) \le \frac{1}{2}\ (1+\sigma_\tau)\ [[(v^m)]]_n^2\ +\ \frac{\tau_1}{2}\ \|\nabla v^0\|^2. \tag{16}$$

□

In order to state the a priori error estimate, we observe that the family $(e^n)_{0\le n\le N}$, with $e^n = u(t_n) - u^n$, satisfies $e^0 = 0$ and also, by integrating $\partial_t u$ between t_{n-1} and t_n and using equation (10) at time $t = t_n$,

$$\forall\ \ v \in H_0^1(\Omega), \quad (e^n, v) + \tau_n(\nabla e^n, \nabla v) = (e^{n-1}, v) + \tau_n(\varepsilon^n, v),$$

where the consistency error ε^n is given by

$$(\varepsilon^n, v) = (\frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} (\partial_t u)(s)ds - (\partial_t u)(t_n), v).$$

By applying (15) to this new problem and evaluating the consistency error thanks to a Taylor expansion, we derive the estimate: if the solution u is such that $\partial_t^2 u$ belongs to $L^2(0,T;H^{-1}(\Omega))$ and for $1\le n\le N$,

$$[[u(t_m) - u^m]]_n \le (\max_{1\le m\le n} \tau_m)\|\partial_t^2 u\|_{L^2(0,t_n;H^{-1}(\Omega))}. \tag{17}$$

2.2 Space discretisation

We now describe the space discretization of problem (10)-(11). Let Λ the interval $]-1,1[$. For each $n, 0\le n\le N$, we introduce a family of reals numbers a_k such that

$$-1 = a_0 \le a_1 \le .... \le a_{K-1} \le a_K = 1.$$

We denote by Λ_k the interval $]a_{k-1}, a_k[, 1\le k\le K$, and h_k his length. With each interval $\Lambda_k, 1\le k\le K$, we agree to associate positive integer $N_k \ge 2$. The parameter of discretization δ is a K-tuple of couples (h_k, N_k). To define the discrete form associated with problem (1), we construct on each interval Λ_k a quadrature formula.

First of all, we recall the formulas which we will use. Let $\xi_0 < ... < \xi_N$ be the zeros of the polynomial $(1-x^2)L_N^{'}$ and ρ_j the associated weights. The Gauss-Lobatto quadrature formula on the interval $\Lambda =]-1,1[$ is written

$$\forall \phi \in \mathbb{P}_{2N-1}(\Lambda); \qquad \int_{-1}^{1} \phi(x)dx = \sum_{j=0}^{N} \phi(\xi_j^N)\rho_j^N \tag{18}$$

where $\mathbb{P}_N(\Lambda)$ is the space of polynomials, defined on Λ, with degree $\leq N$.
We introduce an approximation of the scalar product in $L^2(\Lambda)$, so we define a discrete scalar product on any u and v continuous on $\overline{\Lambda}$ by

$$(u,v)_N = \sum_{j=0}^{N} u(\xi_j^N)v(\xi_j^N)\rho_j^N.$$

By translation and homothety, we define on each $\Lambda_k =]a_{k-1}, a_k[$ a Gauss-Lobatto quadrature formula of N_k+1 nodes $\xi_j^{N_k}$ and of weights $\rho_j^{N_k}$, then we set:

$$(u,v)_\delta = \sum_{k=1}^{K}\sum_{j=0}^{N_k} u(\xi_j^{N_k})v(\xi_j^{N_k})\rho_j^{N_k}. \tag{19}$$

We note by i_δ the Lagrange interpolation operator on the set of nodes $\xi_j^{N_k}$ with values in

$$Y_\delta = \Big\{ v_\delta \in H^1(\Lambda);\quad v_{\delta|\Lambda_k} \in \mathbb{P}_{N_k}(\Lambda_k), 1 \leq k \leq K \Big\}.$$

Equivalently, for each function φ continuous on $\overline{\Lambda}_k$, $i_\delta(\varphi)_{|\Lambda_k}$ in $\mathbb{P}_{N_k}(\Lambda_k)$ and verify

$$i_\delta(\varphi)_{|\Lambda_k}(\xi_j^{N_k}) = \varphi_{|\Lambda_k}(\xi_j^{N_k}). \tag{20}$$

We recall the following property, which is useful in what follows

$$\forall\, u_\delta \in Y_\delta, \qquad \|u_\delta\|^2_{L^2(\Lambda)} \leq (u_\delta, u_\delta)_\delta \leq 3\|u_\delta\|^2_{L^2(\Lambda)}. \tag{21}$$

For each δ, the discrete spaces are defined by

$$X_\delta = \Big\{ v_\delta \in H^1_0(\Lambda);\quad v_{\delta|\Lambda_k} \in \mathbb{P}_{N_k}(\Lambda_k), 1 \leq k \leq K \Big\}. \tag{22}$$

We suppose that f is continuous on $\overline{\Lambda} \times [0,T]$ and u_0 continuous on $\overline{\Lambda}$, the fully discrete problem now reads

Find $(u_\delta^n)_{0\leq n\leq N}$ in $Y_\delta \times \prod_{n=1}^{N} X_\delta$ such that for $1 \leq n \leq N$,

$$(u_\delta^n, v_\delta)_\delta + \tau_n(\frac{\partial u_\delta^n}{\partial x}, \frac{\partial v_\delta}{\partial x})_\delta = (u_\delta^{n-1}, v_\delta)_\delta + \tau_n(f^n, v_\delta)_\delta \quad \forall v_\delta \in X_\delta, \tag{23}$$

$$u_\delta^0 = i_\delta u_0 \quad \text{in } \Omega, \tag{24}$$

The form $a_\delta(.,.)$ is defined by

$$a_\delta(u_\delta^n, v_\delta) = (u_\delta^n, v_\delta)_\delta + \tau_n(\frac{\partial u_\delta^n}{\partial x}, \frac{\partial v_\delta}{\partial x})_\delta. \qquad (25)$$

The existence and uniqueness of a solution $(u_\delta^n)_{0\le n\le N}$ for any data f in $C^0(0,T;H^{-1}(\Lambda))$ and $u_0 \in C(\overline{\Lambda})$, follows from the Lax-Milgram Lemma. Moreover exactly the same arguments as for (15) leads to the estimate

$$[[u_\delta^m]]_n \le c\Big(\|i_\delta u_0\|^2_{L^2(\Lambda)} + \sum_{m=1}^{n} \tau_m \|i_\delta f^m\|^2_{L^2(\Lambda)}\Big)^{\frac{1}{2}}. \qquad (26)$$

3 Error indicators for the linear heat equation

We are now interested in exhibiting error indicators and studying their equivalence with the error. We first describe the two types of indicators. Next we prove an upper bound for the error as a function of the Hilbertian bound for each indicator.

3.1 The error indicator

As already hinted, we work with two types of indicators, the first one being linked to time discretization and the second ones to space discretization. The first ones are local in time but global in space while the second ones are local with respect to both the time and space variables.
For each n, $1 \le n \le N$, we define the time error indicator

$$\eta_n = (\frac{\tau_n}{3})^{\frac{1}{2}} \|\frac{\partial}{\partial x}(u_\delta^n - u_\delta^{n-1})\|_{L^2(\Lambda)}. \qquad (27)$$

We refer to [8] for analogous time error indicators, however leading to estimates in different norms.
For the technical reasons, we need to introduce the space

$$X_{\delta-} = \Big\{v_\delta \in H_0^1(\Lambda); \quad v_{\delta|\Lambda_k} \in \mathbb{P}_{N_k-1}(\Lambda_k), 1 \le k \le K\Big\}. \qquad (28)$$

For each n, $1 \le n \le N$ and each interval Λ_k in Λ, we define the space error indicator

$$\eta_{n,k} = N_k^{-1}\|\Big(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}(x)\Big)(x-a_{k-1})^{\frac{1}{2}}(a_k - x)^{\frac{1}{2}}\|_{L^2(\Lambda_k)}. \qquad (29)$$

These indicators are local with respect to both the time and spaces variables and be computed explicitly.

3.2 An upper bound for the error

We now intend to bound the norm introduced in (5) by the error indicators and some further terms involving the data (where of course $u_{\delta\tau}$ denotes the piecewise affine function equal to u_δ^n in each t_n. Here we use the triangular inequality

$$[[u-u_{\delta\tau}]](t_n) \leq [[u-u_\tau]](t_n) + [[u_\tau - u_{\delta\tau}]](t_n),$$

and we begin by evaluating $[[u-u_\tau]](t_n)$. The proof of the estimate is rather technical.

Let π_τ denote the interpolation operator with values in piecewise constant functions on $[0,T]$ defined as follows: for any function v continuous on $[0,T]$, $\Pi_\tau v$ is constant on each interval $[t_{n-1},t_n]$, $1 \leq n \leq N$, equal to $v(t_n)$.

Proposition 3.1. *Assume that the data f is continuous on $[0,T]$ with values in $H^{-1}(\Lambda)$ and the function u_0 belongs to $H^1(\Lambda)$. The following a posteriori error estimate holds between the solution u of problem (1) and the solution $(u^n)_{0\leq n\leq N}$ of problem (9), for all t_n, $1 \leq n \leq N$,*

$$\begin{aligned}[][[u-u_\tau]](t_n) \leq c\Big(\ &(1+\sigma_\tau)^{\frac{1}{2}}[[u_\tau - u_{\delta\tau}]](t_n) + (\sum_{m=1}^{n}\eta_m^2)^{\frac{1}{2}} \\ &+\|f-\pi_\tau f\|_{L^2(0,t_n,H^{-1}(\Lambda))}\Big).\end{aligned} \tag{30}$$

□

Proof:
When "applying" equation (2) to the function u_τ, we obtain for all $t \in [t_{n-1},t_n]$ and $v \in H_0^1(\Lambda)$

$$(\partial_t u_\tau(t),v)+(\partial_x u_\tau(t),\partial_x v) = (\frac{u^n-u^{n-1}}{\tau_n},v)+\big(\partial_x(u_\tau(t)-u^n),\partial_x v\big)+(\partial_x u^n,\partial_x v),$$

whence, from (9),

$$(\partial_t u_\tau(t),v)+(\partial_x u_\tau(t),\partial_x v) = (f^n,v)+\big(\partial_x(u_\tau(t)-u^n),\partial_x v\big).$$

Thus, subtracting this line from equation (2) leads to

$$(\partial_t(u-u_\tau)(t),v)+(\partial_x(u-u_\tau)(t),\partial_x v) = (f(t)-f^n,v)+(\partial_x(u^n-u_\tau(t)),\partial_x v).$$

We now take v equal to $(u-u_\tau)(t)$, we obtain

$$\Big(\partial_t(u-u_\tau)(t),(u-u_\tau)(t)\Big)+\Big(\frac{\partial}{\partial x}(u-u_\tau)(t),\frac{\partial}{\partial x}(u-u_\tau)(t)\Big)$$
$$=\Big(f(t)-f^n,(u-u_\tau)(t)\Big)$$
$$-\Big(\frac{\partial}{\partial x}(u_\tau(t)-u^n),\frac{\partial}{\partial x}(u-u_\tau)(t)\Big).$$

Integrate this line on $[t_{n-1},t_n]$ and sum up on the n. By noting that $u-u_\tau$ vanishes at $t=0$, this yields

$$\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}\frac{1}{2}\partial_t\|(u-u_\tau)(s)\|^2_{L^2(\Lambda)}ds+\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial x}(u-u_\tau)(s)\|^2_{L^2(\Lambda)}ds$$
$$=\sum_{m=1}^{n}\Big(\int_{t_{m-1}}^{t_m}\big(f(s)-f^m,(u-u_\tau)(s)\big)ds$$
$$-\int_{t_{m-1}}^{t_m}\big(\frac{\partial}{\partial x}(u_\tau(s)-u^n),\frac{\partial}{\partial x}(u-u_\tau)(s)\big)ds\Big).$$

Then, we have

$$\frac{1}{2}\|(u-u_\tau)(t_n)\|^2_{L^2(\Lambda)}-\frac{1}{2}\|(u-u_\tau)(0)\|^2_{L^2(\Lambda)}+\int_0^{t_n}\|\frac{\partial}{\partial x}(u-u_\tau)(s)\|^2_{L^2(\Lambda)}ds$$
$$=\sum_{m=1}^{n}\Big(\int_{t_{m-1}}^{t_m}\big(f(s)-f^m,(u-u_\tau)(s)\big)ds$$
$$-\int_{t_{m-1}}^{t_m}\big(\frac{\partial}{\partial x}(u_\tau(s)-u^n),\frac{\partial}{\partial x}(u-u_\tau)(s)\big)ds\Big).$$

We obtain

$$\frac{1}{2}[[u-u_\tau]]^2(t_n)\leq\sum_{m=1}^{n}\Big(\int_{t_{m-1}}^{t_m}\big(f(s)-f^m,(u-u_\tau)(s)\big)ds$$
$$-\int_{t_{m-1}}^{t_m}\big(\frac{\partial}{\partial x}(u_\tau(s)-u^n),\frac{\partial}{\partial x}(u-u_\tau)(s)\big)ds\Big).$$

We now evaluate separately each term in the right-hand side of this equation.
1) It follows from the definition of π_τ that

$$\Big|\int_{t_{m-1}}^{t_m}\big(f(s)-f^m,(u-u_\tau)(s)\big)ds\Big|$$
$$\leq\Big(\int_{t_{m-1}}^{t_m}\|(f-\Pi_\tau f)(s)\|^2_{H^{-1}(\Lambda)}ds\Big)^{\frac{1}{2}}\Big(\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial x}(u-u_\tau)(s)\|^2_{L^2(\Lambda)}ds\Big)^{\frac{1}{2}}.$$

Moreover, note that

$$\Big(\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial x}(u-u_\tau)(s)\|^2_{L^2(\Lambda)}ds\Big)^{\frac{1}{2}}\leq[[u-u_\tau]](t_n).$$

2) Concerning the second term, we have

$$\Big|\int_{t_{m-1}}^{t_m}\Big(\frac{\partial}{\partial x}(u_\tau(s)-u^n),\frac{\partial}{\partial x}(u-u_\tau)(s)\Big)ds\Big|$$
$$\leq\Big(\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial x}(u_\tau(s)-u^m)\|^2_{L^2(\Lambda)}ds\Big)^{\frac12}\Big(\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial x}(u-u_\tau)(s)\|^2_{L^2(\Lambda)}ds\Big)^{\frac12}.$$

By using the definition of the function u_τ, we obtain

$$\frac{\partial}{\partial x}(u_\tau-u^m)=\Big(-\frac{t_m-s}{\tau_m}\Big)\frac{\partial}{\partial x}(u^m-u^{m-1}), \tag{31}$$

then,

$$\begin{aligned}\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial x}(u_\tau(s)-u^m)\|^2_{L^2(\Lambda)}ds &= \|\frac{\partial}{\partial x}(u^m-u^{m-1})\|^2_{L^2(\Lambda)}\int_{t_{m-1}}^{t_m}\frac{(s-t_m)^2}{\tau_m^2}ds\\ &= \frac{\tau_m}{3}\|\frac{\partial}{\partial x}(u^m-u^{m-1})\|^2_{L^2(\Lambda)}.\end{aligned}$$

By using again (31), we obtain

$$\Big(\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial x}(u_\tau(s)-u^m)\|^2_{L^2(\Lambda)}ds\Big)^{\frac12}\leq\Big(\frac{\tau_m}{3}\Big)^{\frac12}\|\frac{\partial}{\partial x}(u^m-u^m_\delta)\|_{L^2(\Lambda)}$$
$$+\Big(\frac{\tau_m}{3}\Big)^{\frac12}\|\frac{\partial}{\partial x}(u^m_\delta-u^{m-1}_\delta)\|_{L^2(\Lambda)}+\Big(\frac{\tau_m}{3}\Big)^{\frac12}\|\frac{\partial}{\partial x}(u^{m-1}-u^{m-1}_\delta)\|_{L^2(\Lambda)}.$$

By summing the previous line on n, we have

$$\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}\|\frac{\partial}{\partial x}(u_\tau(s)-u^m)\|^2_{L^2(\Lambda)}ds$$
$$\leq\ c\Big(\sum_{m=1}^{n}\eta_m^2+\sum_{m=1}^{n}\frac{\tau_m}{3}\big(\|\frac{\partial}{\partial x}(u^m-u^m_\delta)\|^2_{L^2(\Lambda)}+\|\frac{\partial}{\partial x}(u^{m-1}-u^{m-1}_\delta)\|^2_{L^2(\Lambda)}\big)\Big).$$

Using (16) yields that the sum over n of the square of the last two terms can be bounded by $[[u_\tau-u_{\tau\delta}]]^2(t_n)$ times a constant depending on the regularity parameter σ_τ, we have

$$\sum_{m=1}^{n}\frac{\tau_m}{3}\Big(\|\frac{\partial}{\partial x}(u^m-u^m_\delta)\|^2_{L^2(\Lambda)}+\|\frac{\partial}{\partial x}(u^{m-1}-u^{m-1}_\delta)\|^2_{L^2(\Lambda)}\Big)$$
$$\leq\frac{\tau_1}{3}\|\frac{\partial}{\partial x}(u^0-u^0_\delta)\|^2_{L^2(\Lambda)}+\sum_{m=1}^{n}\frac{\tau_m}{3}\|\frac{\partial}{\partial x}(u^m-u^m_\delta)\|^2_{L^2(\Lambda)}$$
$$+\sum_{m=2}^{n}\frac{\tau_{m-1}\sigma_\tau}{3}\|\frac{\partial}{\partial x}(u^{m-1}-u^{m-1}_\delta)\|^2_{L^2(\Lambda)}.$$

then, we have

$$\sum_{m=1}^{n} \frac{\tau_m}{3}\Big(\|\frac{\partial}{\partial x}(u^m - u_\delta^m)\|^2_{L^2(\Lambda)} \quad + \|\tfrac{\partial}{\partial x}(u^{m-1} - u_\delta^{m-1})\|^2_{L^2(\Lambda)}\Big) \leq c(1+\sigma_\tau)[[u_\tau - u_{\tau\delta}]]^2(t_n).$$

Combining all this yields the desired result. □

Now, we evaluate the norms $[[u_\tau - u_{\tau\delta}]](t_n)$,$1 \leq n \leq N$. For this, we will need to introduce the following property for the operator $\Pi_N^{1,0}$ in this lemma, where $\Pi_N^{1,0}$ is the orthogonal projection operator from $H_0^1(\Lambda)$ to $\mathbb{P}_N(\Lambda) \bigcap H_0^1(\Lambda)$ ([5] [7]).

Lemma 3.1. *Let s a real ≥ 1. For each function $v \in H^s(\Lambda) \cap H_0^1(\Lambda)$, we have the following estimate*

$$\Big(\int_{-1}^{1} (v - \Pi_N^{1,0} v)^2(\zeta)(1-\zeta^2)^{-1} d\zeta\Big)^{\frac{1}{2}} \leq c\ N^{-s} |v|_{H^s(\Omega)}. \tag{32}$$

□

Proposition 3.2. *Assume the data f continuous on $[0,T]$ with values in $H^{-1}(\Lambda)$ and $u_0 \in H^1(\Lambda)$. Then, the following a posteriori error estimate holds between the solution $(u^n)_{0\leq n\leq N}$ of problem (9) and the solution $(u_\delta^n)_{0\leq n\leq N}$ of problem(23)-(24), for all $t_n, 1 \leq n \leq N$,*

$$\begin{aligned}[[u_\tau - u_{\tau\delta}]](t_n) \leq \quad & c\Big(\textstyle\sum_{m=1}^{n} \tau_m \sum_{k=1}^{K} (\eta_{n,k}^2 + \|f^m - i_\delta f^m\|^2_{L^2(\Lambda_k)})\Big)^{\frac{1}{2}} \\ & + \|u_0 - i_\delta u_0\|_{L^2(\Lambda)}.\end{aligned} \tag{33}$$

□

Proof:
By taking v equal to $v_\delta \in X_{\delta-}$ in (10) we have

$$(u^n, v_\delta) + \tau_n(\frac{\partial u^n}{\partial x}, \frac{\partial v_\delta}{\partial x}) = (u^{n-1}, v_\delta) + \tau_n(f^n, v_\delta).$$

By taking $v_\delta \in X_{\delta-}$ in (24), the property of exactness of the quadrature formula implies,

$$(u_\delta^n, v_\delta) + \tau_n(\frac{\partial u_\delta^n}{\partial x}, \frac{\partial v_\delta}{\partial x}) = (u_\delta^{n-1}, v_\delta) + \tau_n(f^n, v_\delta)_\delta.$$

The difference of the last two equations yields

$$(u^n - u_\delta^n, v_\delta) + \tau_n(\frac{\partial}{\partial x}(u^n - u_\delta^n), \frac{\partial v_\delta}{\partial x}) = (u^{n-1} - u_\delta^{n-1}, v_\delta) + \tau_n(f^n, v_\delta) - \tau_n(f^n, v_\delta)_\delta.$$

By addition and substraction of the terms $(u^n - u_\delta^n, v)$ and $\tau_n(\frac{\partial}{\partial x}(u^n - u_\delta^n), \frac{\partial v}{\partial x})$, we obtain

$$\begin{aligned}(u^n - u_\delta^n, v) \quad &+\tau_n(\frac{\partial}{\partial x}(u^n - u_\delta^n), \frac{\partial v}{\partial x}) \\ &= (u^n - u_\delta^n, v - v_\delta) + \tau_n(\frac{\partial}{\partial x}(u^n - u_\delta^n), \frac{\partial}{\partial x}(v - v_\delta)) \\ &+(u^{n-1} - u_\delta^{n-1}, v_\delta) + \tau_n(f^n, v_\delta) - \tau_n(f^n, v_\delta)_\delta.\end{aligned}$$

Then,

$$\begin{aligned}a(u^n - u_\delta^n, v) = a(u^n - u_\delta^n, v - v_\delta) + (u^{n-1} - u_\delta^{n-1}, v_\delta) \\ + (u^{n-1} - u_\delta^{n-1}, v_\delta) + \tau_n(f^n, v_\delta) - \tau_n(f^n, v_\delta)_\delta. \qquad (34)\end{aligned}$$

We interested now to the term $a(u^n - u_\delta^n, v - v_\delta)$, for that we can write

$$\begin{aligned}a(u^n - u_\delta^n, v - v_\delta) &= a(u^n, v - v_\delta) - a(u_\delta^n, v - v_\delta) \\ &= (u^{n-1}, v - v_\delta) + \tau_n(f^n, v - v_\delta) - (u_\delta^n, v - v_\delta) \\ &\quad -\tau_n(\frac{\partial u_\delta^n}{\partial x}, \frac{\partial}{\partial x}(v - v_\delta)).\end{aligned}$$

Adding and subtracting the term $(u_\delta^{n-1}, v - v_\delta)$, we note that

$$\begin{aligned}a(u^n - u_\delta^n, v - v_\delta) = \quad &(u^{n-1} - u_\delta^{n-1}, v - v_\delta) - (u_\delta^n - u_\delta^{n-1}, v - v_\delta) \\ &+\tau_n(f^n, v - v_\delta) - \tau_n(\frac{\partial u_\delta^n}{\partial x} u_\delta^n, \frac{\partial}{\partial x}(v - v_\delta)).\end{aligned}$$

So

$$\begin{aligned}a(u^n - u_\delta^n, v - v_\delta) = (u^{n-1} - u_\delta^{n-1}, v - v_\delta) - (u_\delta^n - u_\delta^{n-1}, v - v_\delta) \\ +\tau_n(f^n, v - v_\delta) - \tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \frac{\partial u_\delta^n}{\partial x}(x) \frac{\partial}{\partial x}(v - v_\delta)(x) dx\end{aligned}$$

By integrating by parts the last term of the previous equation, we obtain

$$\begin{aligned}a(u^n - u_\delta^n, v - v_\delta) = (u^{n-1} - u_\delta^{n-1}, v - v_\delta) - (u_\delta^n - u_\delta^{n-1}, v - v_\delta) + \tau_n(f^n, v - v_\delta) \\ -\tau_n \sum_{k=1}^{K-1} \left[\frac{\partial u_\delta^n}{\partial x}\right](a_k)(v - v_\delta)(a_k) + \tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \frac{\partial^2 u_\delta^n}{\partial x^2}(x)(v - v_\delta)(x) dx.\end{aligned}$$

So,

$$a(u^n - u_\delta^n, v - v_\delta) = (u^{n-1} - u_\delta^{n-1}, v - v_\delta)$$

$$+\tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \Big(f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}(x)\Big)(x)(v - v_\delta)(x)dx$$
$$-\tau_n \sum_{k=1}^{K-1} \Big[\frac{\partial u_\delta^n}{\partial x}\Big](a_k)(v - v_\delta)(a_k)$$

By inserting the previous equation to (34), we obtain

$$\begin{aligned} a(u^n - u_\delta^n, v) &= (u^{n-1} - u_\delta^{n-1}, v) \\ &+\tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \Big(f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}(x)\Big)(x)(v - v_\delta)(x)dx \\ &-\tau_n \sum_{k=1}^{K-1} \Big[\frac{\partial u_\delta^n}{\partial x}\Big](a_k)(v - v_\delta)(a_k) + \tau_n(f^n, v_\delta) - \tau_n(f^n, v_\delta)_\delta. \end{aligned}$$

By using (19), we have

$$\begin{aligned} a(u^n - u_\delta^n, v) &= (u^{n-1} - u_\delta^{n-1}, v) \\ &+\tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \Big(f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}(x)\Big)(x)(v - v_\delta)(x)dx \\ &+\tau_n \sum_{k=1}^{K} \Big(\int_{a_{k-1}}^{a_k} f^n(x) v_\delta(x)dx - \sum_{j=1}^{N_k} f^n(\xi_j^{N_k}) v_\delta(\xi_j^{N_k})\rho_j^{N_k}\Big) \\ &-\tau_n \sum_{k=1}^{K-1} \Big[\frac{\partial u_\delta^n}{\partial x}\Big](a_k)(v - v_\delta)(a_k). \end{aligned}$$

By using (20), we can write

$$\begin{aligned} a(u^n - u_\delta^n, v) = \tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \Big(f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}(x)\Big)(x)(v - v_\delta)(x)dx \\ +(u^{n-1} - u_\delta^{n-1}, v) - \tau_n \sum_{k=1}^{K-1} \Big[\frac{\partial u_\delta^n}{\partial x}\Big](a_k)(v - v_\delta)(a_k) \\ + \tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \Big(f^n - i_\delta f^n\Big)(x) v_\delta(x)dx. \quad (35) \end{aligned}$$

The idea is now to associate for each v in $H^1(\Lambda)$ the function (see [4])

$$w_\delta = \sum_{k=1}^{K} \Pi_{N_k-1}^{1,0}\Big(v - v(a_{k-1})\varphi_{k-1} - v(a_k)\varphi_k\Big) + \sum_{k=0}^{K} v(a_k)\varphi_k \qquad (36)$$

where φ_k are continuous functions, affixes on each Λ_k, equal to 1 in a_k and to 0 in the other $a_{k'}$, $k' \neq k$. The function w_δ is in Y_δ (in X_δ when v in $H_0^1(\Lambda)$). By taking v_δ equal to w_δ of the formula (36) in (35), we note that the last term disappears

$$a(u^n - u_\delta^n, v) = (u^{n-1} - u_\delta^{n-1}, v) + \tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \Big(f^n - i_\delta f^n\Big)(x) v(x) dx$$

$$+ \tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \Big(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}(x)\Big)(x)(v - v_\delta)(x) dx. \quad (37)$$

By apparition of the expression $(x - a_{k-1})^{\frac{1}{2}}(a_k - x)^{\frac{1}{2}}$ in the equation (37), we have

$$\begin{aligned}(u^n - u_\delta^n, v) \quad & +\tau_n\Big(\frac{\partial}{\partial x}(u^n - u_\delta^n), \frac{\partial}{\partial x} v\Big) \\ & = \tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \Big(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}\Big)(x) \\ & (x - a_{k-1})^{\frac{1}{2}}(a_k - x)^{\frac{1}{2}}(x - a_{k-1})^{-\frac{1}{2}}(a_k - x)^{-\frac{1}{2}}(v - w_\delta)(x) dx \\ & +(u^{n-1} - u_\delta^{n-1}, v) + \tau_n \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \Big(f^n - i_\delta f^n\Big)(x) v(x) dx.\end{aligned}$$

By applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}(u^n - u_\delta^n, v) + \tau_n\Big(\frac{\partial}{\partial x}(u^n - u_\delta^n), \frac{\partial}{\partial x} v\Big) & \leq (u^{n-1} - u_\delta^{n-1}, v) \\ +\tau_n \sum_{k=1}^{K} \Big(\int_{a_{k-1}}^{a_k} (f^n - i_\delta f^n)^2(x) dx\Big)^{\frac{1}{2}} & \Big(\int_{a_{k-1}}^{a_k} v^2(x) dx\Big)^{\frac{1}{2}} \\ +\tau_n \sum_{k=1}^{K} \Big(\int_{a_{k-1}}^{a_k} (i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2})^2(x) & (x - a_{k-1})(a_k - x) dx\Big)^{\frac{1}{2}} \\ \Big(\int_{a_{k-1}}^{a_k} (v - w_\delta)^2(x)(x - a_{k-1})^{-1}(a_k - x)^{-1} dx\Big)^{\frac{1}{2}}.\end{aligned}$$

By applying Lemma 3.1,we obtain

$$(u^n - u_\delta^n, v) + \tau_n\Big(\frac{\partial}{\partial x}(u^n - u_\delta^n), \frac{\partial}{\partial x} v\Big) \leq (u^{n-1} - u_\delta^{n-1}, v)$$

$$+\tau_n \sum_{k=1}^{K} \eta_{n,k} \|\frac{\partial v}{\partial x}\|_{L^2(\Omega_k)} + \tau_n \sum_{k=1}^{K} \|f^n - i_\delta f^n\|_{L^2(\Omega_k)} \|v\|_{L^2(\Omega_k)}. \quad (38)$$

By taking v equal to $u^n - u^n_\delta$ and using $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ we have

$$\|u^n - u^n_\delta\|^2_{L^2(\Lambda)} + \tau_n \|\frac{\partial}{\partial x}(u^n - u^n_\delta)\|^2_{L^2(\Lambda)} \le \frac{\|u^{n-1} - u^{n-1}_\delta\|^2_{L^2(\Lambda)}}{2} + \frac{\|u^n - u^n_\delta\|^2_{L^2(\Lambda)}}{2} + \tau_n \sum_{k=1}^{K} \Big(\eta_{n,k} + \|f^n - i_\delta f^n\|_{L^2(\Omega_k)}\Big) \|\frac{\partial}{\partial x}(u^n - u^n_\delta)\|^2_{L^2(\Lambda)}.$$

By applying Cauchy-Schwarz inequality, we obtain

$$\frac{\|u^n - u^n_\delta\|^2}{2} + \tau_n \|\frac{\partial}{\partial x}(u^n - u^n_\delta)\|^2 \le \frac{\|u^{n-1} - u^{n-1}_\delta\|^2}{2} + \tau_n \Big(\sum_{k=1}^{K} (\eta_{n,k} + \|f^n - i_\delta f^n\|_{L^2(\Omega_k)})^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{K} \|\frac{\partial}{\partial x}(u^n - u^n_\delta)\|^2\Big)^{\frac{1}{2}}.$$

We again use $ab \le \frac{a^2}{2} + \frac{b^2}{2}$, we obtain

$$\frac{\|u^n - u^n_\delta\|^2}{2} + \tau_n \|\frac{\partial}{\partial x}(u^n - u^n_\delta)\|^2 \le \frac{\|u^{n-1} - u^{n-1}_\delta\|^2}{2} + \frac{\tau_n}{2} \sum_{k=1}^{K} (\eta_{n,k} + \|f^n - i_\delta f^n\|_{L^2(\Omega_k)}) + \frac{\tau_n}{2} \sum_{k=1}^{K} \|\frac{\partial}{\partial x}(u^n - u^n_\delta)\|^2.$$

So,

$$\frac{\|u^n - u^n_\delta\|^2}{2} + \frac{\tau_n}{2} \|\frac{\partial}{\partial x}(u^n - u^n_\delta)\|^2 \le \frac{\|u^{n-1} - u^{n-1}_\delta\|^2}{2} + c\ \tau_n \sum_{k=1}^{K} \Big(\eta^2_{n,k} + \|f^n - i_\delta f^n\|^2_{L^2(\Omega_k)}\Big).$$

By summing up this inequality on n and using (16), we obtain the desired result. □

Combining the results of propositions 3.1 and 3.2 leads to the full a posteriori estimate.

Theorem 3.1. *Assume the data f continuous on $[0,T]$ with values in $H^{-1}(\Lambda)$ and $u_0 \in H^1(\Lambda)$. If moreover the regularity parameter σ_τ is bounded by a constant independent of τ, the following a posteriori error holds between the solution u of problem (1) and the solution $(u_\delta^n)_{0\le n\le N}$ of problem (23)-(24), for all $t_n, 1 \le n \le N$,*

$$[[u - u_{\delta\tau}]](t_n) \le c\Big(\sum_{m=1}^{n}\big(\eta_m^2 + \tau_m \sum_{k=1}^{K}(\eta_{n,k}^2 + \|f^m - i_\delta f^m\|^2_{L^2(\Lambda_k)})\big)\Big)^{\frac{1}{2}} + c\Big(\|u_0 - i_\delta u_0\|_{L^2(\Lambda)} + \|f - \Pi f\|_{L^2(0,t_n,H^{-1}(\Lambda))}\Big).$$

□

3.3 An upper bound for the error indicators.

The idea is now to prove separate bounds for each indicators η_n and $\eta_{n,k}$. We begin with the η_n.

Proposition 3.3. *Assume the data f continous on $[0,T]$ with values in $H^{-1}(\Lambda)$ and the function $u_0 \in H_0^1(\Lambda)$. The following estimate holds for the indicator η_n defined in (27), $1 \le n \le N$*

$$\eta_n \le [[u^n - u_\delta^n]] + (\sigma_\tau)^{\frac{1}{2}}[[u^{n-1} - u_\delta^{n-1}]] + c\ (\tau_n)^{\frac{1}{2}}\|f - \pi_\tau f\|_{L^2(t_{n-1},t_n;H^{-1}(\Lambda))} + c\ (\tau_n)^{\frac{1}{2}}\Big(\|\partial_t(u - u_\tau)\|_{L^2(t_{n-1},t_n;H^{-1}(\Lambda))} + \|\partial_x(u - u_\tau)\|_{L^2(t_{n-1},t_n;L^2(\Lambda)}\Big).$$

□

Proof:
As previously, we use the triangular inequality

$$\eta_n \le (\frac{\tau_n}{3})^{\frac{1}{2}}\Big(\|\partial_x(u_\delta^n - u^n)\|_{L^2(\Lambda)} + \|\partial_x(u^n - u^{n-1})\|_{L^2(\Lambda)} + \|\partial_x(u^{n-1} - u_\delta^{n-1})\|_{L^2(\Lambda)}\Big). \tag{39}$$

By using (12), we have

$$[[u^n - u_\delta^n]] = \Big(\|u^n - u_\delta^n\|^2_{L^2(\Lambda)} + \tau_n\|\partial_x(u^n - u_\delta^n)\|^2_{L^2(\Lambda)}\Big)^{\frac{1}{2}}.$$

So, we derive

$$(\frac{\tau_n}{3})^{\frac{1}{2}}\|\partial_x(u^n - u_\delta^n)\|_{L^2(\Lambda)} \le [[u^n - u_\delta^n]]. \tag{40}$$

Similary, we use (12), we obtain

$$[[u^{n-1}-u_\delta^{n-1}]] = \Big(\|u^{n-1}-u_\delta^{n-1}\|^2_{L^2(\Lambda)} + \tau_{n-1}\|\partial_x(u^{n-1}-u_\delta^{n-1})\|^2_{L^2(\Lambda)}\Big)^{\frac{1}{2}}.$$

We also have

$$\tau_n\|\partial_x(u^{n-1}-u_\delta^{n-1})\|^2_{L^2(\Lambda)} \le \tau_{n-1}\sigma_\tau\|\partial_x(u^{n-1}-u_\delta^{n-1})\|^2_{L^2(\Lambda)}.$$

So, we have

$$(\frac{\tau_n}{3})^{\frac{1}{2}}\|\partial_x(u^{n-1}-u_\delta^{n-1}\|_{L^2(\Lambda)} \le (\sigma_\tau)^{\frac{1}{2}}[[u^{n-1}-u_\delta^{n-1}]]. \tag{41}$$

For estimate the term $(\frac{\tau_n}{3})^{\frac{1}{2}}\|\partial_x(u^n-u^{n-1}\|_{L^2(\Lambda)}$, we can use the same technical of proposition 3.1

When "applying" equation (2) to the function u_τ, we obtain for all $t \in [t_{n-1}, t_n]$ and $v \in H_0^1(\Lambda)$

$$\begin{aligned}(\partial_t u_\tau(t), v) + (\partial_x u_\tau(t), \partial_x v) = \ & (\frac{u^n-u^{n-1}}{\tau_n}, v) + \big(\partial_x(u_\tau(t)-u^n), \partial_x v\big) \\ & +(\partial_x u^n, \partial_x v),\end{aligned}$$

By injection the equation (10) in the last equation, we obtain

$$(\partial_t u_\tau(t), v) + (\partial_x u_\tau(t), \partial_x v) = (f^n, v) + \big(\partial_x(u_\tau(t)-u^n), \partial_x v\big).$$

Thus, subtracting this line from equation (2) leads to

$$(\partial_t(u-u_\tau)(t), v) + (\partial_x(u-u_\tau)(t), \partial_x v) = (f(t)-f^n, v) + (\partial_x(u^n-u_\tau(t)), \partial_x v).$$

By taking v equal u^n-u^{n-1} and integrating between t_{n-1} and t_n, we obtain

$$\int_{t_{n-1}}^{t_n}(\partial_t(u-u_\tau)(t), u^n-u^{n-1})ds + \int_{t_{n-1}}^{t_n}(\partial_x(u-u_\tau)(t), \partial_x(u^n-u^{n-1}))ds$$

$$= \int_{t_{n-1}}^{t_n}(f(s)-f^n, u^n-u^{n-1})ds - \int_{t_{n-1}}^{t_n}(\partial_x(u_\tau(t)-u^n, \partial_x(u^n-u^{n-1}))ds. \tag{42}$$

By using (31), the last term of (42) can be written

$$\begin{aligned}-\int_{t_{n-1}}^{t_n} & (\partial_x(u_\tau(t)-u^n), \partial_x(u^n-u^{n-1}))ds \\ &= \int_{t_{n-1}}^{t_n}\frac{t_n-s}{\tau_n}\|\partial_x(u^n-u^{n-1})\|^2_{L^2(\Lambda)}ds \\ &= \|\partial_x(u^n-u^{n-1})\|^2_{L^2(\Lambda)}\int_{t_{n-1}}^{t_n}\frac{t_n-s}{\tau_n}ds \\ &= \frac{\tau_n}{2}\|\partial_x(u^n-u^{n-1})\|^2_{L^2(\Lambda)}.\end{aligned}$$

Then the equation (42) becomes

$$\begin{aligned}\frac{\tau_n}{2}\|\partial_x(u^n-u^{n-1})\|^2_{L^2(\Lambda)} = & \int_{t_{n-1}}^{t_n}(\partial_t(u-u_\tau)(t),u^n-u^{n-1})ds \\ & +\int_{t_{n-1}}^{t_n}(\partial_x(u-u_\tau)(t),\partial_x(u^n-u^{n-1}))ds \\ & -\int_{t_{n-1}}^{t_n}(f-\pi_\tau f)(s),u^n-u^{n-1})ds.\end{aligned} \tag{43}$$

Next, in order to evaluate the terms in the right of the equation (43), we use divers Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned}\frac{\tau_n}{2}\|\partial_x(u^n-u^{n-1})\|^2 \le (\tau_n)^{\frac{1}{2}}\Big(&\|\partial_t(u-u_\tau)\|_{L^2(t_{n-1},t_n;H^{-1}(\Lambda))} \\ &+\|\partial_x(u-u_\tau)\|_{L^2(t_{n-1},t_n;L^2(\Lambda))}+\|f-\pi_\tau f\|_{L^2(t_{n-1},t_n;H^{-1}(\Lambda))}\Big).\end{aligned} \tag{44}$$

Combining all this yields the desired result. □

We evaluate now the indicator η. We need to introduce the following Lemma (see [4]).

Lemma 3.2.
For all $\varphi_N \in \mathbb{P}_N(\Lambda)$, we have

$$\int_{-1}^{1}\varphi_N'^2(\zeta)(1-\zeta^2)^2d\zeta \le \; c\; N^2\int_{-1}^{1}\varphi_N^2(\zeta)(1-\zeta^2)d\zeta \tag{45}$$

and

$$\int_{-1}^{1}\varphi_N^2(\zeta)d\zeta \le \; c\; N^2\int_{-1}^{1}\varphi_N^2(\zeta)(1-\zeta^2)d\zeta. \tag{46}$$

□

Proposition 3.4. *Assume the data f continuous on $[0,T]$ with values in $H^{-1}(\Lambda)$ and the function $u_0 \in H_0^1(\Lambda)$. The following estimate holds for the indicator $\eta_{n,k}$ defined in (29) for all $1 \le k \le K$ and $1 \le n \le N$:*

$$\begin{aligned}\eta_{n,k} \le c\Big(\|\frac{\partial}{\partial x}(u^n-u^n_\delta)\|_{L^2(\Lambda_k)}+\|\frac{(u^n-u^n_\delta)-(u^{n-1}-u^{n-1}_\delta)}{\tau_n}\|_{H^{-1}(\Lambda_k)} \\ +N_k^{-1}h_k\|f^n-i_\delta f^n\|_{L^2(\Lambda_k)}\Big).\end{aligned} \tag{47}$$

□

Proof:
We use equation (35) with $v_\delta = 0$ and v chosen as

$$\begin{cases} v = \Big(i_\delta f^n - \dfrac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \dfrac{\partial^2 u_\delta^n}{\partial x^2}\Big)(x - a_{k-1})(a_k - x) & \text{sur } \Lambda_k \\ v = 0 & \text{sur } \Lambda \setminus \Lambda_k \end{cases}$$

We obtain

$$\begin{aligned}\int_{a_{k-1}}^{a_k} &\Big(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}\Big)^2 (x - a_{k-1})(a_k - x)dx \\ &= a(\frac{u^n - u_\delta^n}{\tau_n}, v) - \int_{a_{k-1}}^{a_k} (f^n - i_\delta f^n) v(x) dx - (\frac{u^{n-1} - u_\delta^{n-1}}{\tau_n}, v) \\ &= \Big(\frac{\partial}{\partial x}(u^n - u_\delta^n), \frac{\partial v}{\partial x}\Big) - \int_{a_{k-1}}^{a_k} (f^n - i_\delta f^n) v(x) dx \\ &\quad + \Big(\frac{(u^n - u_\delta^n) - (u^{n-1} - u_\delta^{n-1})}{\tau_n}, v\Big).\end{aligned}$$

By using Cauchy-schwarz inequality, we have

$$\begin{aligned}\int_{a_{k-1}}^{a_k} &\Big(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}\Big)^2 (x - a_{k-1})(a_k - x)dx \\ &\leq \|\frac{(u^n - u_\delta^n) - (u^{n-1} - u_\delta^{n-1})}{\tau_n}\|_{H^{-1}(\Lambda_k)} \|\frac{\partial v}{\partial x}\|_{L^2(\Lambda_k)} \\ &+ \|\frac{\partial}{\partial x}(u^n - u_\delta^n)\|_{L^2(\Lambda_k)} \|\frac{\partial v}{\partial x}\|_{L^2(\Lambda_k)} + \|f^n - i_\delta f^n\|_{L^2(\Lambda_k)} \|v\|_{L^2(\Lambda_k)}. \quad (48)\end{aligned}$$

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ yields

$$\begin{aligned}\|\frac{\partial v}{\partial x}\|_{L^2(\Lambda_k)} &\leq 2\int_{a_{k-1}}^{a_k} \Big(\frac{\partial}{\partial x}(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2})\Big)^2 (x - a_{k-1})^2 (a_k - x)^2 dx \\ &+ \ 2\int_{a_{k-1}}^{a_k} \Big(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2}\Big)^2 (a_k + a_{k-1} - 2x)^2 dx\end{aligned}$$

By applying (45) et (46), we obtain

$$\|\frac{\partial v}{\partial x}\|_{L^2(\Omega_k)} \leq \ c\, N_k \|(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2})(x - a_{k-1})^{\frac{1}{2}} (a_k - x)^{\frac{1}{2}}\|_{L^2(\Omega_k)}.$$

Similarly, we have

$$\|v\|_{L^2(\Omega_k)} \leq \ c\, h_k \|(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2})(x - a_{k-1})^{\frac{1}{2}} (a_k - x)^{\frac{1}{2}}\|_{L^2(\Omega_k)}.$$

By substitution of the last two inequalities in (48), we derive

$$\begin{aligned}
&\|(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2})(x-a_{k-1})^{\frac{1}{2}}(a_k - x)^{\frac{1}{2}}\|^2_{L^2(\Lambda_k)}\\
\leq\; & c\; N_k \| \frac{(u^n - u_\delta^n) - (u^{n-1} - u_\delta^{n-1})}{\tau_n} \|_{H^{-1}(\Lambda_k)}\\
&\quad \|(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2})(x-a_{k-1})^{\frac{1}{2}}(a_k - x)^{\frac{1}{2}}\|_{L^2(\Lambda_k)}\\
+\; & c\; N_k \|(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2})(x-a_{k-1})^{\frac{1}{2}}(a_k - x)^{\frac{1}{2}}\|_{L^2(\Lambda_k)}\\
&\quad \|\frac{\partial}{\partial x}(u^n - u_\delta^n)\|_{L^2(\Lambda_k)}\\
+\; & c\; h_k \|(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2})(x-a_{k-1})^{\frac{1}{2}}(a_k - x)^{\frac{1}{2}}\|_{L^2(\Lambda_k)}\\
&\quad \|f^n - i_\delta f^n\|_{L^2(\Lambda_k)}.
\end{aligned}$$

Simplifying by $\|(i_\delta f^n - \frac{u_\delta^n - u_\delta^{n-1}}{\tau_n} + \frac{\partial^2 u_\delta^n}{\partial x^2})(x-a_{k-1})^{\frac{1}{2}}(a_k - x)^{\frac{1}{2}}\|_{L^2(\Lambda_k)}$ and multiplying by N_k^{-1}, we derive the desired bound. □

4 Numerical experiments

We present some numerical experiments in order to test the obtained time and space indicators. In the following $\eta(t)$ and $\eta(N)$ denote respectively the time and the space errors indicators defined in (27) and (29). We consider the domain $\Lambda =]-1,1[$ and the solution $u(x,t) = e^t \sin(\pi\, x)$. All presented figures are plotted in logarithmic scale for the error axis.

Figure 1 (a) plots the time error and the time indicator (N fixed and t variable). We remark that we obtain the same slope of convergence and that the error is upper bounded by the time indicator $\eta(t)$ which is in coherence with proposition 3.3.

In figure 1 (b) we show the time error and the space indicator (t fixed and N variable). We remark that we obtain the same slope of convergence and that the error is lower bounded by the space indicator $\eta(N)$ which is in coherence with theorem 3.1.

In the second test we study the influence of time t and space N on the indicators $\eta(t)$ and $\eta(N)$. For N fixed equal to 30 and t varying between 0.1 and 10^{-5}, figure 2 (a) presents the error $\|u - u_N^t\|_{H^1(\Omega)}$ (plain red line), the error indicator $\eta(t)$ (dashed dotted blue line) and $\eta(N)$ (dashed blach line). It can thus be checked that the error indicator $\eta(N)$ is fully independent of t. Moreover the error indicator $\eta(t)$ descreases with exactly the same slope as the error until the discretization error becomes larger than the error. Similarly,

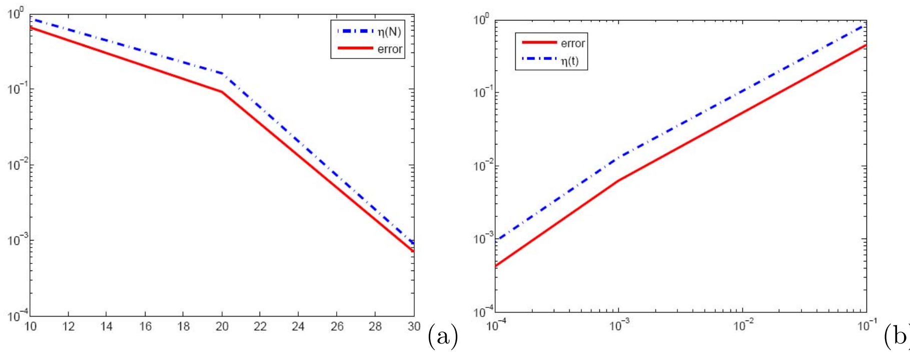

Figure 1: Error and indicators slope.

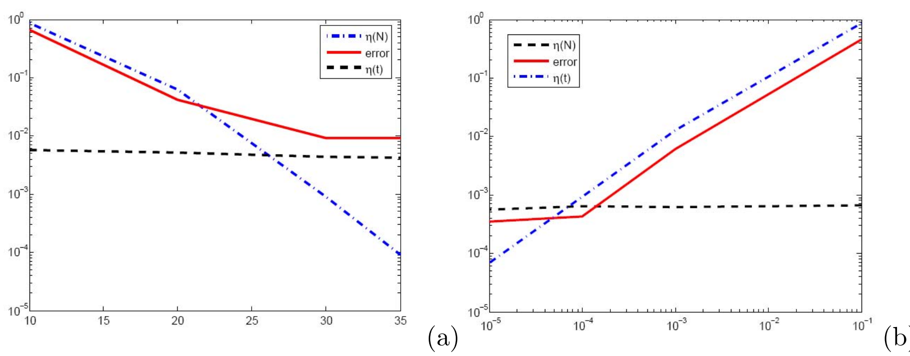

Figure 2: Influence of time and space on the indicators.

for t fixed equal to 10^{-5} and N varying between 10 and 35. Figure 2 (b) presents the full error $\|u - u_N^t\|_{H^1(\Omega)}$ (plain red line), the error indicator $\eta(t)$ (dashed black line) and $\eta(N)$ (dashed dotted blue line). Here, the error $\eta(t)$ are completely independent of N.

5 Conclusion

In this paper, we are interested in the posteriori analysis of the discretization of the one-dimensional heat equation by spectral element method which is the very efficient tool for mesh adaptivity. More precisely, we are constructed two kinds of indicators of residual type and we proved optimal upper and lower error bounds, in the sense of [6]. As usual and up to some terms concerning the approximation of the data, the global error between the solution of the exact equation and the solution of the time-space discrete problem at each discrete time is bounded by the Hilbertian sum of the indicators while each indicator is bounded by the local error.

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