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ON A CLASS OF FUNCTIONS WITH THE GRAPH BOX DIMENSION s

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To Professor Dan Pascali, at his 70's anniversary

Abstract

In our previous papers [1] – [3] the Hausdorff $h-$measures of a class of functions have been studied . In the present paper, we prove that this class of functions has the graph Box dimension s.

1. Introduction

The most important attributes of fractals are the dimensions. One of these is Box counting dimension.

Definition 1. *Let $\mathbf{R}^n$ be the Euclidean n - dimensional space.*

If $r_0 > 0$ is a given number, then, a continuous function $h(r)$, defined on $[0, r_0)$, nondecreasing and such that $\lim_{r\to 0} h(r) = 0$ is called a *measure function.*

If E is a nonempty and bounded subset of $\mathbf{R}^n$, $\delta > 0$ and h is a measure function, then, the Hausdorff h-measure of E is defined by:

$$H_h(E) = \lim_{\delta\to 0} \left\{ \inf_i \sum h(\rho_i) \right\}.$$

inf *being taken over all covers of E with a countable number of spheres of radii $\rho_i < \delta$.*

Particularly, when $h(r) = r^s$, the given measure is called the s-dimensional Hausdorff measure and is denoted by H_s.

Definition 2. *The Hausdorff dimension of a nonempty set $E \subset \mathbf{R}^n$ is the number defined by*

$$\dim_H E = \inf \{s : H_s(E) = 0\} = \sup \{s : H_s(E) = \infty\}$$

Key Words: Hausdorff h-measure; fractals.

It is known that the graph of a function $f : D \to \mathbf{R}$ is the set

$$\Gamma(f) = \{(x,\, f(x)) : x \in D\}.$$

In our papers ([1] − [3]), the following functions were introduced:

$$g(x) = \begin{cases} 2x & ,0 \le x < \frac{1}{2} \\ -2(x-1), & \frac{1}{2} \le x < \frac{3}{2} \\ 2(x-2) & ,\frac{3}{2} \le x < 2 \end{cases}, \tag{1}$$

$$f(x) = \sum_{i=1}^{\infty} \lambda_i^{s-2} g(\lambda_i x), (\forall) x \in [0,1], \tag{2}$$

where g is given in (1) and $\{\lambda_i\}_{i\in\mathbf{N}^*}$ is a sequence such that

$$(\exists)\, \varepsilon > 1 : \lambda_{i+1} \ge \varepsilon \lambda_i > 0, (\forall)\ i \in \mathbf{N}^*. \tag{3}$$

Theorem 1 ([3]) *Let h be a measure function, such that*

$$h(t)\tilde{\ } P(t) e^{T(t)},\ t \ge 0,$$

where P and T are polynomials:

$$P(t) = a_1 t + a_2 t^2 + ... + a_p t^p,\ p \ge 1,$$

$$T(t) = b_0 + b_1 t + ... + a_m t^m,$$

with the property

$$P'(t) + P(t) \cdot T(t) > 0,\ t \ge 0.$$

If f the function defined in (1)*, $s \in [0\,,2)$, $\{\lambda_i\}_{i\in\mathbf{N}^*} \in \mathbf{R}_+$ is a sequence that satisfies* (3)*, then: $H_h(\Gamma(f)) < +\infty$.*

In what follows we shall determine the Box dimension of the graph of the function given in (2), with a stronger restriction than (3).

There are many equivalent definitions ([6]) for the Box dimension, but we shall use the following one.

Definition 3. *Let β be a positive number and let E be a nonempty and bounded subset of $\mathbf{R}^2$. Consider the $\beta-$mesh of $\mathbf{R}^2$,*

$$\{[i\beta,\, (i+1)\beta] \times [j\beta,\, (j+1)\beta] : i,\, j \in \mathbf{Z}\}.$$

If $N_\beta(E)$ is the number of $\beta-$ mesh squares that intersect E, then the upper and lower Box dimension of E are defined by:

$$\overline{\dim}_B E = \overline{\lim_{\beta\to 0}} \frac{\log N_\beta(E)}{-\log \beta};\quad \underline{\dim}_B E = \underline{\lim_{\beta\to 0}} \frac{\log N_\beta(E)}{-\log \beta}.$$

If these limits are equal, the common value is called Box dimension of E *and is denoted by* $\dim_B E$.

For any given function $f : [0,1] \to \mathbf{R}$ and $[t_1, t_2] \subset [0,1]$, we shall denote by $R_f\,[t_1,\, t_2]$ *the oscillation of* f *on the interval* $[t_1, t_2]$, that is

$$R_f\,[t_1,\, t_2] = \sup_{t_1 \le t,\, u \le t_2} |f\,(t) - f\,(u)|\,.$$

In the second part of the paper we shall use the following results:

Lemma 1 ([6]). *Let* $f \in C\,[0,1]$, $0 < \beta < 1$ *and* m *be the least integer greater than or equal to* $1/\beta$. *If* N_β *is the number of the squares of the* $\beta-$*mesh that intersects* $\Gamma\,(f)$, *then*

$$\beta^{-1}\sum_{j=0}^{m-1} R_f\,[j\beta,\,(j+1)\,\beta] \le N_\beta \le 2m + \beta^{-1}\sum_{j=0}^{m-1} R_f\,[j\beta,\,(j+1)\,\beta]\,.$$

Lemma 2 ([6]). *If* E *is a set in* $\mathbf{R}^2$, *then*

$$\dim_H E \le \underline{\dim}_B E \le \overline{\dim}_B E.$$

For briefly, any C in this paper indicates a positive constant that may have different values.

2. Results

Theorem 2. *If* f *is the function given in* (2), $s \in [1,\,2)$ *and* $\{\lambda_i\}_{i\in\mathbf{N}^*} \in \mathbf{R}_+$ *is a sequence that satisfies* (3), *then* $\overline{\dim}_B \Gamma\,(f) \le s$.

Proof. Let us consider $0 < \beta < 1$, small enough, and $k \in \mathbf{N}^*$ such that:

$$\lambda_{k+1}^{-1} \le \beta < \lambda_k^{-1}. \tag{4}$$

Then for every $0 \le x \le 1 - \beta$:

$$|f(x+\beta) - f(x)| = \left|\sum_{i=1}^{\infty} \lambda_i^{s-2}\left\{g(\lambda_i(x+\beta)) - g(\lambda_i x)\right\}\right| \le$$

$$\le \sum_{i=1}^{k} \lambda_i^{s-2}\,|g(\lambda_i(x+\beta)) - g(\lambda_i x)| + \sum_{i=k+1}^{\infty} \lambda_i^{s-2}\,|g(\lambda_i(x+\beta)) - g(\lambda_i x)|$$

Since

$$|g(\lambda_i(x+\beta)) - g(\lambda_i x)| \le 2,$$

then

$$|f(x+\beta)-f(x)|=2\left[\beta\sum_{i=1}^{k}\lambda_i^{s-1}+\sum_{i=k+1}^{\infty}\lambda_i^{s-2}\right].$$

Using the condition (3) it can be deduced that

$$\sum_{i=1}^{k}\lambda_i^{s-1}<C_1\lambda_k^{s-1};\ \sum_{i=k=1}^{\infty}\lambda_i^{s-2}<C_2\lambda_{k+1}^{s-2},$$

so

$$|f(x+\beta)-f(x)|\leq 2\beta C_1\lambda_k^{s-1}+2C_2\lambda_{k+1}^{s-2}. \tag{5}$$

Using (4) and (5) we obtain

$$|f(x+\beta)-f(x)|\leq 2\beta C_1\left(\beta^{-1}\right)^{s-1}+2C_2\beta^{2-s}\Leftrightarrow$$

$$|f(x+\beta)-f(x)|\leq C\beta^{2-s},$$

From lemma 1 we deduce that

$$N_\beta\leq 2m+\beta^{-1}\sum_{j=0}^{m-1}R_f\left[j\beta,\,(j+1)\,\beta\right]\leq 2\beta^{-1}+\beta^{-1}\left(\beta^{-1}C\beta^{2-s}\right)\Rightarrow$$

$$N_\beta\leq 2\beta^{-1}+C\beta^{-s}.$$

Since $\beta\in(0,1)$ and $s\in[1,\,2)$, then $\beta^{-1}<\beta^{-s}$ and $N_\beta\leq C\beta^{-s}$. Therefore

$$\overline{\dim}_B\Gamma\left(f\right)=\overline{\lim_{\beta\to 0}}\frac{\log N_\beta}{-\log\beta}\leq\overline{\lim_{\beta\to 0}}\frac{\log C-s\log\beta}{-\log\beta}=s,$$

so, $\overline{\dim}_B\Gamma\left(f\right)\leq s$.

Theorem 3. *In the hypotheses of the theorem* 2*, if* $\varepsilon>2$, $\lambda_1>1$ *and* $\lambda_{i+1}\lambda_{i-1}>\lambda_i^2$, *for every* $i\in\mathbf{N}^*-\{1\}$, *then* $\dim_H\Gamma\left(f\right)=s$.

Proof. The proof follows that of the theorem 8.2 from [5].

Let S be a square with sides of length h, parallel to the coordinates axes. Let I be the interval of projection of S onto the x-axis. We show that the Lebesgue measure of the set $E=\{x:(x,\ f\left(x\right))\in S\}$ can not be too big.

Let us define

$$f_k\left(x\right)=\sum_{i=k+1}^{\infty}\lambda_i^{s-2}g\left(\lambda_i x\right).$$

Since $|g\left(\lambda_i x\right)|\leq 1$, $s-2<0$ and $\lambda_{i+1}\geq\varepsilon\lambda_i>2\lambda_i$, $(\forall)\,i\in\mathbf{N}^*$, we have

$$|f\left(x\right)-f_k\left(x\right)|\leq\sum_{i=k+1}^{\infty}\lambda_i^{s-2}\left|g\left(\lambda_i x\right)\right|\leq\sum_{i=k+1}^{\infty}\lambda_i^{s-2}<\frac{\lambda_{k+1}^{s-2}}{1-\varepsilon^{s-2}}<2\lambda_{k+1}^{s-2}. \tag{6}$$

Indeed,

$$s \in [1,2)\,,\ \varepsilon > 2 \Rightarrow \frac{1}{2} < \frac{1}{\varepsilon} \leq \varepsilon^{s-2} \Rightarrow \frac{1}{1-\varepsilon^{s-2}} < 2.$$

A point x is called an exceptional point for g if the derivative $g'(x)$ doesn't exist.

For non-exceptional x,

$$|f_k'(x)| = \left|\sum_{i=1}^{k} \lambda_i^{s-2} g'(\lambda_i x)\right| \geq \left|\lambda_k^{s-1}\right| |g'(\lambda_i x)| - \sum_{i=1}^{k-1} \lambda_i^{s-1} |g'(\lambda_i x)| =$$

$$= 2\lambda_k^{s-1} - \sum_{i=1}^{k-1} \lambda_i^{s-1} |g'(\lambda_i x)|\,.$$

Using Holder inequalities, it can be proved that there is $k \in \mathbf{N}^*$ such that

$$|f_k'(x)| \geq \lambda_k^{s-1}.$$

First suppose that the square S has the side $h = \lambda_k^{-1}$, for such k. Let m be a natural number such that

$$\lambda_{k+m}^{s-2} \leq h = \lambda_k^{-1} < \lambda_{k+m-1}^{s-2}.$$

$$\lambda_{i+1}\lambda_{i-1} > \lambda_i^2,\ (\forall)\, i \geq 2 \Rightarrow \frac{\lambda_k}{\lambda_{k-1}} < \frac{\lambda_{k+1}}{\lambda_k} < \ \ldots\ < \frac{\lambda_{k+m-1}}{\lambda_{k+m-2}} \Rightarrow$$

$$\left(\frac{\lambda_{k+1}}{\lambda_k}\right)^{(m-1)(2-s)} \lambda_k^{2-s} \leq \left[\lambda_k \frac{\lambda_{k+1}}{\lambda_k}\frac{\lambda_{k+2}}{\lambda_{k+1}} \cdots \frac{\lambda_{k+m-1}}{\lambda_{k+m-2}}\right]^{2-s} = \lambda_{k+m-1}^{2-s}. \quad (7)$$

But,

$$\lambda_k^{-1} < \lambda_{k+m-1}^{s-2} \Rightarrow \lambda_{k+m-1}^{2-s} < \lambda_k \Rightarrow \left(\frac{\lambda_{k+1}}{\lambda_k}\right)^{(m-1)(2-s)} \lambda_k^{2-s} < \lambda_k \Rightarrow$$

$$\left(\frac{\lambda_{k+1}}{\lambda_k}\right)^{(m-1)(2-s)} < \lambda_k^{s-1} = \left(\frac{\lambda_k}{\lambda_{k-1}} \cdots \frac{\lambda_2}{\lambda_1}\lambda_1\right)^{s-1} \Rightarrow$$

$$\left(\frac{\lambda_{k+1}}{\lambda_k}\right)^{(m-1)(2-s)} < \left(\frac{\lambda_{k+1}}{\lambda_k}\right)^{(k-1)(s-1)} \lambda_1^{s-1},$$

because, by hypothesis, the sequence $\left\{\frac{\lambda_{i+1}}{\lambda i}\right\}_{i\in\mathbf{N}^*}$ is increasing.

Hence, taking logarithm, we obtain

$$(m-1)(2-s)\log\frac{\lambda_{k+1}}{\lambda_k} < (k-1)(s-1)\log\frac{\lambda_{k+1}}{\lambda_k} + (s-1)\log\lambda_1 \Rightarrow$$

$$(m-1)\log\frac{\lambda_{k+1}}{\lambda_k} < \frac{(k-1)(s-1)}{2-s}\log\frac{\lambda_{k+1}}{\lambda_k} + \frac{s-1}{2-s}\log\lambda_1.$$

$$\frac{\lambda_{k+1}}{\lambda_k} > 2 \Rightarrow \log\frac{\lambda_{k+1}}{\lambda_k} > 1 \Rightarrow m < \frac{(k-1)(s-1)}{2-s} + \frac{s-1}{2-s}\frac{\log\lambda_1}{\log\frac{\lambda_{k+1}}{\lambda_k}} + 1 \Leftrightarrow$$

$$m < k\frac{s-1}{2-s} + \frac{3-2s}{2-s} + \frac{s-1}{2-s}\frac{\log\lambda_1}{\log\frac{\lambda_{k+1}}{\lambda_k}} \Leftrightarrow$$

$$m < \frac{k}{2-s}\left[s-1+\frac{3-2s}{k}+\frac{s-1}{k}\frac{\log\lambda_1}{\log\frac{\lambda_{k+1}}{\lambda_k}}\right] \Rightarrow$$

$$m < \frac{k}{2-s}\left[(s-1)\left(1+\frac{1}{k}\frac{\log\lambda_1}{\log\frac{\lambda_{k+1}}{\lambda_k}}\right)+\frac{3-2s}{k}\right].$$

But,

$$s \in [1,\, 2) \Rightarrow |3-2s| < 1 \Rightarrow \frac{3-2s}{k} < 1,\, k \in \mathbf{N}^* \Rightarrow$$

$$m < \frac{k}{2-s}\left[(s-1)\left(1+\frac{\log\lambda_1}{\log\frac{\lambda_2}{\lambda_1}}\right)+1\right] \Rightarrow$$

$$m < k\left[\frac{s-1}{2-s}\left(1+\frac{\log\lambda_1}{\log\frac{\lambda_2}{\lambda_1}}\right)+\frac{1}{2-s}\right] \Rightarrow m < ka,$$

$$m < k\left[C\cdot\frac{s-1}{2-s}+1\right] = ka,$$

where $C = 1 + \frac{\log\lambda_1}{\log\frac{\lambda_2}{\lambda_1}}$ and $a = C + 1$ don't depend on k.

If $m = 1$, then $(x,\, f(x)) \in S$ if $(x,\, f_k(x)) \in S_1$, where S_1 is a rectangle obtained by extending S at a distance $2\lambda_{k+1}^{s-2}$ above and below. The derivative changes sign at most once in the interval. On each section on which $f_k'(x)$ is of constant sign, $|f_k'(x)| > \lambda_k^{s-1}$. Thus, $(x,\, f_k(x)) \in S_1$ if x lies in an interval of length at most $\frac{1}{2}\lambda_k^{1-s}$ times the height of S_1.

$$L^1(E) \leq 2\cdot\frac{1}{2}\lambda_k^{1-s}\cdot 5h = 5h^s$$

If $m > 1$, dividing I in subintervals that satisfies the conditions that $f_k', \dots,$ f_{k+m-1}' have constant signs and using (7), E can be covered by at most $\frac{1}{4} \cdot 2^{m-1} \left(\frac{\lambda_{k+m-1}}{\lambda_k}\right)^{s-1}$ intervals of height less than $5h$. So,

$$L^1(E) \leq 2 \cdot 2^{m-1} \left(\frac{\lambda_{k+m-1}}{\lambda_k}\right)^{s-1} \cdot 5h \cdot \frac{1}{2} \lambda_{k+m-1}^{1-s} \leq 5 \cdot 2^{m-1} h^s \leq 5 \cdot 2^{ak} h^s.$$

Thus, there exists constants b and c such that $L^1(E) \leq cb^k h^s$ if $h = \lambda_k^{-1}$.

Analogous, if S is a square of side h, where $\lambda_{k+1}^{-1} < h < \lambda_k^{-1}$, then, $L^1(E) \leq c_1 h^t$, $t < s$.

If $\{U_i\}$ is any cover of $\Gamma(f)$, and we consider $U_i \subset S_i$, where S_i is a square with the side equal to $|U_i|$, then $[0, 1] \subset \bigcup_i E_i$, with $E_i = \{x : (x, f(x)) \in S_i\}$. Then

$$\sum_i |U_i|^t = \sum_i 2^{-\frac{1}{2}t} |S_i|^t \geq c_1^{-1} \sum_i L^1(E_i) \geq c_1^{-1} \Rightarrow H_s(\Gamma(f)) \geq c_1^{-1} > 0,$$

if $t < s \Rightarrow \dim_H \Gamma(f) = s$.

Theorem 4. *In the hypotheses of the theorem* 3, $\dim_B \Gamma(f) = s$.

Proof. Using the theorems 2 and 3 and lemma 2, it results that

$$s = \dim_H \Gamma(f) \leq \underline{\dim}_B \Gamma(f) \leq \overline{\dim}_B \Gamma(f) \leq s \Rightarrow \dim_B \Gamma(f) = s.$$

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