An. Şt. Univ. Ovidius Constanţa Vol. 12(2), 2004, 51–64

MULTILEVEL SCHWARZ METHOD FOR THE MINIMIZATION WITH CONSTRAINTS OF NON-QUADRATIC FUNCTIONALS

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To Professor Dan Pascali, at his 70's anniversary

Abstract

We succinctly present the results in [2] and [3] on the convergence rate of a multilevel method for the constrained minimization of non-quadratic functionals. The main goal of this paper is to check up the dependence of this convergence rate on the mesh and overlapping parameters by numerical tests concerning the solution of the two-obstacle problem of a nonlinear elastic membrane.

AMS subject classification: 65N55, 65N30, 65J15

1 Introduction

The literature on the domain decomposition methods is very large. We can see, for instance, the papers in the proceedings of the annual conferences on domain decomposition methods starting with [8], or those cited in the books [12] and [13]. The multilevel or multigrid methods can be viewed as domain decomposition methods and we can cite, for instance, the results obtained by [9], [10], and [13]. Evidently, this list is not exhaustive and it can be completed with a lot of other papers.

In [1], the convergence of a Schwarz method for variational inequalities coming from the minimization of a quadratic functional has been proved. In

Key Words: domain decomposition methods, variational inequalities, non-quadratic minimization, multigrid and multilevel methods, finite element methods, nonlinear obstacle problems.

that paper, the convex set is not assumed to be decomposed as a sum of convex subsets. This method has been extended to the one- and two-level methods in [4]. Also, its convergence for the constrained minimization of the non-quadratic convex functionals in a reflexive Banach space is proved in [2]. This result extends to variational inequalities that given in [15] for nonlinear equations. Using the general convergence theorem in [2], errors estimates for the one- two- and multilevel methods are given in [3]. These error estimates are similar with those which are obtained for the minimization of quadratic functionals in [4] or [14]. The main goal of this paper is to confirm by numerical examples the dependence on the mesh and overlapping parameters of the convergence rate given in [3].

The paper is organized as follows. In Section 2, we succinctly present the convergence result in [2]. In Section 3, we give the convergence rate for the multilevel method in [3], and, as some particular cases, we obtain the dependence of the convergence rate on the mesh and overlapping parameters for the multigrid and two-level methods. Finally, in Section 4, we illustrate and compare the convergence rates of the one- and two-level methods using numerical tests concerning the solution of the two-obstacle problem for a nonlinear elastic membrane.

2 General convergence result

In this section, a general algorithm and an error estimate theorem for it are given. This general theory, the proof of the theorem included, are given in detail in [2].

We consider a reflexive Banach space V, and some closed subspaces of it, $V_1, \cdots, V_m$. Also, let $K \subset V$ be a non empty closed convex set which satisfies together with the subspaces $V_1, \cdots, V_m$ the following

Assumption 2.1 *There exist two constants $C_0 > 0$ and $p > 1$ such that for any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K \textit{ for } i = 1, \cdots, m, \tag{2.1}$$

$$v - w = \sum_{i=1}^{m} v_i, \tag{2.2}$$

and

$$\sum_{i=1}^{m} ||v_i||^p \leq C_0^p \left(||v - w||^p + \sum_{i=1}^{m} ||w_i||^p \right). \tag{2.3}$$

We point out that we do not assume that the space V is written as $V = V_1 + \cdots + V_m$, as usually it is supposed in order to prove the convergence of the Schwarz method. Also, in the above assumption, even if it looks rather complicated, we do not assume that the convex set K should be written as a sum of convex subsets, as it is supposed for the solution of the obstacle problems. Moreover, we can easily check that if we impose the condition $K = K_1 + \cdots + K_m$, $K_i \subset V_i$, $i = 1, \cdots, m$, then equations (2.1) and (2.2) are verified.

Let $F : K \to R$ be a Gâteaux differentiable functional which will be assumed to be coercive if K is not bounded. We assume that for any real number $M > 0$ there exist two functions

$$\alpha_M(\tau) = A_M \tau^p, \quad \beta_M(\tau) = B_M \tau^{q-1}, \tag{2.4}$$

such that

$$< F'(v) - F'(u), v - u > \geq \alpha_M(||v-u||), \text{ for any } u, v \in K, \ ||u||, ||v|| \leq M, \tag{2.5}$$

and

$$\beta_M(||v-u||) \geq ||F'(v) - F'(u)||_{V'}, \text{ for any } u, v \in K, \ ||u||, ||v|| \leq M, \tag{2.6}$$

where F' is the Gâteaux derivative of F, and $A_M > 0$, $B_M > 0$ and $q > 1$ are some real constants. We have marked here that the constants A_M and B_M depend on M.

It is well known (see [6]) that if V and F satisfy the above assumptions, then the minimization problem

$$u \in K : F(u) \leq F(v), \text{ for any } v \in K \tag{2.7}$$

has an unique solution, and it also is the unique solution of the problem

$$u \in K : \ < F'(u), v - u > \geq 0, \text{ for any } v \in K. \tag{2.8}$$

The proposed algorithm corresponding to the subspaces $V_1, \cdots, V_m$ and the convex set K is written as follows

Algorithm 2.1 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, $w_i^{n+1} \in V_i$ satisfying*

$$w_i^{n+1} = \arg \min_{\substack{u^{n+\frac{i-1}{m}} + v_i \in K \\ v_i \in V_i}} G(v_i), \ \textit{with } G(v_i) = F(u^{n+\frac{i-1}{m}} + v_i), \tag{2.9}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

As for problem (2.7), since the subspaces V_i are reflexive Banach spaces, problem (2.9) has a unique solution and it also satisfies the variational inequality

$$\begin{array}{l} w_i^{n+1} \in V_i,\ u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K : \\ < F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1} > \geq 0, \\ \text{for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K. \end{array} \tag{2.10}$$

Concerning the convergence of Algorithm 1, we have the following

Theorem 2.1 *We consider that V is a reflexive Banach space, $V_1, \cdots, V_m$ are some closed subspaces of V, K is a non empty closed convex subset of V, and F is a Gâteaux differentiable functional on K which is assumed to be coercive if K is not bounded. We assume that the functional F satisfies (2.5) and (2.6), and we make Assumption 2.1. On these conditions, if u is the solution of problem (2.7) and u^n, $n \geq 0$, are its approximations obtained from Algorithm 2.1, then we have the following error estimations:*

(i) if $p = q$ we have

$$\begin{array}{l} F(u^n) - F(u) \leq \left(\frac{\hat{C}}{\hat{C}+1}\right)^n \left[F(u^0) - F(u)\right], \\ ||u^n - u||^p \leq \frac{\hat{C}+1}{\bar{C}} \left(\frac{\hat{C}}{\hat{C}+1}\right)^n \left[F(u^0) - F(u)\right]. \end{array} \tag{2.11}$$

(ii) if $p > q$ we have

$$\begin{array}{l} F(u^n) - F(u) \leq \dfrac{F(u^0)-F(u)}{\left[1+n\tilde{C}(F(u^0)-F(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}, \\ ||u - u^n||^p \leq \frac{\hat{C}}{\bar{C}} \dfrac{\left(F(u^0)-F(u)\right)^{\frac{q-1}{p-1}}}{\left[1+(n-1)\tilde{C}(F(u^0)-F(u))^{\frac{p-q}{q-1}}\right]^{\frac{(q-1)^2}{(p-1)(p-q)}}}. \end{array} \tag{2.12}$$

The constants $\hat{C}$, $\bar{C}$ and $\tilde{C}$ are written as

$$\hat{C} = \hat{C}(m, C_0, u^0) = B_M(\tfrac{p}{A_M})^{\frac{q}{p}} |\varepsilon_{ij}| \left[(1 + 2C_0) \left(F(u^0) - F(u)\right)^{\frac{p-q}{p(p-1)}} + \left(B_M(\tfrac{p}{A_M})^{\frac{q}{p}} |\varepsilon_{ij}|\right)^{\frac{1}{p-1}} C_0^{\frac{p}{p-1}} / \eta^{\frac{1}{p-1}} \right] / (1 - \eta), \tag{2.13}$$

$$\bar{C} = \frac{(2-\eta)A_M}{(1-\eta)p}, \tag{2.14}$$

$$\tilde{C} = \frac{p - q}{(p-1)\left(F(u^0) - F(u)\right)^{\frac{p-q}{q-1}} + (q-1)\hat{C}^{\frac{p-1}{q-1}}}. \tag{2.15}$$

The value of η can be arbitrary in $(0, 1)$.

The above algorithm can be viewed as a multiplicative Schwarz method, in a subspace correction variant, if we use the Sobolev spaces. In this way, we consider for a domain Ω in $\mathbf{R}^d$, $d \geq 1$, with Lipschitz continuous boundary $\partial\Omega$, an overlapping decomposition $\Omega = \cup_{i=1}^{m} \Omega_i$ in which the subdomains Ω_i have Lipschitz continuous boundary, too. We associate with the domain Ω the space $V = W_0^{1,s}(\Omega)$, $1 < s < \infty$, and with the subdomains Ω_i the subspaces $V_i = W_0^{1,s}(\Omega_i)$, $i = 1, \cdots, m$. For a convex sets $K \subset V$ satisfying

Property 2.1 *If $v, w \in K$, and if $\theta \in C^1(\Omega)$ with $0 \leq \theta \leq 1$, then $\theta v + (1 - \theta)w \in K$*

it has been proved in [2] that Assumption 2.1 holds. Consequently, provided that the functional F satisfies (2.5) and (2.6), Algorithm 2.1 converges and we can apply Theorem 2.1 to get the convergence rate. The above Sobolev spaces $W_0^{1,s}$ correspond to Dirichlet boundary conditions. Similar results can be obtained if we consider appropriate subspaces of $W^{1,s}$ for the mixed boundary conditions.

The constants $\hat{C}$ and $\bar{C}$ in the error estimations in Theorem 2.1 depend on the domain decomposition parameters through C_0. For the multilevel multiplicative Schwarz method, we show in the next section that Assumption 1 holds for any closed convex set K satisfying a certain property. In this case we are able to explicitly write the dependence of C_0 on the domain decomposition and mesh parameters.

3 Multilevel multiplicative Schwarz method

The framework and details concerning the proofs of the results in this section can be found in [3]. Over the domain $\Omega \subset \mathbf{R}^d$ we consider a family of L regular simplicial meshes $\mathcal{T}_{h_j}$, of mesh sizes h_j, such that $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, $j = 1, \cdots, L-1$. We write

$$\Omega_j = \bigcup_{\tau \in \mathcal{T}_{h_j}} \tau \tag{3.1}$$

and we assume that $\Omega = \Omega_L$. Also, we assume that if a node of $\mathcal{T}_{h_j}$ lies on $\partial\Omega_j$ then it lies on $\partial\Omega_{j+1}$, too, that is, it lies on $\partial\Omega$. Also, for the nodes $x_j \in \partial\Omega$ of $\mathcal{T}_{h_j}$, $j = 1, \cdots, L-1$, we consider the set ω_j defined as the union of the all $\tau \in \mathcal{T}_{h_j}$ having x_j as a vertex, and we define the set S_{x_j} as the union of ω_j with all $\tau \in \mathcal{T}_{h_{j+1}}$, $\tau \not\subset \Omega_j$, which are contained in the smallest sphere which is centered at x_j and contains ω_j. We assume that

$$\Omega_{j+1} \backslash \Omega_j \subset \bigcup_{x_j \text{ node of } \mathcal{T}_{h_j},\, x_j \in \partial\Omega} S_{x_j} \text{ for } j = 1, \cdots, L-1. \tag{3.2}$$

Since the mesh $\mathcal{T}_{h_{j+1}}$ is a refinement of $\mathcal{T}_{h_j}$, we have $h_{j+1} \leq h_j$, and we assume that there exists a constant γ, independent of the number of meshes, L, such that

$$1 < \gamma \leq \frac{h_j}{h_{j+1}}, \quad j = 1, \cdots, L-1. \tag{3.3}$$

At each level $j = 1, \cdots, L$, we consider an overlapping decomposition $\{O_j^i\}_{1 \leq i \leq M_j}$ of Ω_j, and we assume that the mesh partition $\mathcal{T}_{h_j}$ of Ω_j supplies a mesh partition for each O_j^i, $1 \leq i \leq M_j$. Also, we assume that the overlapping size for the domain decomposition at the level $1 \leq j \leq L$ is δ_j, i.e.,

$$O_j^i \cap \partial(\bigcup_{l \neq i} O_j^l) \neq \emptyset \text{ and } \mathrm{dist}(\partial O_j^i \backslash \partial\Omega_j, O_j^i \cap \partial(\bigcup_{l \neq i} O_j^l) \geq \delta_j \tag{3.4}$$

is satisfied. In addition, we suppose that there exists a constant C such that

$$\mathrm{diam}(O_{j+1}^i) \leq C h_j, \; j = 1, \cdots, L-1, \; i = 1, \cdots, M_j. \tag{3.5}$$

Now, at each level $j = 1, \cdots, L$, we color the subdomains O_j^i, $i = 1, \cdots, M_j$, such that the subdomains with the same color do not intersect with each other, and the union of the subdomains O_j^l having the color i will be denoted by Ω_j^i, $i = 1, \cdots, m_j$. Finally, we assume that $m_1 = 1$, and let us write

$$m = \max_{j=1,\cdots,L} m_j. \tag{3.6}$$

At each level $j = 1, \cdots, L$, we introduce the linear finite element spaces,

$$V_{h_j} = \{v \in C^0(\bar{\Omega}_j) : v|_\tau \in P_1(\tau), \; \tau \in \mathcal{T}_{h_j}, \; v = 0 \text{ on } \partial\Omega_j\}, \tag{3.7}$$

and, for $i = 1, \cdots, m_j$, we write

$$V_{h_j}^i = \{v \in V_{h_j} : v = 0 \text{ in } \Omega_j \backslash \Omega_j^i\} \tag{3.8}$$

The spaces V_{h_j} and $V_{h_j}^i$, $j = 1, \cdots, L$, $i = 1, \cdots, m_j$, will be considered as subspaces of $W^{1,s}$, $1 \leq s \leq \infty$. We denote by $||\cdot||_{0,s}$ the norm in L^s, and by $||\cdot||_{1,s}$ and $|\cdot|_{1,s}$ the norm and seminorm in $W^{1,s}$, respectively. The convex set will be a subset K_{h_L} of V_{h_L} satisfying

Property 3.1 *If $v, w \in K_{h_L}$, and if $\theta \in C^1(\Omega)$ with $0 \leq \theta \leq 1$, then $L_{h_L}(\theta v + (1-\theta)w) \in K_{h_L}$.*

Above, L_{h_L} is the P_1–Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_{h_L}$.

It is proved in [3] an inequality of Friedrichs-Poincaré type for the finite element spaces. In general, the constant in this inequality depends on how complicated is the shape of the domain. Since our meshes are regular, we give here the following simplified result

Lemma 3.1 *Let $\omega \subset \mathbf{R}^d$ be a domain of diameter H, and $\mathcal{T}_h$ a simplicial regular mesh partition of it. If v is a continuous function which is linear on each $\tau \in \mathcal{T}_h$, and $x^0 \in \bar{\omega}_0$ is a node of $\mathcal{T}_h$ such that $v(x^0) = 0$, then*

$$||v||_{0,s,\omega} \leq CHC_{d,s}(H,h)|v|_{1,s,\omega},$$

where $C_{d,s}(H,h) = \begin{cases} 1 & \text{if } d = s = 1 \text{ or } 1 \leq d < s \leq \infty \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-s}{s}} & \text{if } 1 \leq s < d < \infty, \end{cases}$
and the constant C is independent of domain and mesh.

The above lemma can be very useful in the various error estimations. In the proof of the following proposition we use some operators $I_{h_j} : V_{h_{j+1}} \to V_{h_j}$, whose properties are found using the above lemma.

Proposition 3.1 *Let, for each level $j = 1, \cdots, L$, $\Omega_j^1, \cdots, \Omega_j^{m_j}$ be the overlapping decomposition of the domain Ω_j defined in this section with $\Omega_L = \Omega$ and $m_1 = 1$. Then Assumption 2.1 is verified for the piecewise linear finite element spaces, $V = V_{h_L}$ and $V_j^i = V_{h_j}^i$, $j = 1, \cdots, L$, $i = 1, \cdots, m_j$ defined in (3.7) and (3.8), respectively, and any convex set $K = K_{h_L} \subset V_{h_L}$ with Property 3.1. The constant in (2.3) of Assumption 2.1 can be taken of the form*

$$C_0 = Cm^2(L+1)^{2-\frac{1}{p}-\frac{1}{s}} \sum_{j=1}^{L} [1 + (m-1)\frac{h_{j-1}}{\delta_j}] C_{d,s}(h_{j-1}, h_L) \qquad (3.9)$$

in which we take $h_0 = h_1$. The constant C is independent of the mesh and domain decomposition parameters.

The multigrid method is obtained from the multilevel method by taking the subsets O_j^i as the supports of the basis functions associated with the nodes of $\mathcal{T}_{h_j}$. Evidently, all the previous assumptions on the domain decompositions are satisfied and we can take $\delta_j = h_j$. In the multigrid methods, the construction of a finer mesh from a coarse one, is made following the same procedure of division of the simplexes at each level. Therefore, we can assume that $1 < \gamma \leq \frac{h_j}{h_{j+1}} \leq C\gamma, \quad j = 1, \cdots, L-1$. From (3.9), if we write $h = h_L$ and $H = h_1$, we get

$$C_0 = CL^{3-\frac{1}{p}-\frac{1}{s}} \gamma C_{d,s}(H,h). \qquad (3.10)$$

In the case of the two-level method, if we denote by $H = h_1$, $h = h_2$ and $\delta = \delta_2$, from (3.9), we get

$$C_0 = Cm^2[1 + (m-1)\frac{H}{\delta}] C_{d,s}(H,h) \qquad (3.11)$$

This expression of C_0 fits with that one given in [4] for the case of the minimization of quadratic functionals. Also, it is proved in [3] that for the one-level method the constant C_0 can be taken of the form

$$C_0 = C(m+1)(1+\frac{m-1}{\delta}). \tag{3.12}$$

4 Numerical example

We illustrate the error estimations for the one- and two- level methods given in the previous sections, by a numerical example concerning the two-obstacle problem of a nonlinear elastic membrane without exterior forces: find $u \in K$ such that

$$\int_\Omega |\nabla u|^{s-2}\nabla u \nabla(v-u) \geq 0 \text{ for any } v \in K.$$

Here, $\Omega \subset \mathbf{R}^2$, $K = W_0^{1,s}(\Omega) \cap [a,b]$, $a, b \in L^\infty(\Omega)$, $a \leq b$, and $1 < s < \infty$. Evidently, we take $V = W_0^{1,s}$, and our problem is equivalent with

$$u \in K : F(u) = \min_{v\in K} F(v), \text{ with } F(v) = \frac{1}{s}\int_\Omega |\nabla v|^s. \tag{4.1}$$

Using [7], we can show that if $1 < s \leq 2$, then we can take $\alpha_M(\tau) = \frac{\alpha}{(2M)^{2-s}}\tau^2$ and $\beta_M(\tau) = \beta\tau^{s-1}$ in (2.4). If $s \geq 2$, we get (see [5]) $\alpha_M(\tau) = \alpha\tau^s$ and $\beta_M(\tau) = \beta(2M)^{s-2}\tau$. The convex set K having Property 3.1, we can conclude that Algorithm 1 can be applied for the solution of problem (4.1).

In our numerical tests, the domain Ω is the rectangle $(0,4)\times(0,3)$, and the two obstacles of the convex set K are given by (see Figure 4.1.b): $a(x,y) = 3+\sqrt{\left(\frac{1}{6}\right)^2 - (x-2)^2 - (y-1.5)^2}$ if $(x-2)^2+(y-1.5)^2 \leq \left(\frac{1}{6}\right)^2$, else $a(x,y) = 0$, and $b(x,y) = 1/6-\sqrt{\left(\frac{1}{6}\right)^2 - (x-4/3)^2 - (y-3/4)^2}$ if $(x-4/3)^2+(y-3/4)^2 \leq \left(\frac{1}{6}\right)^2$, else $b(x,y) = \frac{19}{6}$. The meshes $\mathcal{T}_H$ and $\mathcal{T}_h$ contain right triangles, which are obtained through a rectangular uniform refinement of Ω. In Figure 4.1.a, the fine mesh $\mathcal{T}_h$ contains 30×30 rectangles, ie. 1800 triangles, and the coarse mesh $\mathcal{T}_H$ contains 6×6 rectangles, ie. 72 triangles. The obstacles a and b in Figure 4.1.b are plotted for a mesh $\mathcal{T}_h$ coming from a 60×60 rectangular uniform partition of Ω.

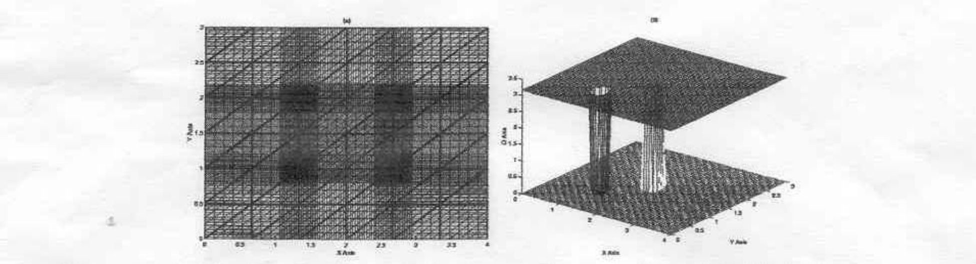

Figure 4.1. **(a) Meshes $\mathcal{T}_H$, $\mathcal{T}_h$, and the domain decomposition, (b) Obstacles a and b.**

The domain decomposition on the first level contains only one subdomain $O_1^1 = \Omega_1^1 = \Omega$, $M_1 = m_1 = 1$. The subdomains O_2^i, $i = 1, \ldots, M_2$, on the second level, are obtained from an uniform rectangular partition of Ω. In Figure 4.1.a we have $M_2 = 9$, and evidently, the number of the subdomains Ω_i is $m_2 = 4$. The width of the overlaps in this figure is of 2 triangles in $\mathcal{T}_h$.

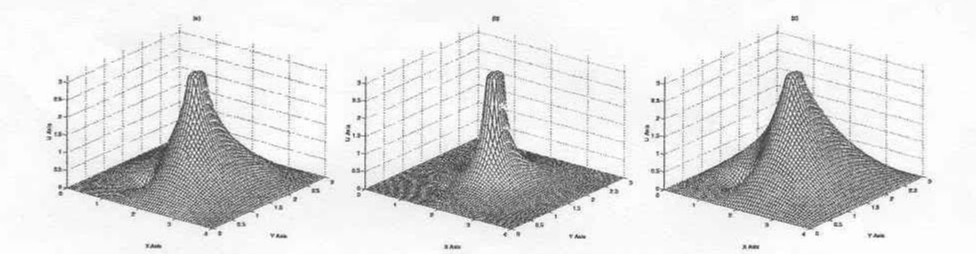

Figure 4.2. **Solutions for: (a) s=2, (b) s=1.5, (c) s=3.**

The computed solutions for $s = 2.0$, $s = 1.5$ and $s = 3.0$ are plotted in Figure 4.2 for a mesh $\mathcal{T}_h$ coming from a 60×60 rectangular uniform partition of Ω.

We have seen in the previous section that the constant C_0 depends on $1/\delta$ in equation (3.12), in the case of the one-level method, and on H/h and H/δ in equation (3.11), for the two-level method. We have tried to verify it by numerical tests for the nonlinear membrane problem taking various values of

H, h and δ. In all the numerical tests the calculus has been stopped at a relative error of 1.E-03 at the nodes of $\mathcal{T}_h$ between two consecutive computed solutions. The solution on the subdomains have been calculated by the relaxation method, which is a particular case of the Schwarz domain decomposition method. The computing of the solutions on subdomains has been stopped at a relative error of 1.E-05 at the nodes of $\mathcal{T}_h$ between two consecutive computed subsolutions. For the results in Figure 4.3, $H/h = 6$ and $H/\delta = 2$ stay unchanged while the coarse mesh size H varies and it corresponds to 2, 4, ... , 18, 20 segments on each side of the rectangular domain Ω. The number of the iterations is bounded for the two-level method, and it is in concordance with the fact that C_0 in (3.11) is constant. Also, we see that the number of iterations is an decreasing function of H for the one-level method. Since H/δ is constant, it follows that the number of iterations is an increasing function of $1/\delta$, and it is in concordance with C_0 in (3.12). For the results in Figure 4.4, we have taken $H = 5.0/12$, $h = 5.0/120$ and $\delta = 1h, 2h, \cdots, 10h$. We see that, in both cases, the number of iterations is a decreasing function of δ, and it is concordance with the expressions of C_0 in (3.12) and (3.11). For the results in Figure 4.5, $H = 5.0/6$, $\delta = 5.0/12$, and h corresponds to partitions $\mathcal{T}_h$ with $12, 24, 36, \cdots, 120$ segments on each side of the rectangular domain Ω. For the one-level method, the number of iterations is constant for $h \leq 5/24$, and it is in concordance with C_0 in (3.12). In the case of the two-level method, the number of iterations is a decreasing function of h for $s = 1.5$ and $s = 2$, and it is also in concordance with C_0 in (3.11). For $s = 3 > d = 2$, $C_{d,s}(H, h) = 1$, and the number of iterations should be bounded. In Figure 4.5.b, the number of iterations for $s = 3$ becomes constant for values of h less than 5.0/60.

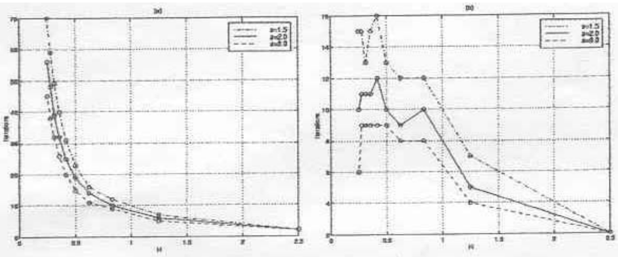

Figure 4.3. **Iterations for H/h and H/δ constant: (a) one level, (b) two levels.**

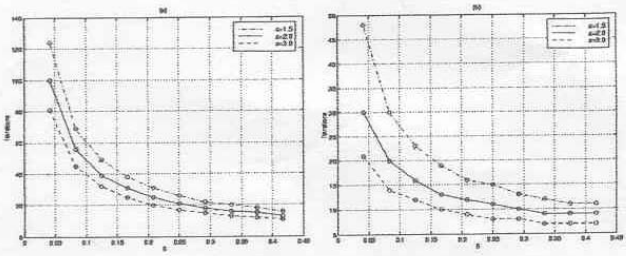

Figure 4.4. **Iterations for H and h constant, and δ variable: (a) one level, (b) two levels.**

In the tests in Figure 4.6 we have taken $h = 5.0/120$, $\delta = 5.0/20$ and $H = 5.0/20$, $5.0/12$, $5.0/10$, $5.0/8$ and $5.0/6$. In the case of the two-level method, the number of iterations is an increasing function of H which is in concordance with our constant C_0 in (3.11).
Finally, we see from the above numerical tests that the number of iterations for the two-level method is significantly less than that for the one-level method.

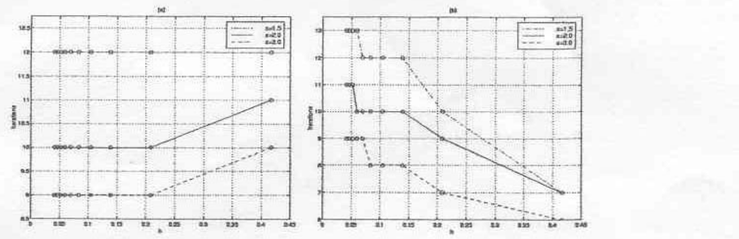

Figure 4.5. **Iterations for H and δ constant, and h variable: (a) one level, (b) two levels.**

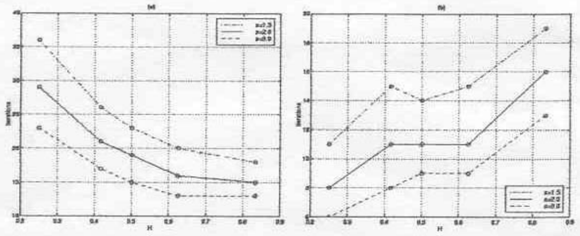

Figure 4.6. **Iterations for h and δ constant, and H variable: (a) one level, (b) two levels.**

Since the number of iterations is less in the two-level method than that one in the one-level method, even if the projection for the two-level method is a bit more complicated than that in the one-level method, the two-level method is more efficient in point of view of the computing time. For instance, we see in Figure 4.3 that for $H = 5.0/10$, $h = 5.0/60$ and $\delta = 5.0/20$, the number of iteration is: 23 for $s = 1.5$, 19 for $s = 2.0$, and 15 for $s = 3.0$, in the case of the one-level method, and 13 for $s = 1.5$, 10 for $s = 2.0$, and 9 for $s = 3.0$, in the case of the two-level method. The computing time obtained on a PC with one processor Intel Pentium III of 600MHz was: 18min45sec for $s = 1.5$, 6min16sec for $s = 2.0$, and 17min8sec for $s = 3.0$, in the case of the one-level method, and 13min54sec for $s = 1.5$, 4min43sec for $s = 2.0$, and 14min27sec for $s = 3.0$, in the case of the two-level method. Naturally, the computing time for $s = 2.0$ is less than that one for $s = 1.5$ or $s = 3.0$, since, in this case, we solve linear equations in the relaxation method. This case corresponds to the minimization of a quadratic functional. The finite element problem in these computing time tests have had 3481 unknowns.

Acknowledgment. The author acknowledges the financial support of IMAR under the contracts CNCSIS nr. 33079/2004 and CERES 4-187/2004.

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