An. Şt. Univ. Ovidius Constanţa Vol. 11(2), 2003, 137–144

ON MULTIVARIATE INTERPOLATION BY WEIGHTS

Dana Simian and Corina Simian

Abstract

The aim of this paper is to study a particular bivariate interpolation problem, named interpolation by weights. A minimal interpolation space is derived for these interpolation conditions. An integral formula for the remainder is given, as well as a superior bound for it. An expression for $g(D)(L_n(f))$ is obtained.

1 Introduction

Multivariate interpolation is a problem situated in the field of interest of many mathematicians. To find out an interpolation space for certain interpolation conditions, to derive the form of interpolation operator and a formula for the remainder are some of the problems in multivariate interpolation. The aim of this paper is to solve the problems enumerated above, for a particular multivariate interpolation scheme, named interpolation by weights.

To do this we need some preliminary notions, that we will present next.

Let Λ be a set of linear independent functionals and $\mathcal{F}$ be a space of functions which includes polynomials. Multivariate polynomial interpolation problem consists in finding a polynomial subspace $\mathcal{P}(\Lambda)$, such that, for a given function $f \in \mathcal{F}$ there exists a unique polynomial $p \in \mathcal{P}(\Lambda)$ satisfying the conditions:

$$\lambda(f) = \lambda(p), \ \forall \lambda \in \Lambda \tag{1}$$

In this case we say that the interpolation problem is well-posed in $\mathcal{P}(\Lambda)$, the space $\mathcal{P}(\Lambda)$ is an interpolation space for Λ or the pair $(\Lambda, \mathcal{P}(\Lambda))$ is correct.

Kergin proved that always exists an interpolation space for a set of conditions Λ, but we are interested in finding a minimal interpolation space, that is $\mathcal{P}(\Lambda) \subset \Pi_n^d$ with n the minimum of all possible values, or equivalent, the interpolation problem with respect to Λ is not well-posed in any subspaces of

Π^d_{n-1}.

To express the interpolation operator from a minimal interpolation space we can use a Newton formula. In multivariate case, Newton basis can be defined using a sequence of nested sets of multiindices :

$$I_0 \subset I_1 \subset \ldots \subset I_n;\ I_{-1} = \Phi;\ I_k \setminus I_{k-1} \subset \{\alpha:\ |\alpha| = k\};\ k = \overline{0,n} \quad (2)$$

$$I'_k = \{\alpha \in N^d:\ |\alpha| \le k\} \setminus I_k;\ k = \overline{0,n} \quad (3)$$

$$I_n \setminus I_{n-1} \ne \Phi; \qquad card\ I_n = dim\ \mathcal{P}(\Lambda); \quad (4)$$

and such that the functionals in Λ can be reindexed in the blocks:
$\Lambda^{(k)} = \{\lambda_\alpha:\ \lambda_\alpha \in \Lambda;\ \alpha \in I_k \setminus I_{k-1}\},\ k = 0,\ldots,n; \quad \Lambda = \{\lambda_\beta:\ \beta \in I_n\}$

The Newton polynomials $p_\alpha \in \Pi_{|\alpha|},\ \alpha \in I_n$ of $\mathcal{P}(\Lambda)$ have the properties $\lambda_\beta(p_\alpha) = \delta_{\alpha,\beta};\ \beta \in I_n;\ |\beta| \le |\alpha|$ and there exists the complementary polynomials $p^\perp_\alpha \in \Pi_{|\alpha|},\ \alpha \in I'_n$ such that $\Lambda(p^\perp_\alpha) = 0$ and $\Pi_n = span\{p_\alpha:\ \alpha \in I_n\} \oplus span\{p^\perp_\alpha:\ \alpha \in I'_n\}$.

The number of functionals in the block $\Lambda^{(k)}$ is $n_k = dim\ \mathcal{P}^0_k \le k+1$; $\mathcal{P}^0_k = \mathcal{P}(\Lambda) \cap \Pi^0_k$.

If $ker(\Lambda)$ is a polynomial ideal, then we say that Λ defines an ideal interpolation scheme.

Definition 1 *A subspace $\mathcal{P}(\Lambda) \subset \Pi^d_n$ is called a minimal interpolation space of order n with respect to Λ if:*
1. The pair $(\Lambda, \mathcal{P}(\Lambda))$ is correct.
2. Λ defines an ideal interpolation scheme.
3. The interpolation scheme $(\Lambda, \mathcal{P}(\Lambda))$, $\mathcal{P}(\Lambda) \subset \Pi^d_n$ is degree reducing (or equivalent, the interpolation problem with respect to Λ is not posed in any subspaces of Π^d_{n-1}).

Theorem 1 *Let Λ be a set of linear independent functionals. The polynomial subspace $\mathcal{P}(\Lambda)$ is a minimal interpolation space of order n, with respect to Λ, if and only if there exists a Newton basis of order n for $\mathcal{P}(\Lambda)$ with respect to Λ.*

The Newton basis for a minimal interpolation space can be derived using an inductive algorithm (see [4], [6]).

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two minimal interpolation spaces for the set of functionals Λ, then

$$dim(\mathcal{P}_1 \cap \Pi_k) = dim(\mathcal{P}_2 \cap \Pi_k).$$

Moreover, the Newton basis is unique iff $\mathcal{P}(\Lambda) = \Pi_n$(see [1], [2]).

We introduce a general divided difference, named λ- divided difference:

Definition 2 *Let* (p_α), $\alpha \in I_n$ *be the Newton basis for the minimal interpolation space of order n* $\mathcal{P}(\Lambda)$ *and* $\Lambda^{(k)}$ *be the proper blocks of functionals. The* λ*-divided difference is defined recursively by:*
$d_0[\lambda; f] = \lambda(f)$,
$d_{k+1}[\Lambda^{(0)}, \ldots, \Lambda^{(k)}, \lambda; f] = d_k[\Lambda^{(0)}, \ldots, \Lambda^{(k-1)}, \lambda; f] -$
$- \sum_{\alpha \in J_k} d_k[\Lambda^{(0)}, \ldots, \Lambda^{(k-1)}, \lambda_\alpha; f]\lambda(p_\alpha)$, *with* $J_k = I_k \setminus I_{k-1}$.

Taking $\lambda = \delta_x$ and $\Lambda = \{\delta_\theta : \ \theta \in \Theta \subset R^d\}$, we obtain the divided difference uses by T. Sauer in [3], from which, in the univariate case, we obtain the classical divided difference multiplied with the knots polynomial:

$$d_{n+1}[\theta_0, \ldots, \theta_n, x; f] = [\theta_0, \ldots, \theta_n, x; f] \cdot (x - \theta_0) \cdots (x - \theta_n)$$

Theorem 2 *([5]) With the notations in the Definition 2 and considering* L_n *as the corresponding interpolation operator, the following equalities hold:*

$$\lambda(L_n(f)) = \sum_{\alpha \in I_n} d_{|\alpha|}[\Lambda^{(0)}, \ldots, \Lambda^{(|\alpha|-1)}, \lambda_\alpha; f] \cdot \lambda(p_\alpha); \ \lambda \in (\Pi^d)', \quad (5)$$

$$\lambda(f - L_n(f)) = R_{\Lambda,\lambda}(f) = d_{n+1}[\Lambda^{(0)}, \ldots, \Lambda^{(n)}, \lambda; f]. \quad (6)$$

We call λ*- remainder the value* $R_{\Lambda,\lambda}$*, for a certain linear functional* Λ.

In [1] C. de Boor and A. Ron proves that, for a given set of points, Θ, always exists a minimal interpolation space,

$$\Pi_\Theta = (Exp_\Theta)\!\downarrow = span\{g\!\downarrow; \ g \in Exp_\Theta\}, \quad (7)$$

with $Exp_\Theta = span\{e_\theta; \ \theta \in \Theta\}$ and $f\!\downarrow = T_j f$, with j the smallest integer for which $T_j f \not= 0$, $T_j f$ being the Taylor polynomial of degree $\le j$ and $e_\theta(z) = e^{\theta \cdot z}$.

In the case of an arbitrary set of functionals, Λ, a minimal interpolation space is given by

$$H_\Lambda\!\downarrow = span\{g\!\downarrow; g \in H_\Lambda\}, \quad \text{with } H_\Lambda = span\{\lambda^\nu; \ \lambda \in \Lambda\}. \quad (8)$$

We denoted by λ^ν the generating function of the functional $\lambda \in \Lambda$.

2 Interpolation by weights

Definition 3 *Let* $X = \{x_1, \ldots, x_N\} \subset R^2$ *be a set of different points and*

$$W = \{(w_1^1, \ldots, w_N^1), \ldots, (w_1^N, \ldots, w_N^N)\} \subset Z_+^N \quad (9)$$

be a set of weights, such that the of functionals

$$\Lambda = \{\lambda_k \mid \lambda_k = \delta_{y_k},\ y_k = \sum_{i=1}^{N} w_k^i x_i;\ k = 1, \ldots, N\}, \tag{10}$$

be linear independent.

We name the interpolation problem given by the set of conditions Λ *interpolation by weights.*

Our aim is to find a minimal interpolation space for the conditions (10) and to derive a formula for the remainder in the interpolation by weights.

Proposition 1 *The weights used in the interpolation by weights satisfy the equality:*

$$\sum_{i=1}^{N}(w_i^k - w_l^k)x_i \neq 0,\ \forall k \neq l, k, l \in \{1, \ldots, N\}. \tag{11}$$

The conditions (7) express linear independence of functionals from (10).

Proposition 2 *The interpolation by weights scheme is an ideal interpolation scheme.*

Theorem 3 *A minimal interpolation space for the conditions of the interpolation by weights is*

$$H_\Lambda \downarrow = \Pi_Y = span\{e_{y_k} = \ \mid y_k \in Y;\ k = 1, \ldots, N\}, \tag{12}$$

$$Y = \{y_k = \sum_{i=1}^{N} w_k^i x_i;\ k = 1, \ldots, N\}. \tag{13}$$

Proof: We use the relations (8) and the fact that, for the functionals in the interpolation by weights, the generating function is $\lambda_k^\nu = e_{y_k}$.

Let us denote by n the order of the minimal interpolation space Π_Y. For this minimal interpolation space, there exists a Newton basis, that is, the functionals in Λ can be reindexed and put into blocks, using the sequence of index sets $\{I_0, \ldots, I_n\}$. Let $\Lambda^{(k)} = \{\lambda_r^{[k]}\}$, $k \in \{0, \ldots, n\}$, $r \in \{1, \ldots, n_k\}$, $n_k = \#I_k$, the functionals corresponding to the set of multiindices I_k. We associate to the blocks $\Lambda^{(k)}$ the corresponding blocks of points

$$Y^{(k)} = \{y_r^{[k]}\},\ k \in \{0, \ldots, n\},\ r \in \{1, \ldots, n_k\}. \tag{14}$$

Polynomials p_α, with $\alpha \in J_k = I_k \backslash I_{k-1}$, are denoted by $p_i^{[k]},\ k \in \{0,\ldots,n\};\ i \in \{1,\ldots,n_k\}$.
Then the followings relations hold:

$$\lambda_i^{[k]}(p_j^{[k]}) = \delta_{i,j} \ \Rightarrow\ \delta_{y_i^{[k]}}(p_j^{[k]}) = \delta_{i,j},\ \forall k = \overline{0,n};\ i,j = \overline{1,n_k}, \quad (15)$$

$$\lambda_i^{[l]}(p_j^{[k]}) = 0 \ \Rightarrow\ \delta_{y_i^{[l]}}(p_j^{[k]}) = 0,\ \forall l < k;\ i = \overline{1,n_k};\ j = \overline{1,n_l}. \quad (16)$$

Proposition 3 *If we reindex the weights from (9), such that*

$$y_i^{[k]} = \sum_{j=1}^{N} c_{i,j}^{[k]} x_j, \quad (17)$$

and denote by

$$C^{[k]} = (c_{j,i}^{[k]});\ P^{[k]} = (p_i^{[k]}(x_j));\ k = 0,\ldots n;\ j = 1,\ldots,N;\ i = 1,\ldots,n_k$$

the blocks matrix $C^{[k]}$ and $P^{[k]}$, with n_k columns, then the matrix

$$M = C^T \cdot P \quad (18)$$

is a block matrix of the same type with C and P, left triangular and with unitary diagonal .

Proof: Obviously results from (15), (16) and (17).

Taking in Definition 2, $\Lambda = \{\delta_{y_\lambda} : \lambda \in \Lambda\}$ and $\lambda = \delta_x$, we can formally write

$$d_{k+1}[\Lambda^{(0)},\ldots,\Lambda^{(k)},\lambda;f] = d_{k+1}[Y^{(0)},\ldots,Y^{(k)},x;f]. \quad (19)$$

We want to give an integral form for the divided difference given in Definition 2, that is for the remainder in the interpolation by weights.

Using the model in [3] we introduce the notion of path in a multiindices set and consider the following elements:

1. A path, μ 'in I_n is $\mu = (\mu_0,\ldots,\mu_n);\ \mu_k \in J_k = I_k \setminus I_{k-1};\ k = \overline{0,n}$;
2. $\mathcal{C}_n$ is the set of all paths. The number of path is $N_c = \prod_{k=0}^{n} n_k,\ n_k = card\ J_k$;
3. $\mathcal{C}_n(\alpha)$ is the set of paths $\mu \in \mathcal{C}_n$ having the property that $\mu_n = \alpha$;
4. The set of functionals according to a path $\mu \in \mathcal{C}_n$: $\Lambda^\mu = \{\lambda_{\mu_0},\ldots,\lambda_{\mu_n}\}$; $(\mu_0,\ldots,\mu_n) \in \mathcal{C}_n$;
5. The set $Y^\mu = \{y_{\lambda_{\mu_0}},\ldots,y_{\lambda_{\mu_n}}\} = \{y_{\mu_0},\ldots,y_{\mu_n}\};\ (\mu_0,\ldots,\mu_n) \in \mathcal{C}_n$;

6. The points blocks $Y^{(k)} = \{y_\lambda |\ \lambda \in \Lambda^{(k)}\}$;

7. The number $\Pi_\mu(\Lambda^\mu) = \Pi_\mu(Y^\mu) = \prod_{i=0}^{n-1} p_{\mu_i}(y_{\mu_{i+1}})$;

8. The differential operator $D^n_{Y^\mu} = D_{y_{\mu_n} - y_{\mu_{n-1}}} \dots D_{y_{\mu_1} - y_{\mu_0}}$.

We will need the application $f \to \int\limits_\Theta f$, $\Theta = \{\theta_0, \dots, \theta_k\} \subset R^2$, where

$$\int\limits_\Theta f = \int\limits_0^1 \int\limits_0^{s_1} \cdots \int\limits_0^{s_{k-1}} f\left(\theta_0 + s_1(\theta_1 - \theta_0) + \dots + s_k(\theta_k - \theta_{k-1})\right) \cdot ds_k \dots ds_1.$$

Theorem 4 *Let Y be the set of the associated points given in (14). Then*

$$d_{n+1}[Y^{(0)}, \dots, Y^{(n)}, x; f] = \sum_{\mu \in \mathcal{C}_n} p_{\mu_n}(x) \Pi_\mu(Y^\mu) \int\limits_{[Y^\mu, x]} D_{x - y_{\mu_n}} D^n_{Y^\mu} f \tag{20}$$

$$+ \sum_{j=1}^{n} \sum_{\mu \in \mathcal{C}_{j-1}} \sum_{\beta \in I'_j} p^\perp_\beta(x) \Pi_\mu(Y^\mu) \int\limits_{[Y^\mu, x]} D_{d_{\mu_{j-1}, \beta}} D^{j-1}_{Y^\mu} f,\ \forall n \in N,\ x \in R^2.$$

Proof: We use the definition of the interpolation by weights, the Theorem 3 from [3] and the equality:

$$\sum_{j=1}^{n} \sum_{\mu \in \mathcal{C}_{j-1}} \sum_{\beta \in I'_j} p^\perp_\beta(x) \Pi_\mu(Y^\mu) \int\limits_{[Y^\mu, x]} D_{d_{\mu_{j-1}, \beta}} D^{j-1}_{Y^\mu} f =$$

$$= \sum_{\beta \in I'_n} p^\perp_\beta(x) \sum_{j=|\beta|} \sum_{\mu \in \mathcal{C}_{j-1}} \Pi_\mu(Y^\mu) \int\limits_{[Y^\mu, x]} D_{d_{\mu_{j-1}, \beta}} D^{j-1}_{Y^\mu} f,\ \forall n \in N,\ x \in R^2.$$

Corollary 1 *Let $g \in \Pi^2$ and $g(D)$ be the differential operator with constant coefficients associated to it. The following equality holds*

$$\left(g(D)\left(L_n(f)\right)\right)(x) = \tag{21}$$

$$= \sum_{j=1}^{n} \sum_{\alpha \in I_j} \sum_{\mu \in \mathcal{C}_{j-1}} (g(D)p_\alpha)(x) p_{\mu_{j-1}}(y_\alpha) \Pi_\mu(Y^\mu) \int\limits_{[Y^\mu, y_\alpha]} D_{y_\alpha - y_{\mu_{j-1}}} D^{j-1}_{Y^\mu} f.$$

Proof: We take in Theorem 2 the functional λ by $\lambda_g = g(D)$, we apply Theorem 4 , we replace $n+1$ by $|\alpha|$, x with y_α and take into account that $p^\perp_\beta(y_\alpha) = 0,\ \forall y\alpha \in Y$. We obtain:

$$(g(D)\,(L_n(f)))\,(x) = \sum_{\alpha\in I_n} (g(D)p_\alpha)(x) \sum_{\mu\in\mathcal{C}_{|\alpha|-1}} p_{\mu_{|\alpha|-1}}(y_\alpha)\Pi_\mu(Y^\mu)\cdot$$
$$\cdot \int_{[Y^\mu,y_\alpha]} D_{y_\alpha-y_{\mu_{|\alpha|-1}}} D^{|\alpha|-1}_{Y^\mu} f.$$

Rearranging the sums, we obtain (21).

Corollary 2 $f(y_\gamma) = \sum_{|\alpha|=|\gamma|+1} p_\alpha(y_\gamma) d_{|\alpha|}[Y^{(0)},\ldots,Y^{(|\alpha|-1)},y_\alpha;f] +$

$$+d_{|\gamma|}[Y^{(0)},\ldots,Y^{(|\gamma|-1)},y_\gamma;f]. \qquad (22)$$

Proof: $p_\alpha(y_\beta)=0,\ \forall\alpha\neq\beta,\ |\beta|\le|\alpha|$ and $p_\alpha(y_\alpha)=1$.
Using the equality $(L_n(f))(y_\gamma)=f(y_\gamma)$, $\forall y_\gamma\in Y$, and (5) we obtain (22).

Theorem 5 *Let $f\in C^{n+1}(R^2)$ and $\Omega\subset R^2$ be a convex domain containing the associated points y_k, $k\in\{1,\ldots,n\}$ in the interpolation by weights. Then, for every $x\in\Omega$, the following inequality holds:*

$$\begin{aligned}|(f-L_n(f))(x)| \quad &\le \quad \frac{\|f\|_{n+1,\Omega}}{(n+1)!}\sum_{\alpha\in J_n}\sum_{i=1}^{2}|p_\alpha(x)(\xi_i-(\xi_\alpha)_i)|\,c_\alpha + \qquad (23)\\ &+ \quad \sum_{j=1}^{n}\frac{\|f\|_{j,\Omega}}{j!}\sum_{\beta\in I'_j}|p^\perp_\beta(x)|b_{j,\beta};\ x=(\xi_1,\xi_2)\in\Omega,\end{aligned}$$

$c_\alpha,\ b_{j,\beta}\in R$ are constants independent of x, given by:

$$c_\alpha = \sum_{\mu\in\mathcal{C}_n(\alpha)}|\Pi_\mu(Y^\mu)|\sum_{(\beta_1,\ldots,\beta_n)\in\{1,2\}^n}\left|(y_{\mu_n}-y_{\mu_{n-1}})_{\beta_n}\ldots(y_{\mu_1}-y_{\mu_0})_{\beta_1}\right|$$

$$b_{j,\beta} = \sum_{\mu\in\mathcal{C}_{j-1}}|\Pi_\mu(Y^\mu)|\cdot$$

$$\cdot\sum_{(\gamma_1,\ldots,\gamma_j)\in\{1,2\}^j}\left|(d_{\mu_{j-1},\beta})_{\gamma_j}(y_{\mu_{j-1}}-y_{\mu_{j-2}})_{\gamma_{j-1}}\ldots(y_{\mu_1}-y_{\mu_0})_{\gamma_1}\right|$$

and

$$\|f\|_{j,\Omega} = \sup_{y\in\Omega}\max_{|\beta|=j}\left|\frac{\partial^j}{\partial y^\beta}f(x)\right|,\ \beta\in N^2.$$

Proof:

$$\sum_{\mu\in\mathcal{C}_n} p_{\mu_n}(x)\Pi_\mu(Y^\mu)\int_{[Y^\mu,x]} D_{x-y_{\mu_n}}D^n_{Y^\mu}f+$$

$$\sum_{j=1}^{n} \sum_{\mu \in \mathcal{C}_{j-1}} \sum_{\beta \in I'_j} p_\beta^\perp(x) \Pi_\mu(Y^\mu) \int\limits_{[Y^\mu, x]} D_{d_{\mu_{j-1},\beta}} D_{Y^\mu}^{j-1} f =$$

$$= \sum_{\alpha \in J_n} \sum_{i=1}^{2} p_\alpha(x) \left(\xi_i - (\xi_\alpha)_i\right) \sum_{\mu \in \mathcal{C}_n(\alpha)} \Pi_\mu(Y^\mu) \int\limits_{[Y^\mu, x]} D_{e^i} D_{Y^\mu}^{n} f +$$

$$+ \sum_{\beta \in I'_n} p_\beta^\perp(x) \sum_{j=|\beta|}^{n} \sum_{\mu \in \mathcal{C}_{j-1}} \Pi_\mu(Y^\mu) \int\limits_{[Y^\mu, x]} D_{d_{\mu_{j-1},\beta}} D_{Y^\mu}^{j-1} f.$$

But, for $x = (\xi_1, \xi_2) \in R^2$, we have:

$$D_{Y^\mu}^{n} f = \sum_{(\beta_1, \ldots, \beta_n) \in \{1,2\}^n} (y_{\mu_n} - y_{\mu_{n-1}})_{\beta_n} \ldots (y_{\mu_1} - y_{\mu_0})_{\beta_1} \cdot \frac{\partial^n f}{\partial \xi_{\beta_1} \ldots \partial \xi_{\beta_n}}.$$

We can act similarly for $D_{Y^\mu}^{j-1} f$. Taking into account $\int\limits_{\Theta} f = \frac{1}{k!} f(\xi)$, we obtain (23).

References

[1] de Boor C. (1992), *Polynomial interpolation in several variables,* Math. Z., **210**, 347-378.

[2] Gasca M., Sauer T. (1999), *Polynomial interpolation in several variables,* Advances in Computational Mathematics.

[3] Sauer T. (1997), *Polynomial interpolation of minimal degree,* Numer. Math., **78**, 59-85.

[4] Sauer T. (1995), *Computational aspects of multivariate polynomial interpolation,* Advances in Comp. Math., **3**, 219-238.

[5] Simian D. (2001), *Ideal interpolation schemes,* Proceedings of "The 9-th Symposium of Mathematics and its Applications", Timişoara, 145-153.

[6] Simian D., Ene M. (2003), *Algorithmical Aspects of Multivariate Interpolation,* Proceedings of the 7-th Annual Conference of the Romanian Society of Mathematical Sciences, Bistriţa, to appear.

"Lucian Blaga" University
5-7 dr. Ratiu St., 550012, Sibiu, Romania

"Babes -Bolyai" University
1 M. Kogalniceanu St., 400084, Cluj-Napoca, Romania