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# RELIABILITY ESTIMATION FROM LUCENO DISTRIBUTED DATA

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*To Professor Silviu Sburlan, at his 60's anniversary*

**Abstract**

This paper considers the problem of estimating the mixture parameter $\theta$, the scale parameters $\lambda_1, \lambda_2$, the location parameters $u, \nu$ and the probability $R = \Pr(Y < X)$ when $X$ and $Y$ are independent, identically Luceno random variables with common scale and mixture parameters and unequal location parameters. The unbiased maximum likelihood estimators (MLE) of $\theta, \lambda_1, \lambda_2, u, \nu$ and $R$ are obtained. The performance of these estimators for moderate sample sizes is studied by Monte Carlo simulation. It appears that the procedure used to estimate the parameters, at least for the mixture parameter $\theta$ and the reliability $R$, is reasonable.

## 1. INTRODUCTION

The probability $R = \Pr Y < X)$ is often used as a measure of the mechanical reliability or performance of an item of strength $X$ subject to a stress $Y$. Awad and Fayoumi in [1] derived the MLE of $R$ when $X$ and $Y$ have a double exponential distribution with unequal location parameters and common sample sizes. Bai and Hong in [2] obtained the uniformly minimum variance unbiased estimation of $R$ with unequal sample sizes when $X$ and $Y$ are independent two-parameter exponential random variables with an unknown common location parameter.

A distribution of theoretical and some practical importance, because it allow data analysis to be performed in the original scale, is that of a random variable $X$ with probability density function

$$g(x) = \begin{cases} g_1(x; \theta, \lambda_1, \nu) \equiv \theta\lambda_1 \exp(-\lambda_1 (x-\nu)) & x \geq \nu \\ g_2(x; \theta, \lambda_2, \nu) \equiv (1-\theta)\lambda_2 \exp(-\lambda_2(\nu - x)) & x < \nu \end{cases}$$

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with $\lambda_1, \lambda_2 \geq 0 \quad \lambda_1 + \lambda_2 > 0$ the scale parameters, $\nu \in \mathbb{R}$ the location parameter and $\theta \in [0,1]$ the mixture parameter. This distribution is called the Luceno distribution; a survey of the properties and practical importance of this distribution is given in [3]. We note that $X$ is Luceno distributed with $X \sim Luc(\theta, \lambda_1, \lambda_2, \nu)$. Observe that, in general, the Luceno distribution has a jump discontinuity at the point $\nu$. The case where $\nu$ is considered as a change-point is of particular interest in the context of changepoint models in reliability.

Consider a system, with stress $Y$ and strength $X$. Assume that $X$ and $Y$ are independent random variables distributed $X \sim Luc(\theta, \lambda_1, \lambda_2, \nu)$ and $Y \sim Luc(\theta, \lambda_1, \lambda_2, u)$ with unknown parameters and $u \neq \nu$. In this paper we obtain the MLE of $\theta, \lambda_1, \lambda_2, u, \nu$ and $R$ for the case of unequal selection sizes when $X$ and $Y$ are distributed as above. Monte Carlo simulations with small sample and moderate equal sample are made to compare the estimators. Finally, we conclude that the suggested procedure appears to work well, at least for $\theta$ and $R$, because the estimated biases, in the worsted situation, are of the hundredth and thousandth order, respectively.

## 2. MAXIMUM LIKELIHOOD ESTIMATION

Let be $n, m \in \mathbb{N}$ given sample sizes. Consider two independent samples $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_m)$ from $Luc(\theta, \lambda_1, \lambda_2, \nu)$ and $Luc(\theta, \lambda_1, \lambda_2, u)$ respectively, with unknown scale parameters $\lambda_1$ and $\lambda_2$, location parameters $\nu$ and $u$ and mixture parameter $\theta$. Assume that in this samples, the last $n_1 > 0$ and $m_1 > 0$ values are greater than or equal to $\nu$ and $u$, respectively, and, consequently, the first $n_2 = n - n_1 > 0$ and $m_2 = m - m_1 > 0$ values are less than $\nu$ and $u$, respectively.

The likelihood function can be written as

$$\begin{aligned} L(\theta, \lambda_1, \lambda_2, \nu, u) &= \theta^{n_1+m_1} (1-\theta)^{n_2+m_2} \lambda_1^{n_1+m_1} \lambda_2^{n_2+m_2} \times \\ & \exp\left[-\lambda_1 \left(\sum_{i=1}^{n_1} (X_i - \nu) + \sum_{i=1}^{m_1} (Y_i - u)\right)\right] \times \\ & \exp\left[-\lambda_2 \left(\sum_{i=1}^{n_2} (\nu - X_i) + \sum_{i=1}^{m_2} (u - Y_i)\right)\right] \end{aligned}$$

Let be

$$\widehat{\theta}(n_1, m_1) = \frac{n_1 + m_1}{n + m},$$

$$\widehat{\nu}(n_1) = \frac{\left(\max_{1\le i\le n_2}(X_i) + \min_{1\le i\le n_1}(X_i)\right)}{2},$$

$$\widehat{u}(m_1) = \frac{\left(\max_{1\le i\le m_2}(Y_i) + \min_{1\le i\le m_1}(Y_i)\right)}{2}$$

$$\widehat{\lambda}_1(n_1, m_1) = \frac{n_1 + m_1}{T_1^1(n_1) + T_2^1(m_1)} \qquad \text{and} \qquad \widehat{\lambda}_2 = \frac{n_2 + m_2}{T_1^2(n_1) + T_2^2(m_1)}$$

where

$$T_1^1(n_1) = \sum_{i=1}^{n_1}(X_i - \nu); \quad T_2^1(m_1) = \sum_{i=1}^{m_1}(Y_i - u);$$

$$T_1^2(n_1) = \sum_{i=1}^{n_2}(\nu - X_i); \quad T_2^2(m_1) = \sum_{i=1}^{m_2}(u - Y_i).$$

The MLE's of $\theta, \nu, u, \lambda_1$, and $\lambda_2$ are given, respectively, by

$$(\widehat{n}_1, \widehat{m}_1) =$$

$$\arg \max_{3\le m_1\le m, 3\le n_1\le n} L\left(\theta(n_1, m_1), \lambda_1(n_1, m_1), \lambda_2(n_1, m_1), \nu(n_1), u(m_1), n_1, m_1\right),$$

$$\widehat{\theta} = \theta(\widehat{n}_1, \widehat{m}_1),$$

$$\widehat{\nu} = \nu(\widehat{n}_1) \qquad \widehat{u} = u(\widehat{m}_1) \qquad \Rightarrow \quad \widehat{\rho} = \widehat{u} - \widehat{\nu}$$

$$\widehat{\lambda}_1 = \lambda_1(\widehat{n}_1, \widehat{m}_1), \qquad \widehat{\lambda}_2 = \lambda_2(\widehat{n}_1, \widehat{m}_1)$$

In the above formula we consider the maximum only for $n_1 \in \{3, \ldots, n-3\}$ and $m_1 \in \{3, \ldots, m-3\}$ because, in order to obtain consistent estimators, $n_1$ must be greater than or equal to $[\theta n] + 1$ and $m_1$ must be greater than or equal to $[\theta m] + 1$, where $[\bullet]$ is the function "integer part" (observing that $\theta$ is the probability to generate a Luceno distributed value greater than or equal to $\nu$ and $u$, respectively, for $\theta = 0.1$ and $n = 25 = m$ we obtain the above bounds).

Finally, the MLE of $R$ is $\widehat{R} \equiv p\left(\widehat{\rho}; \widehat{\theta}, \widehat{\lambda}_1, \widehat{\lambda}_2\right)$ where

$$p(\rho; \theta, \lambda_1, \lambda_2) \equiv \Pr(Y < X) =$$

$$\begin{cases} \dfrac{\theta\left(-2\lambda_2+\theta\left(\lambda_1+\lambda_2\right)\right)}{2\left(\lambda_1-\lambda_2\right)}\exp\left(-\lambda_1\rho\right)+ & \\ & \text{if } \rho\equiv u-v\geq 0 \\ \dfrac{(1-\theta)\left((1+\theta)\lambda_1-(1-\theta)\lambda_2\right)}{2\left(\lambda_1-\lambda_2\right)}\exp\left(-\lambda_2\rho\right) & \\ & \\ 1-\dfrac{\theta\left(-2\lambda_2+\theta\left(\lambda_1+\lambda_2\right)\right)}{2\left(\lambda_1-\lambda_2\right)}\exp\left(\lambda_1\rho\right)- & \\ & \text{if } \rho<0 \\ \dfrac{(1-\theta)\left((1+\theta)\lambda_1-(1-\theta)\lambda_2\right)}{2\left(\lambda_1-\lambda_2\right)}\exp\left(\lambda_2\rho\right) & \end{cases}$$

(see [4]).

## 3. A SIMULATION STUDY

In this section we present some numerical results regarding the sample mean square error - $MSE[f]=\frac{1}{N}\sum_{i=1}^{N}\left(f-\widehat{f}\right)^2$ - of $f$, where $f\in\{\theta,\lambda_1,\lambda_2,\nu,u,R\}$.

Samples of sizes

$$n=25=m,\quad n=50=m,\quad n=100=m$$

from $X\sim Luc\left(\theta,\lambda_1=1.0,\lambda_2,\nu=1.\right)$, $Y\sim Luc\left(\theta,\lambda_1=1.0,\lambda_2,u\right)$ respectively, with $\theta\in\{0.1,0.5,0.9\}$, $\lambda_2\in\{0.2,1.01,5.0\}$ and $u\in\{1.0,1.2,1.5,1.8,2.0,6.0,11.0,21.0\}$ were generated.

To estimate the selection means the experiment was repeated $N=100$ times.

The main results are given in Tables 1-2.

**Table 1** ($\lambda_1 = 1.00, \quad \lambda_2 = 0.20, \quad \nu = 1.0$ )

| $\theta \rightarrow$ | | **0.1** | | | **0.5** | | | **0.9** | |
|---|---|---|---|---|---|---|---|---|---|
| $u \rightarrow$ | 1.0 | 11.0 | 21.0 | 1.0 | 11.0 | 21.0 | 1.0 | 11.0 | 21.0 |
| $R \rightarrow$ | 0.5000 | 0.0700 | 0.0095 | 0.5000 | 0.0592 | 0.0080 | 0.5000 | 0.0159 | 0.0022 |
| MSE $\downarrow$ | | | | | | | | | |
| | | | | | *n=25=m* | | | | |
| $\widehat{\theta}$ | 0.0359 | 0.0464 | 0.0472 | 0.0199 | 0.0092 | 0.0154 | 0.0043 | 0.0057 | 0.0070 |
| $\widehat{\lambda}_1$ | 0.5627 | 1.1786 | 0.6464 | 0.3447 | 0.1168 | 0.7505 | 0.0484 | 0.0300 | 0.0347 |
| $\widehat{\lambda}_2$ | 0.0013 | 0.0016 | 0.0009 | 0.0022 | 0.0020 | 0.0032 | 0.6880 | 7.2196 | 4.1454 |
| $\widehat{\nu}$ | 1.2650 | 2.6707 | 1.7487 | 0.2756 | 0.2808 | 0.1773 | 0.0140 | 0.0285 | 0.0292 |
| $\widehat{u}$ | 1.4126 | 1.8008 | 1.7676 | 0.3975 | 0.1029 | 0.4058 | 0.0236 | 0.0373 | 0.0231 |
| $\widehat{R}$ | 0.0056 | 0.0060 | 0.0001 | 0.0037 | 0.0004 | 0.0000 | 0.0022 | 0.0002 | 0.0000 |
| | | | | | *n=100=m* | | | | |
| | 0.0148 | 0.0189 | 0.0135 | 0.0016 | 0.0016 | 0.0013 | 0.0007 | 0.0005 | 0.0004 |
| | 0.1095 | 0.1491 | 0.1044 | 0.0111 | 0.0100 | 0.0150 | 0.0055 | 0.0062 | 0.0062 |
| | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0005 | 0.0030 | 0.0052 | 0.0025 |
| | 0.4939 | 0.5003 | 0.4396 | 0.0085 | 0.0058 | 0.0094 | 0.0007 | 0.0005 | 0.0011 |
| | 0.4465 | 0.6349 | 0.5082 | 0.0071 | 0.0129 | 0.0162 | 0.0008 | 0.0003 | 0.0005 |
| | 0.0011 | 0.0020 | 0.0000 | 0.0003 | 0.0001 | 0.0000 | 0.0002 | 0.0001 | 0.0000 |

**Table 2** ($\lambda_1 = 1.00, \quad \lambda_2 = 5.00, \quad \nu = 1.0$ )

| $\theta \rightarrow$ | | **0.1** | | | **0.5** | | | **0.9** | |
|---|---|---|---|---|---|---|---|---|---|
| $u \rightarrow$ | 1.0 | 11.0 | 21.0 | 1.0 | 11.0 | 21.0 | 1.0 | 11.0 | 21.0 |
| $R \rightarrow$ | 0.5000 | 0.0000 | 0.0000 | 0.5000 | 0.0000 | 0.0000 | 0.5000 | 0.0000 | 0.0000 |
| MSE $\downarrow$ | | | | | | | | | |
| | | | | | *n=25=m* | | | | |
| $\widehat{\theta}$ | 0.0073 | 0.0047 | 0.0143 | 0.0131 | 0.0184 | 0.0117 | 0.0429 | 0.0375 | 0.0417 |
| $\widehat{\lambda}_1$ | 13.946 | 76.221 | 9.9803 | 0.1071 | 0.0678 | 0.0503 | 0.0235 | 0.0427 | 0.0261 |
| $\widehat{\lambda}_2$ | 0.6928 | 0.7957 | 0.9596 | 6.1591 | 4.6956 | 4.7726 | 27.939 | 24.685 | 14.213 |
| $\widehat{\nu}$ | 0.0009 | 0.0008 | 0.0018 | 0.0100 | 0.0175 | 0.0107 | 0.0638 | 0.0563 | 0.0605 |
| $\widehat{u}$ | 0.0009 | 0.0008 | 0.0013 | 0.0105 | 0.0110 | 0.0079 | 0.0615 | 0.0567 | 0.0636 |
| $\widehat{R}$ | 0.0020 | 0.0000 | 0.0000 | 0.0031 | 0.0000 | 0.0000 | 0.0049 | 0.0000 | 0.0000 |
| | | | | | *n=100=m* | | | | |
| | 0.0005 | 0.0005 | 0.0005 | 0.0012 | 0.0014 | 0.0013 | 0.0118 | 0.0159 | 0.0108 |
| | 0.1127 | 0.1700 | 0.1485 | 0.0157 | 0.0127 | 0.0090 | 0.0068 | 0.0074 | 0.0050 |
| | 0.1715 | 0.1276 | 0.1854 | 0.3418 | 0.3161 | 0.3880 | 1.9885 | 2.0159 | 2.0780 |
| | 0.0000 | 0.0000 | 0.0000 | 0.0003 | 0.0005 | 0.0004 | 0.0167 | 0.0182 | 0.0137 |
| | 0.0000 | 0.0000 | 0.0000 | 0.0004 | 0.0003 | 0.0005 | 0.0164 | 0.0210 | 0.0141 |
| | 0.0001 | 0.0000 | 0.0000 | 0.0002 | 0.0000 | 0.0000 | 0.0014 | 0.0000 | 0.0000 |

The simulation study showed that:

a) $\widehat{R}$ estimate $R$ with an error of the thousandth order, in the 'worsted' situation. $\widehat{\theta}$ estimate $\theta$ with an error of the hundredth order, in the 'worsted' situation. The rest of the parameters $\lambda_1, \lambda_2, \nu$ and $u$ are estimated by the MLE's with an error, in general, of the tenth order;

b) The exceptions from the above last affirmation are doing to the following facts:

- the moderate size of $N$ (only 100) for the exceptions from Table 1;

- the small sizes of $n$ and $m$ and the fact that only 10% of the sample is used to estimate $\lambda_1$ and $\lambda_2$ for $\theta = 0.1$ and $\theta = 0.9$, respectively, in the exception cases from Tables 2 and 3;

c) The MSE is a decreasing function in the sample size;

d) For a fixed $n$ the MSE decrease when $\rho > 1$. This means that, when the two distributions become well separated, we can estimate the reliability $R$ more precisely.

In conclusion, it appears that the estimators works well for Luceno distributed data.

## References

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