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BL-ALGEBRA OF FRACTIONS RELATIVE TO AN $\wedge$-CLOSED SYSTEM

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

The aim of this paper is to introduce the notion of BL-algebra of fractions relative to an $\wedge$-closed system. For the case of Hilbert algebras, MV-algebras and pseudo MV-algebras see [2], [3] and [10].

1 Definitions and first properties

Definition 1.1 *A BL-algebra ([7]-[11]) is an algebra*

$$(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$$

of type (2,2,2,2,0,0) satisfying the following:

(a_1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice,*

(a_2) $(A, \odot, 1)$ *is a commutative monoid,*

(a_3) $\odot$ *and* $\rightarrow$ *form an adjoint pair, i.e.* $c \leq a \rightarrow b$ *iff* $a \odot c \leq b$ *for all* $a, b, c \in A$,

(a_4) $a \wedge b = a \odot (a \rightarrow b)$,

(a_5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$, *for all* $a, b \in A$.

The origin of BL-algebras is in Mathematical Logic; they were invented by Hájek in [7] in order to study the „Basic Logic" (BL, for short) arising from the continuous triangular norms, familiar in the framework of fuzzy set theory.

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They play the role of Lindenbaum algebras from classical Propositional calculus. Apart from their logical interest, BL-algebras have important algebraic properties (see [8]-[11]).

Examples

(E_1) Define on the real unit interval $I = [0,1]$ binary operations $\odot$ and $\to$ by

$$x \odot y = \max\{0, x + y - 1\}$$

$$x \to y = \min\{1, 1 - x + y\}.$$

Then $(I, \leq, \min, \max, \odot, \to, 0, 1)$ is a BL-algebra (called *Lukasiewicz structure*).

(E_2) Define on the real unit interval I

$$x \odot y = \min\{x, y\}$$

$$x \to y = 1 \text{ iff } x \leq y \text{ and } y \text{ otherwise.}$$

Then $(I, \leq, \min, \max, \odot, \to, 0, 1)$ is a BL-algebra (called *Gődel structure*).

(E_3) Let $\odot$ be the usual multiplication of real numbers on the unit interval I and $x \to y = 1$ iff $x \leq y$ and y/x otherwise. Then $(I, \leq, \min, \max, \odot, \to, 0, 1)$ is a BL-algebra (called *Products structure* or *Gaines structure*).

Remark 1.1 *Not every residuated lattice, however, is a BL-algebra (see [11], p.16). Consider , for example a residuated lattice defined on the unit interval, for all $x, y, z \in I$, such that*

$$x \odot y = 0, \text{ if } x + y \leq \frac{1}{2} \text{ and } x \wedge y \text{ elsewhere,}$$

$$x \to y = 1 \text{ if } x \leq y \text{ and } \max\{\frac{1}{2} - x, y\} \text{ elsewhere.}$$

Let $0 < y < x$, $x + y < \frac{1}{2}$. Then $y < \frac{1}{2} - x$ and $0 \neq y = x \wedge y$, but $x \odot (x \to y) = x \odot (\frac{1}{2} - x) = 0$. Therefore (a_4) does not hold.

(E_4) If $(A, \wedge, \vee, \rceil, 0, 1)$ is a Boolean algebra, then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra where the operation $\odot$ coincide with $\wedge$ and $x \to y = \rceil x \vee y$ for all $x, y \in A$.

(E_5) If $(A, \wedge, \vee, \to, 0, 1)$ is a *relative Stone lattice* (see [1], p.176), then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra where the operation $\odot$ coincide with $\wedge$.

(E_6) If $(A, \oplus, ^*, 0)$ is a *MV-algebra* (see [3], [4], [11]), then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra, where for $x, y \in A$:

$$x \odot y = (x^* \oplus y^*)^*,$$

$$x \to y = x^* \oplus y, 1 = 0^*,$$

$$x \vee y = (x \to y) \to y = (y \to x) \to x \text{ and } x \wedge y = (x^* \vee y^*)^*.$$

A BL-algebra is *nontrivial* if $0 \neq 1$. For any BL-algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. For any $a \in A$, we define $a^* = a \to 0$ and denote $(a^*)^*$ by a^{**}. We denote the set of natural numbers by ω and define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in \omega \backslash \{0\}$.

In [4], [7]-[11] it is proved that if A is a BL-algebra and $a, b, c, b_i \in A$, ($i \in I$) then we have the following rules of calculus:

(c_1) $a \odot b \leq a, b$, hence $a \odot b \leq a \wedge b$ and $a \odot 0 = 0$,

(c_2) $a \leq b$ implies $a \odot c \leq b \odot c$,

(c_3) $a \leq b$ iff $a \to b = 1$,

(c_4) $1 \to a = a, a \to a = 1, a \leq b \to a, a \to 1 = 1$,

(c_5) $a \odot a^* = 0$,

(c_6) $a \odot b = 0$ iff $a \leq b^*$,

(c_7) $a \vee b = 1$ implies $a \odot b = a \wedge b$,

(c_8) $a \to (b \to c) = (a \odot b) \to c = b \to (a \to c)$,

(c_9) $(a \to b) \to (a \to c) = (a \wedge b) \to c$,

(c_{10}) $a \to (b \to c) \geq (a \to b) \to (a \to c)$,

(c_{11}) $a \leq b$ implies $c \to a \leq c \to b, b \to c \leq a \to c$ and $b^* \leq a^*$,

(c_{12}) $a \leq (a \to b) \to b$, $((a \to b) \to b) \to b = a \to b$,

(c_{13}) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$,

(c_{14}) $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$,

(c_{15}) $a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a)$,

(c_{16}) $(a \wedge b)^n = a^n \wedge b^n, (a \vee b)^n = a^n \vee b^n$, hence $a \vee b = 1$ implies $a^n \vee b^n = 1$ for any $n \in \omega$,

(c_{17}) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$,

(c_{18}) $(b \wedge c) \to a = (b \to a) \vee (c \to a)$,

(c_{19}) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$,

(c_{20}) $a \to b \leq (b \to c) \to (a \to c)$,

(c_{21}) $a \to b \leq (c \to a) \to (c \to b)$,

(c_{22}) $a \to b \leq (a \odot c) \to (b \odot c)$,

(c_{23}) $a \odot (b \to c) \leq b \to (a \odot c)$,

(c_{24}) $(b \to c) \odot (a \to b) \leq a \to c$,

(c_{25}) $(a_1 \to a_2) \odot (a_2 \to a_3) \odot ... \odot (a_{n-1} \to a_n) \leq a_1 \to a_n$,

(c_{26}) $a, b \leq c$ and $c \to a = c \to b$ implies $a = b$,

(c_{27}) $a \vee (b \odot c) \geq (a \vee b) \odot (a \vee c)$, hence $a^m \vee b^n \geq (a \vee b)^{mn}$, for any $m, n \in \omega$,

(c_{28}) $(a \to b) \odot (a' \to b') \leq (a \vee a') \to (b \vee b')$,

(c_{29}) $(a \to b) \odot (a' \to b') \leq (a \wedge a') \to (b \wedge b')$,

(c_{30}) $(a \to b) \to c \leq ((b \to a) \to c) \to c$,

(c_{31}) $a \odot (\bigwedge_{i \in I} b_i) \leq \bigwedge_{i \in I} (a \odot b_i)$,

$a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i)$,

$a \to (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \to b_i)$,

$(\bigvee_{i \in I} b_i) \to a = \bigwedge_{i \in I} (b_i \to a)$

$\bigvee_{i \in I} (b_i \to a) \leq (\bigwedge_{i \in I} b_i) \to a$,

$\bigvee_{i \in I} (a \to b_i) \leq a \to (\bigvee_{i \in I} b_i)$,
$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$; if A is an BL-chain then $a \vee (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \vee b_i)$,
(whenever the arbitrary meets and unions exist)

(c_{32}) $a \leq a^{**}$, $1^* = 0$, $0^* = 1, a^{***} = a, a^{**} \leq a^* \to a$,

(c_{33}) $(a \wedge b)^* = a^* \vee b^*$ and $(a \vee b)^* = a^* \wedge b^*$,

(c_{34}) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$, $(a \vee b)^{**} = a^{**} \vee b^{**}, (a \odot b)^{**} = a^{**} \odot b^{**}$, $(a \to b)^{**} = a^{**} \to b^{**}$,

(c_{35}) If $a^{**} \leq a^{**} \to a$, then $a^{**} = a$,

(c_{36}) $a = a^{**} \odot (a^{**} \to a)$,

(c_{37}) $a \to b^* = b \to a^* = a^{**} \to b^* = (a \odot b)^*$,

(c_{38}) $(a^{**} \to a)^* = 0, (a^{**} \to a) \vee a^{**} = 1$,

(c_{39}) $b^* \leq a$ implies $a \to (a \odot b)^{**} = b^{**}$.

For any BL-algebra A, $B(A)$ denotes the Boolean algebra of all complemented elements in $L(A)$ (hence $B(A) = B(L(A))$).

Proposition 1.1 *([7]-[11])For $e \in A$, the following are equivalent:*

(i) $e \in B(A)$,

(ii) $e \odot e = e$ *and* $e = e^{**}$,

(iii) $e \odot e = e$ *and* $e^* \to e = e$,

(iv) $e \vee e^* = 1$.

Remark 1.2 *If $a \in A$, and $e \in B(A)$, then $e \odot a = e \wedge a, a \to e = (a \odot e^*)^* = a^* \vee e$; if $e \leq a \vee a^*$, then $e \odot a \in B(A)$.*

Proposition 1.2 *([4]) For $e \in A$, the following are equivalent:*

(i) $e \in B(A)$,

(ii) $(e \to x) \to e = e$, *for every* $x \in A$.

Lemma 1.1 *If $e, f \in B(A)$ and $x, y \in A$, then:*

(c_{40}) $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$,

(c_{41}) $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$,

(c_{42}) $e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)]$,

(c_{43}) $x \odot (e \to f) = x \odot [(x \odot e) \to (x \odot f)]$,

(c_{44}) $e \to (x \to y) = (e \to x) \to (e \to y)$.

Proof. (c_{40}). We have

$$(e \vee x) \odot (e \vee y) \overset{c_{13}}{=} [(e \vee x) \odot e] \vee [(e \vee x) \odot y] = [(e \vee x) \odot e] \vee [(e \odot y) \vee (x \odot y)]$$
$$= [(e \vee x) \wedge e] \vee [(e \odot y) \vee (x \odot y)] = e \vee (e \odot y) \vee (x \odot y) = e \vee (x \odot y).$$

(c_{41}). We have

$$(e \wedge x) \odot (e \wedge y) = (e \odot x) \odot (e \odot y) = (e \odot e) \odot (x \odot y) = e \odot (x \odot y) = e \wedge (x \odot y).$$

(c_{42}). By (c_{22}) we have

$$x \to y \leq (e \odot x) \to (e \odot y),$$

hence

$$e \odot (x \to y) \leq e \odot [(e \odot x) \to (e \odot y)].$$

Conversely,

$$e \odot [(e \odot x) \to (e \odot y)] \leq e$$

and

$$(e \odot x) \odot [(e \odot x) \to (e \odot y)] \leq e \odot y \leq y$$

so

$$e \odot [(e \odot x) \to (e \odot y)] \leq x \to y.$$

Hence

$$e \odot [(e \odot x) \to (e \odot y)] \leq e \odot (x \to y).$$

(c_{43}). We have $x \odot [(x \odot e) \to (x \odot f)] = x \odot [(x \odot e) \to (x \wedge f)] \overset{c_{31}}{=} x \odot [(x \odot e \to x) \wedge (x \odot e \to f)] = x \odot [1 \wedge (x \odot e \to f)] = x \odot (x \odot e \to f) \overset{c_{8}}{=} x \odot [x \to (e \to f)] = x \wedge (e \to f) = x \odot (e \to f)$.

(c_{44}). Follows from (c_8) and (c_9) since $e \wedge x = e \odot x$.■

Definition 1.2 *([7]-[11])Let A and B be $BL-$algebras. A function $f : A \to B$ is a morphism of $BL-$algebras iff it satisfies the following conditions, for every $x, y \in A$:*

(a_6) $f(0) = 0$,

(a_7) $f(x \odot y) = f(x) \odot f(y)$,

(a_8) $f(x \to y) = f(x) \to f(y)$.

Remark 1.3 *([7]-[11]) It follows that:*

$$f(1) = 1,$$
$$f(x^*) = [f(x)]^*$$
$$f(x \vee y) = f(x) \vee f(y),$$
$$f(x \wedge y) = f(x) \wedge f(y),$$

for every $x, y \in A$.

2 BL-algebra of fractions relative to an $\wedge-$closed system

Definition 2.1 *A nonempty subset $S \subseteq A$ is called $\wedge-$closed system in A if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.*

We denote by $S(A)$ the set of all $\wedge-$closed system of A (clearly $\{1\}, A \in S(A)$).

For $S \in S(A)$, on the BL-algebra A we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap B(A) \text{ such that } x \wedge e = y \wedge e.$$

Lemma 2.1 *θ_S is a congruence on A.*

Proof. The reflexivity (since $1 \in S \cap B(A)$) and the symmetry of θ_S are immediately. To prove the transitivity of θ_S, let $(x, y), (y, z) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $y \wedge f = z \wedge f$. If denote $g = e \wedge f \in S \cap B(A)$, then $g \wedge x = (e \wedge f) \wedge x = (e \wedge x) \wedge f = (y \wedge e) \wedge f = (y \wedge f) \wedge e = (z \wedge f) \wedge e = z \wedge (f \wedge e) = z \wedge g$, hence $(x, z) \in \theta_S$.

To prove the compatibility of θ_S with the operations $\wedge, \vee, \odot$ and $\rightarrow$, let $x, y, z, t \in A$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap B(A)$.

We obtain:

$$(x \wedge z) \wedge g = (x \wedge z) \wedge (e \wedge f) = (x \wedge e) \wedge (z \wedge f) = (y \wedge e) \wedge (t \wedge f) = (y \wedge t) \wedge g,$$

hence $(x \wedge z, y \wedge t) \in \theta_S$ and

$$(x \vee z) \wedge g = (x \vee z) \wedge (e \wedge f) = [(e \wedge f) \wedge x] \vee [(e \wedge f) \wedge z] = [(e \wedge x) \wedge f] \vee [e \wedge (f \wedge z)]$$

$$= [(e \wedge y) \wedge f] \vee [e \wedge (f \wedge t)] = [(e \wedge f) \wedge y] \vee [(e \wedge f) \wedge t] = (y \vee t) \wedge (e \wedge f) = (y \vee t) \wedge g,$$

hence $(x \vee z, y \vee t) \in \theta_S$.

By Remark 1.2 we obtain:

$$(x \odot z) \wedge g = (x \odot z) \odot g = (x \odot e) \odot (z \odot f) = (y \odot e) \odot (t \odot f) = (y \odot t) \odot g = (y \odot t) \wedge g,$$

hence $(x \odot z, y \odot t) \in \theta_S$ and by (c_{42}):

$$(x \rightarrow z) \wedge g = (x \rightarrow z) \odot g = g \odot [(g \odot x) \rightarrow (g \odot z)] =$$

$$g \odot [(g \odot y) \rightarrow (g \odot t)] = (y \rightarrow t) \odot g = (y \rightarrow t) \wedge g,$$

hence $(x \rightarrow z, y \rightarrow t) \in \theta_S$.■

For x we denote by x/S the equivalence class of x relative to θ_S and by

$$A[S] = A/\theta_S.$$

By $p_S : A \to A[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, in $A[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A$,

$$x/S \wedge y/S = (x \wedge y)/S$$

$$x/S \vee y/S = (x \vee y)/S$$

$$x/S \odot y/S = (x \odot y)/S$$

$$x/S \to y/S = (x \to y)/S.$$

So, p_S is an onto morphism of BL-algebras.

Remark 2.1 *Since for every $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$ we deduce that $s/S = 1/S = \mathbf{1}$, hence $p_S(S \cap B(A)) = \{\mathbf{1}\}$.*

Proposition 2.1 *If $a \in A$, then $a/S \in B(A[S])$ iff there exists $e \in S \cap B(A)$ such that $e \wedge a \in B(A)$. So, if $e \in B(A)$, then $e/S \in B(A[S])$.*

Proof. For $a \in A$, we have $a/S \in B(A[S]) \Leftrightarrow a/S \odot a/S = a/S$ and $(a/S)^{**} = a/S$.

From $a/S \odot a/S = a/S$ we deduce that $(a \odot a)/S = a/S \Leftrightarrow$ there exists $g \in S \cap B(A)$ such that $(a \odot a) \wedge g = a \wedge g \Leftrightarrow (a \odot a) \odot g = a \wedge g \Leftrightarrow (a \odot g) \odot (a \odot g) = a \wedge g \Leftrightarrow (a \wedge g) \odot (a \wedge g) = a \wedge g$.

From $(a/S)^{**} = a/S$ we deduce that exists $f \in S \cap B(A)$ such that $a^{**} \wedge f = a \wedge f$. If denote $e = g \wedge f \in S \cap B(A)$, then

$$(a \wedge e) \odot (a \wedge e) = (a \wedge g \wedge f) \odot (a \wedge g \wedge f) \Leftrightarrow (a \odot g) \odot f \odot (a \odot g) \odot f =$$

$$a \odot g \odot f = a \wedge g \wedge f = a \wedge e$$

and

$$a^{**} \wedge e = a^{**} \wedge g \wedge f = (a^{**} \wedge f) \wedge g = (a \wedge f) \wedge g = a \wedge e,$$

hence $a \wedge e \in B(A)$.

If $e \in B(A)$, since $1 \in S \cap B(A)$ and $1 \wedge e = e \in B(A)$ we deduce that $e/S \in B(A[S])$. ■

Theorem 2.1 *If A' is a BL-algebra and $f : A \to A'$ is a morphism of BL-algebras such that $f(S \cap B(A)) = \{\mathbf{1}\}$, then there exists a unique morphism of BL-algebras $f' : A[S] \to A'$ such that the diagram*

$$\begin{tikzcd} A \arrow[rr, "p_S"] \arrow[dr, "f"'] & & A[S] \arrow[dl, "f'"] \\ & A' & \end{tikzcd}$$

is commutative (i.e. $f' \circ p_S = f$).

Proof. If $x, y \in A$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$. Since f is morphism of BL-algebras, we obtain that $f(x \wedge e) = f(y \wedge e) \Leftrightarrow f(x) \wedge f(e) = f(y) \wedge f(e) \Leftrightarrow f(x) \wedge \mathbf{1} = f(y) \wedge \mathbf{1} \Leftrightarrow f(x) = f(y)$.

From this observation we deduce that the map $f' : A[S] \to A'$ defined for $x \in A$ by $f'(x/S) = f(x)$ is correctly defined. Clearly, f' is an morphism of BL-algebras. The unicity of f' follows from the fact that p_S is a onto map.■

Remark 2.2 *Theorem 2.1 allows us to call $A[S]$ the BL-algebra of fractions relative to the $\wedge-$closed system S.*

References

[1] R. Balbes, Ph. Dwinger: *Distributive Lattices*, University of Missouri Press, 1974

[2] D. Buşneag: *Hilbert algebra of fractions relative to an $\veebar$-closed system*, Analele Universităţii din Craiova, Seria Matematica-Fizica-Chimie, vol. XVI, (1998), 34-38

[3] D. Buşneag, D. Piciu: *MV-algebra of fractions relative to an $\wedge$-closed system*, Analele Universităţii din Craiova, Seria Matematica-Informatica, vol. XXXI, (2003), 1-5

[4] D. Buşneag, D. Piciu: *On the lattice of deductive systems of a BL-algebra*, Central European Journal of Mathematics, Volume 1, Number 2, April 2003, 221-238

[5] C. C. Chang: *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc., 88(1958), 467-490

[6] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici: *Algebraic foundation of many -valued Reasoning*, Kluwer Academic Publishers, Dordrecht, 2000

[7] P. Hájek: *Metamathematics of fuzzy logic*, Kluwer Acad. Publ., Dordrecht, 1998

[8] A. Iorgulescu: *Iséki algebras. Connections with BL-algebras* (to appear in Soft Computing)

[9] A. Di Nola, G. Georgescu, A. Iorgulescu: *Pseudo-BL-algebras* (to appear in Multiple Valued Logic)

[10] D. Piciu: *Pseudo MV-algebra of fractions relative to an $\wedge$-closed system*, Analele Universităţii din Craiova, Seria Matematica-Informatica, vol. XXXI, (2003), 6-11

[11] E. Turunen: *Mathematics Behind Fuzzy Logic*, Physica-Verlag, 1999

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