Quasipolar Subrings of 3×3 Matrix Rings

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Abstract

An element a of a ring R is called *quasipolar* provided that there exists an idempotent $p \in R$ such that $p \in comm^2(a)$, $a + p \in U(R)$ and $ap \in R^{qnil}$. A ring R is *quasipolar* in case every element in R is quasipolar. In this paper, we determine conditions under which subrings of 3×3 matrix rings over local rings are quasipolar. Namely, if R is a bleached local ring, then we prove that $\mathcal{T}_3(R)$ is quasipolar if and only if R is uniquely bleached. Furthermore, it is shown that $T_n(R)$ is quasipolar if and only if $T_n\big(R[[x]]\big)$ is quasipolar for any positive integer n.

1 Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Following Koliha and Patricio [11], the *commutant* and *double commutant* of an element $a \in R$ are defined by $comm(a) = \{x \in R \mid xa = ax\}$, $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$, respectively. If $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$ and $a \in R^{qnil}$, then a is said to be *quasinilpotent* [10]. An element $a \in R$ is called *quasipolar* provided that there exists an idempotent $p \in R$ such that $p \in comm^2(a)$, $a + p \in U(R)$ and $ap \in R^{qnil}$. A ring R is *quasipolar* in case every element in R is quasipolar. Properties of quasipolar rings were studied in [6, 7, 14].

For a ring R, let $\mathcal{T}_3(R) = \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \mid a_{11}, a_{21}, a_{22}, a_{23}, a_{33} \in R \right\}$.

Then $\mathfrak{T}_3(R)$ is a ring under the usual addition and multiplication, and so $\mathfrak{T}_3(R)$ is a subring of $M_3(R)$. Motivated by results in [3] and [5], we study quasipolar subrings of 3×3 matrix rings over local rings. We prove that
$\begin{bmatrix} \mathbb{Z}_{(2)} & 0 & 0 \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \\ 0 & 0 & \mathbb{Z}_{(2)} \end{bmatrix}$ is quasipolar but the full matrix ring $M_3(\mathbb{Z}_{(2)})$ is not quasipolar.

In this paper, $M_n(R)$ and $T_n(R)$ denote the ring of all $n \times n$ matrices and the ring of all $n \times n$ upper triangular matrices over R, respectively. We write $R[[x]]$, $U(R)$ and $J(R)$ for the power series ring over a ring R, the set of all invertible elements and the Jacobson radical of R, respectively. For $A \in M_n(R)$, $\chi(A)$ stands for the characteristic polynomial $det(tI_n - A)$.

2 Quasipolar Elements

In [12], Nicholson gives several equivalent characterizations of strongly clean rings through the endomorphism ring of a module. Analogously, we present similar results for quasipolar rings. For convenience, we use left modules and write endomorphisms on the right. For a module $_R M$, we write $E = End_R(M)$ for the ring of endomorphisms of $_R M$.

Lemma 2.1. [12, Lemma 2] Let $\beta, \pi^2 = \pi \in End_R(M)$. Then both $M\pi$ and $M(1 - \pi)$ are β-invariant if and only if $\pi\beta = \beta\pi$.

Similar to [4, Theorem 2.1] we have the following results for quasipolar endomorphisms of a module.

Theorem 2.2. Let $\alpha \in E = \text{End}_R(M)$. The following are equivalent.

(1) α is quasipolar in E.

(2) There exists $\pi^2 = \pi \in E$ such that $\pi \in comm_E^2(\alpha)$, $\alpha\pi$ is a unit in $\pi E \pi$ and $\alpha(1 - \pi)$ is a quasinilpotent in $(1 - \pi)E(1 - \pi)$.

(3) $M = P \oplus Q$, where P and Q are β-invariant for every $\beta \in comm_E(\alpha)$, $\alpha|_P$ is a unit in $\text{End}(P)$ and $\alpha|_Q$ is a quasinilpotent in $\text{End}(Q)$.

(4) $M = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ for some $n \geq 1$, where P_i is β-invariant for every $\beta \in comm_E(\alpha)$, $\alpha|_{P_i}$ is quasipolar in $\text{End}(P_i)$ for each i.

Proof. (1) $\Rightarrow$ (2) Since α is quasipolar in E, there exists an idempotent $\tau \in R$ such that $\tau \in comm_E^2(\alpha)$, $\alpha + \tau = \eta \in U(E)$ and $\alpha\tau \in E^{qnil}$. Let $\pi = 1 - \tau$. Clearly, $\pi^2 = \pi \in comm_E^2(\alpha)$. Note that α, π, η and τ all commute. Now, multiplying $\alpha + \tau = \eta$ by π yields $\alpha\pi = \eta\pi = \pi\eta \in \pi E \pi$. Since $\eta^{-1}\pi \in \pi E \pi$

this gives $(\alpha\pi)(\eta^{-1}\pi) = (\pi\eta)(\eta^{-1}\pi) = \pi$. Similarly, $(\eta^{-1}\pi)(\alpha\pi) = \pi$ so $\alpha\pi$ is a unit in $\pi E\pi$. Let $(1-\pi)\gamma(1-\pi) \in comm_{(1-\pi)E(1-\pi)}(\alpha(1-\pi))$. Then $(1-\pi)\gamma(1-\pi) \in comm_E(\alpha(1-\pi))$. The remaining proof is to show that $(1-\pi) + \alpha(1-\pi)\gamma(1-\pi)$ is a unit in $(1-\pi)E(1-\pi)$. Since $\alpha(1-\pi) \in E^{qnil}$, $1 + \alpha(1-\pi)\gamma(1-\pi)$ is a unit in E and so $(1-\pi) + \alpha(1-\pi)\gamma(1-\pi)$ is a unit $(1-\pi)E(1-\pi)$.

$(2) \Rightarrow (3)$ Given π as in (2), let $P = M\pi$ and $Q = M(1-\pi)$. Then $M = P \oplus Q$. For any $\beta \in comm_E(\alpha)$, the hypothesis $\pi \in comm_E^2(\alpha)$ implies that $\pi\beta = \beta\pi$. By Lemma 2.1, both P and Q are β-invariant. As in the proof [12, Theorem 3], $\alpha\pi = \alpha|_P$ is a unit in $\text{End}(P)$. Let $\gamma \in comm_{\text{End}(Q)}(\alpha|_Q)$. We show that $1_Q + \alpha|_Q\gamma$ is a unit in $\text{End}(Q)$. Clearly, $\gamma \in comm_{(1-\pi)E(1-\pi)}(\alpha(1-\pi))$. Since $\alpha(1-\pi)$ is a quasinilpotent in $(1-\pi)E(1-\pi)$, $(1-\pi) + \alpha(1-\pi)\gamma$ is a unit in $(1-\pi)E(1-\pi)$. Let $[(1-\pi) + \alpha(1-\pi)\gamma]^{-1} = (1-\pi)\tau(1-\pi) = \tau_0 \in \text{End}(Q)$ and let $q \in Q$. Then $(q)[1_Q + \alpha|_Q\gamma]\tau_0 = (q + q(1-\pi)\alpha\gamma)\tau_0 = (q(1-\pi) + q\alpha(1-\pi)\gamma)\tau_0 = q[(1-\pi) + \alpha(1-\pi)\gamma]\tau_0 = (q)1_Q$. Hence $(1_Q + \alpha|_Q\gamma)\tau_0 = 1_Q$. Similarly, $\tau_0(1_Q + \alpha|_Q\gamma) = 1_Q$. Thus $\alpha|_Q$ is a quasinilpotent in $\text{End}(Q)$.

$(3) \Rightarrow (4)$ Suppose $M = P \oplus Q$ as in (3). Since $\alpha|_P$ is a unit in $\text{End}(P)$, $\alpha|_P$ is a quasipolar in $\text{End}(P)$ by [6, Example 2.1]. As $\alpha|_Q$ is a quasinilpotent in $\text{End}(Q)$, $1_Q + \alpha|_Q$ is a unit in $\text{End}(Q)$. Further, $1_Q^2 = 1_Q$ and $1_Q \in comm_{\text{End}(Q)}^2(\alpha|_Q)$ so $\alpha|_Q$ is quasipolar in $\text{End}(Q)$.

$(4) \Rightarrow (1)$ Let $\lambda_i \in \text{End}(P_i)$. Given the situation in (4), extend maps λ_i in $\text{End}(P_i)$ to $\overline{\lambda_i}$ in $\text{End}(M)$ by defining $(\sum_{j=1}^{n} p_j)\overline{\lambda_i} = (p_i)\lambda_i$ for any $p_j \in P_j$. Then $\overline{\lambda_i}\ \overline{\lambda_j} = 0$ if $i \neq j$ while $\overline{\lambda_i}\ \overline{\mu_i} = \overline{\lambda_i\mu_i}$ and $\overline{\lambda_i} + \overline{\mu_i} = \overline{\lambda_i + \mu_i}$ for all $\mu_i \in \text{End}(P_i)$. By hypothesis, there exists $\pi_j^2 = \pi_j \in comm_{\text{End}(P_j)}^2(\alpha|_{P_j})$, $\sigma_j \in U(\text{End}(P_j))$ such that $\alpha|_{P_j} + \pi_j = \sigma_j$ and $\alpha|_{P_j}\pi_j \in \text{End}(P_j)^{qnil}$. If $\pi = \sum_{j=1}^{n} \overline{\pi_j}$ and $\sigma = \sum_{j=1}^{n} \overline{\sigma_j}$ then $\pi^2 = \sum_{j=1}^{n} \overline{\pi_j}^2 = \pi \in \text{End}(M)$ and σ is a unit in E because $\sigma^{-1} = \sum_{j=1}^{n} \overline{\sigma_j}^{-1}$. Since $\alpha = \sum_{j=1}^{n} \overline{\alpha|_{P_j}} = \sum_{j=1}^{n}(\overline{-\pi_j + \sigma_j}) = -\pi + \sigma$, we show that $\pi \in comm_E^2(\alpha)$ and $\alpha\pi \in E^{qnil}$. Since for each $\beta \in comm_E(\alpha)$, P and Q are β-invariant. Hence, $\pi\beta = \beta\pi$ by Lemma 2.1 and so $\pi \in comm_E^2(\alpha)$. For any $\beta \in comm_E(\alpha\pi)$, we only need to show that $1_E + \beta\alpha\pi$ is an isomorphism in E. Note that $\beta|_{P_j} \in comm_{\text{End}(P_j)}(\alpha|_{P_j})$ and $1_{P_j} + \beta|_{P_j}\alpha|_{P_j} = (\pi + \beta\alpha)|_{P_j}$. Since $\alpha|_{P_j}\pi_j \in \text{End}(P_j)^{qnil}$, $1_{P_j} + \beta|_{P_j}\alpha|_{P_j}\pi_j = (\pi + \beta\pi\alpha)|_{P_j}$ is a unit in $\text{End}(P_j)$. Let $\gamma_j \in \text{End}(P_j)$ be such that $(1_{P_j} + \beta|_{P_j}\alpha|_{P_j}\pi_j)\gamma_j = 1_{P_j} = \gamma_j(1_{P_j} + \beta|_{P_j}\alpha|_{P_j}\pi_j)$ and let $m = \sum_{j=1}^{n} p_j$ with $p_j \in P_j$. So $(\sum_{j=1}^{n} p_j)(1_E +$

$$\beta\alpha\pi)\gamma = \big(\sum_{j=1}^{n} p_j + (\sum_{j=1}^{n} p_j)\beta\alpha\pi\big)\gamma = \big(\sum_{j=1}^{n}(p_j)1_{P_j} + (\sum_{j=1}^{n}(p_j)[\beta|_{P_j}\alpha|_{P_j}\pi_j])\big)\gamma =$$

$$\big(\sum_{j=1}^{n}(p_j)[1_{P_j}+\beta|_{P_j}\alpha|_{P_j}\pi_j]\big)\gamma = \big(\sum_{j=1}^{n}(p_j)[1_{P_j}+\beta|_{P_j}\alpha|_{P_j}\pi_j]\gamma_j\big) = \sum_{j=1}^{n}(p_j)[1_{P_j}]{=}m$$

where $\gamma = \sum_{j=1}^{n}\overline{\gamma_j}$. Similarly, we have $(\sum_{j=1}^{n} p_j)\gamma(1_E + \beta\alpha\pi) = \sum_{j=1}^{n}(p_j)[1_{P_j}] = m$.

Therefore $\alpha\pi \in E^{qnil}$, the proof is completed. $\qquad\square$

The following result is a direct consequence of Theorem 2.2.

Corollary 2.3. Let R be a ring. The following are equivalent for $a \in R$.

(1) $a \in R$ is quasipolar.

(2) There exists $e^2 = e \in R$ such that $e \in comm_R^2(a)$, $ae \in U(eRe)$ and $a(1-e) \in (1-e)R(1-e)^{qnil}$.

3   The Rings $\mathfrak{T}_3(R)$

For a ring R, let $a \in R$, $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, for $a, b \in R$, l_a, r_b is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$. A local ring R is called *bleached* [1] if, for any $a \in J(R)$ and any $b \in U(R)$, the abelian group endomorphisms $l_b - r_a$ and $l_a - r_b$ of R are both surjective. A local ring R is called *uniquely bleached* if, for any $a \in J(R)$ and any $b \in U(R)$, the abelian group endomorphisms $l_b - r_a$ and $l_a - r_b$ of R are isomorphic. According to [8, Example 2.1.11], commutative local rings, division rings, local rings with nil Jacobson radicals, local rings for which some power of each element of their Jacobson radicals is central are uniquely bleached. Clearly uniquely bleached local rings are bleached. But so far it is unknown whether a bleach local ring is uniquely bleached. Obviously, $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \in U\big(\mathfrak{T}_3(R)\big)$ if and only if $a_{11}, a_{22}, a_{33} \in U(R)$. Further,

$$J\big(\mathfrak{T}_3(R)\big) = \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \mid a_{11}, a_{22}, a_{33} \in J(R), a_{21}, a_{23} \in R \right\}.$$ Note

that if, for every $A \in \mathfrak{T}_3(R)$, there exists $E^2 = E \in comm^2(A)$ such that $A - E \in U\big(\mathfrak{T}_3(R)\big)$ and $EA \in J\big(\mathfrak{T}_3(R)\big) \subseteq \mathfrak{T}_3(R)^{qnil}$, then $-A$ is quasipolar and so $\mathfrak{T}_3(R)$ is quasipolar. We use this fact in the proof of Theorem 3.1 without mention.

By [13, Example 1] and [9, Remark 3.2.11], $M_3(R)$ is not quasipolar in general. Our next aim is to determine to find conditions under which $\mathcal{T}_3(R)$ is quasipolar. In this direction we can give the following theorem.

Theorem 3.1. Let R be a bleached local ring. The following are equivalent.

(1) R is uniquely bleached.

(2) $\mathcal{T}_3(R)$ is quasipolar.

(3) $T_2(R)$ is quasipolar.

Proof. $(1) \Rightarrow (2)$ Let $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \in \mathcal{T}_3(R)$. Consider the following cases.

Case 1. $a_{11}, a_{22}, a_{33} \in J(R)$. Then $A + I_3 \in U(\mathcal{T}_3(R))$ and $AI_3 = A \in J(\mathcal{T}_3(R)) \subseteq \mathcal{T}_3(R)^{qnil}$. So A is quasipolar.

Case 2. $a_{11}, a_{22}, a_{33} \in U(R)$. Then $A + 0 \in U(\mathcal{T}_3(R))$ and $A0 = 0 \in \mathcal{T}_3(R)^{qnil}$. So A is quasipolar.

Case 3. $a_{11} \in U(R), a_{22}, a_{33} \in J(R)$. There exists a unique element $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$. Let $E = \begin{bmatrix} 0 & 0 & 0 \\ e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $E^2 = E$, $A - E \in U(\mathcal{T}_3(R))$ and $AE \in J(\mathcal{T}_3(R)) \subseteq \mathcal{T}_3(R)^{qnil}$. We show that $E \in comm^2(A)$. Let $X = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \in comm(A)$. Then $XA = AX$ and so

$$a_{11}x_{11} = x_{11}a_{11}, \ a_{22}x_{22} = x_{22}a_{22}, \ a_{33}x_{33} = x_{33}a_{33} \qquad \text{(i)}$$

$$x_{21}a_{11} + x_{22}a_{21} = a_{21}x_{11} + a_{22}x_{21} \qquad \text{(ii)}$$

$$x_{22}a_{23} + x_{23}a_{33} = a_{22}x_{23} + a_{23}x_{33} \qquad \text{(iii)}$$

Since $a_{22}e_{21} - e_{21}a_{11} = a_{21}$, $a_{22}[x_{22}e_{21} - e_{21}x_{11} - x_{21}] - [x_{22}e_{21} - e_{21}x_{11} - x_{21}]a_{11} = 0$ by (i) and (ii). By (1), $l_{a_{22}} - r_{a_{11}}$ is injective and so $x_{22}e_{21} - e_{21}x_{11} = x_{21}$. That is, $XE = EX$. Hence $E \in comm^2(A)$.

Case 4. $a_{11} \in J(R), a_{22} \in U(R), a_{33} \in J(R)$. There exist unique elements $e_{21}, e_{23} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$ and $a_{22}e_{23} - e_{23}a_{11} = -a_{23}$. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ e_{21} & 0 & e_{23} \\ 0 & 0 & 1 \end{bmatrix}$. Then $E^2 = E$, $A - E \in U(\mathcal{T}_3(R))$ and $AE \in J(\mathcal{T}_3(R)) \subseteq \mathcal{T}_3(R)^{qnil}$. We prove $E \in comm^2(A)$. Let $X =$

$$\begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \in comm(A).$$ Then $XA = AX$. Since $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$, $a_{22}[-x_{22}e_{21} + e_{21}x_{11} - x_{21}] - [-x_{22}e_{21} + e_{21}x_{11} - x_{21}]a_{11} = 0$ by (i) and (ii). By (1), $l_{a_{22}} - r_{a_{11}}$ is injective and so $x_{22}e_{21} + x_{21} = e_{21}x_{11}$. Since $a_{22}e_{23} - e_{23}a_{11} = -a_{23}$, $a_{22}[-x_{22}e_{23} + e_{23}x_{33} - x_{23}] - [-x_{22}e_{23} + e_{23}x_{33} - x_{23}]a_{11} = 0$ by (i) and (iii). By (1), $l_{a_{22}} - r_{a_{11}}$ is injective and so $x_{22}e_{23} + x_{23} = e_{23}x_{33}$. That is, $XE = EX$. Hence $E \in comm^2(A)$.

Case 5. $a_{11}, a_{22} \in J(R), a_{33} \in U(R)$. There exists a unique element $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{bmatrix}$. Then $E^2 = E$, $A - E \in U\big(\mathcal{T}_3(R)\big)$ and $AE \in J\big(\mathcal{T}_3(R)\big) \subseteq \mathcal{T}_3(R)^{qnil}$. We show that $E \in comm^2(A)$. Let $X = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \in comm(A)$. Then $XA = AX$. Since $a_{22}e_{23} - e_{23}a_{33} = a_{23}$, $a_{22}[x_{22}e_{23} - e_{23}x_{33} - x_{23}] - [x_{22}e_{23} - e_{23}x_{33} - x_{23}]a_{33} = 0$ by (i) and (iii). By (1), $l_{a_{22}} - r_{a_{33}}$ is injective and so $x_{22}e_{23} - e_{23}x_{33} = x_{23}$. That is, $XE = EX$. Hence $E \in comm^2(A)$.

Case 6. $a_{11} \in J(R), a_{22}, a_{33} \in U(R)$. There exists a unique element $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $E^2 = E$, $A - E \in U\big(\mathcal{T}_3(R)\big)$ and $AE \in J\big(\mathcal{T}_3(R)\big) \subseteq \mathcal{T}_3(R)^{qnil}$. We prove that $E \in comm^2(A)$. Let $X = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \in comm(A)$. Then $XA = AX$. Since $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$, $a_{22}[-x_{22}e_{21} + e_{21}x_{11} - x_{21}] - [-x_{22}e_{21} + e_{21}x_{11} - x_{21}]a_{11} = 0$ by (i) and (ii). By (1), $l_{a_{22}} - r_{a_{11}}$ is injective and so $x_{22}e_{21} + x_{21} = e_{21}x_{11}$. That is, $XE = EX$. Hence $E \in comm^2(A)$.

Case 7. $a_{11} \in U(R), a_{22} \in J(R), a_{33} \in U(R)$. There exist unique elements $e_{21}, e_{23} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$ and $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Let $E = \begin{bmatrix} 0 & 0 & 0 \\ e_{21} & 1 & e_{23} \\ 0 & 0 & 0 \end{bmatrix}$. Then $E^2 = E$, $A - E \in U\big(\mathcal{T}_3(R)\big)$ and $AE \in J\big(\mathcal{T}_3(R)\big) \subseteq \mathcal{T}_3(R)^{qnil}$. To show $E \in comm^2(A)$ let $X = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \in comm(A)$. Then $XA = AX$. Since $a_{22}e_{21} - e_{21}a_{11} = a_{21}$, $a_{22}[x_{22}e_{21} - e_{21}x_{11} - x_{21}] - [x_{22}e_{21} - e_{21}x_{11} - x_{21}]a_{11} = 0$ by (i) and (ii). By (1), $l_{a_{22}} - r_{a_{11}}$ is injective and so $x_{22}e_{21} - e_{21}x_{11} = x_{21}$. Since $a_{22}e_{23} - e_{23}a_{33} = a_{23}$, $a_{22}[x_{22}e_{23} - e_{23}x_{33} -$

$x_{23}] - [x_{22}e_{23} - e_{23}x_{33} - x_{23}]a_{33} = 0$ by (i) and (iii). By (1), $l_{a_{22}} - r_{a_{33}}$ is injective and so $x_{22}e_{23} - e_{23}x_{33} = x_{23}$. That is, $XE = EX$. Hence $E \in comm^2(A)$.

Case 8. $a_{11}, a_{22} \in U(R), a_{33} \in J(R)$. There exists a unique element $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = -a_{23}$. Let $E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{bmatrix}$. Then $E^2 = E$, $A - E \in U\big(\mathfrak{T}_3(R)\big)$ and $AE \in J\big(\mathfrak{T}_3(R)\big) \subseteq \mathfrak{T}_3(R)^{qnil}$. The remaining proof is to show that $E \in comm^2(A)$. Let $X = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \in comm(A)$. Then $XA = AX$. Since $a_{22}e_{23} - e_{23}a_{33} = -a_{23}$, $a_{22}[-x_{22}e_{23} + e_{23}x_{33} - x_{23}] - [-x_{22}e_{23} + e_{23}x_{33} - x_{23}]a_{33} = 0$ by (i) and (iii). By (1), $l_{a_{22}} - r_{a_{33}}$ is injective and so $x_{22}e_{23} + x_{23} = e_{23}x_{33}$. That is, $XE = EX$. Hence $E \in comm^2(A)$.

(2) $\Rightarrow$ (3) Assume that $\mathfrak{T}_3(R)$ is quasipolar. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{T}_3(R)$. Then $T_2(R) \cong E\mathfrak{T}_3(R)E$. Thus $T_2(R)$ is quasipolar by [14, Proposition 3.6].

(3) $\Rightarrow$ (1) It follows from [7, Proposition 2.9]. $\square$

An element $a \in R$ is *strongly rad clean* provided that there exists an idempotent $e \in R$ such that $ae = ea$ and $a - e \in U(R)$ and $ea \in J(eRe)$. A ring R is *strongly rad clean* in case every element in R is strongly rad clean (cf. [8]).

Due to the proof of Theorem 3.1, we have the following.

Corollary 3.2. Let R be a local ring. The following are equivalent.

(1) $\mathfrak{T}_3(R)$ is strongly rad clean.

(2) R is bleached.

For a ring R, let $\mathcal{L}_3(R) = \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \mid a_{11}, a_{31}, a_{22}, a_{33} \in R \right\}$.

Then $\mathcal{L}_3(R)$ is a ring under the usual addition and multiplication, and so $\mathcal{L}_3(R)$ is a subring of $M_3(R)$. Our next endeavor is to find conditions under which $\mathcal{L}_3(R)$ is quasipolar.

Proposition 3.3. Let R be a bleached local ring. The following are equivalent.

(1) R is uniquely bleached.

(2) $\mathcal{L}_3(R)$ is quasipolar.

Proof. Let $\varphi : \mathcal{L}_3(R) \to T_2(R) \oplus R$ given by

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \mapsto \left(\begin{bmatrix} a_{33} & a_{31} \\ 0 & a_{11} \end{bmatrix}, a_{22} \right).$$

Then φ is an isomorphism (see [3, Proposition 2.2]). Since R is local, it is quasipolar. Hence $\mathcal{L}_3(R)$ is quasipolar if and only if $T_2(R)$ is quasipolar. Therefore it follows from Theorem 3.1.   □

Corollary 3.4. Let R be a bleached local ring. The following are equivalent.

(1) R is uniquely bleached.

(2) The ring $\left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mid a_{11}, a_{31}, a_{32}, a_{22}, a_{33} \in R \right\}$ is quasipolar.

(3) The ring $\left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \mid a_{11}, a_{13}, a_{23}, a_{22}, a_{33} \in R \right\}$ is quasipolar.

Proof. $(1) \Leftrightarrow (2)$ Let

$$\varphi : \mathcal{T}_3(R) \to \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mid a_{11}, a_{31}, a_{32}, a_{22}, a_{33} \in R \right\}$$

given by $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{33} & 0 \\ a_{21} & a_{23} & a_{22} \end{bmatrix}$ for any $A \in \mathcal{T}_3(R)$.

Then φ is an isomorphism (see [2, Corollary 3.4]). In view of Theorem 3.1, $\mathcal{T}_3(R)$ is quasipolar if and only if

$$\left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mid a_{11}, a_{31}, a_{32}, a_{22}, a_{33} \in R \right\}$$

is quasipolar, as asserted.

$(1) \Leftrightarrow (3)$ is symmetric.   □

Corollary 3.5. Let R be a bleached local ring. The following are equivalent.

(1) R is uniquely bleached.

(2) The ring $S_1 = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \mid a_{11}, a_{13}, a_{22}, a_{33} \in R \right\}$ is quasipolar.

(3) The ring $S_2 = \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \mid a_{11}, a_{31}, a_{22}, a_{33} \in R \right\}$ is quasipolar.

Proof. $(1) \Leftrightarrow (2)$ As in the proof of Proposition 3.3, R is uniquely bleached if and only if S_1 is quasipolar, as asserted.

$(2) \Leftrightarrow (3)$ Let $\varphi : S_1 \to S_2$ given by

$$A = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \mapsto \begin{bmatrix} a_{22} & 0 & 0 \\ 0 & a_{33} & 0 \\ 0 & a_{13} & a_{11} \end{bmatrix}$$

for any $A \in S_1$. Then φ is an isomorphism. Hence S_1 is quasipolar if and only if S_2 is quasipolar. $\qquad\square$

Let R be a commutative local ring. By Theorem 3.1, Proposition 3.3, Corollary 3.4 and Corollary 3.5, the rings

$$\begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ R & 0 & R \end{bmatrix}, \begin{bmatrix} R & 0 & 0 \\ R & R & R \\ 0 & 0 & R \end{bmatrix}, \begin{bmatrix} R & 0 & R \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix}, \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ R & R & R \end{bmatrix}$$

are all quasipolar.

Remark 3.6. Let $\mathbb{Z}_{(2)} = \{ \frac{m}{n} \mid m, n \in \mathbb{Z}, 2 \nmid n \}$. By [13, Example 1] and [9, Remark 3.2.11], $M_3(\mathbb{Z}_{(2)})$ is not strongly clean and so it is not quasipolar. However, by Theorem 3.1, Proposition 3.3, Corollary 3.4 and Corollary 3.5, the rings $\begin{bmatrix} \mathbb{Z}_{(2)} & 0 & 0 \\ 0 & \mathbb{Z}_{(2)} & 0 \\ \mathbb{Z}_{(2)} & 0 & \mathbb{Z}_{(2)} \end{bmatrix}, \begin{bmatrix} \mathbb{Z}_{(2)} & 0 & 0 \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \\ 0 & 0 & \mathbb{Z}_{(2)} \end{bmatrix}, \begin{bmatrix} \mathbb{Z}_{(2)} & 0 & \mathbb{Z}_{(2)} \\ 0 & \mathbb{Z}_{(2)} & 0 \\ 0 & 0 & \mathbb{Z}_{(2)} \end{bmatrix},$
$\begin{bmatrix} \mathbb{Z}_{(2)} & 0 & 0 \\ 0 & \mathbb{Z}_{(2)} & 0 \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{bmatrix}$ are all quasipolar.

4 Matrices Over Power Series Rings

In this section, we characterize quasipolar matrices over the power series ring of a local ring. In order to prove Theorem 4.2, we need the following lemma.

Lemma 4.1. Let R be a commutative local ring and $A(x) \in M_2\big(R[[x]]\big)$. The following are equivalent.

(1) $\chi\big(A(0)\big)$ has a root in $J(R)$ and a root in $U(R)$.

(2) $\chi\big(A(x)\big)$ has a root in $J\big(R[[x]]\big)$ and a root in $U\big(R[[x]]\big)$.

Proof. $(1) \Rightarrow (2)$ Assume that $\chi\big(A(0)\big) = y^2 - \mu y - \lambda$ has a root $\alpha \in J(R)$ and a root $\beta \in U(R)$. Let $y = \sum\limits_{i=0}^{\infty} b_i x^i$. Then $y^2 = \sum\limits_{i=0}^{\infty} c_i x^i$ where $c_i = \sum\limits_{k=0}^{i} b_k b_{i-k}$. Let $\mu(x) = \sum\limits_{i=0}^{\infty} \mu_i x^i, \lambda(x) = \sum\limits_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$ where $\mu_0 = \mu$ and $\lambda_0 = \lambda$. Then, $y^2 - \mu(x)y - \lambda(x) = 0$ holds in $R[[x]]$ if the following equations are satisfied:

$$b_0^2 - b_0\mu_0 - \lambda_0 = 0;$$
$$(b_0 b_1 + b_1 b_0) - (b_0 \mu_1 + b_1 \mu_0) - \lambda_1 = 0;$$
$$(b_0 b_2 + b_1^2 + b_2 b_0) - (b_0 \mu_2 + b_1 \mu_1 + b_2 \mu_0) - \lambda_2 = 0;$$
$$\vdots$$

Obviously, $\mu_0 = tr A(0) = \alpha + \beta \in U(R)$. Let $b_0 = \alpha$. Since R is commutative local, there exists some $b_1 \in R$ such that

$$b_0 b_1 + b_1(b_0 - \mu_0) = \lambda_1 + b_0 \mu_1.$$

Further, there exists some $b_2 \in R$ such that

$$b_0 b_2 + b_2(b_0 - \mu_0) = \lambda_2 - b_1^2 + b_0 \mu_2 + b_1 \mu_1.$$

By iteration of this process, we get $b_3, b_4, \cdots$. Then $y^2 - \mu(x)y - \lambda(x) = 0$ has a root $\alpha(x) \in J\big(R[[x]]\big)$. If $b_0 = \beta$, analogously, we show that $y^2 - \mu(x)y - \lambda(x) = 0$ has a root $\beta(x) \in U\big(R[[x]]\big)$.

$(2) \Rightarrow (1)$ Suppose that $\chi\big(A(x)\big) = y^2 - \mu(x)y - \lambda(x)$ has a root $\alpha(x) \in J\big(R[[x]]\big)$ and a root $\beta(x) \in U\big(R[[x]]\big)$. Then $\mu(x) = tr A(x)$ and $-\lambda(x) = det A(x)$. Hence $\mu(0) = tr A(0)$ and $-\lambda(0) = det A(0)$. Thus, $\chi\big(A(0)\big) = y^2 - \mu(0)y - \lambda(0)$. Since $\alpha(x)^2 - \mu(x)\alpha(x) - \lambda(x) = 0$ and $\beta(x)^2 - \mu(x)\beta(x) - \lambda(x) = 0$, $\alpha(0)^2 - \mu(0)\alpha(0) - \lambda(0) = 0$ and $\beta(0)^2 - \mu(0)\beta(0) - \lambda(0) = 0$. Then $\chi\big(A(0)\big) = y^2 - \mu(0)y - \lambda(0)$ has a root $\alpha(0) \in J(R)$ and a root $\beta(0) \in U(R)$. □

Theorem 4.2. Let R be a commutative local ring. The following are equivalent.

(1) $A(0) \in M_2(R)$ is quasipolar.

(2) $A(x) \in M_2\big(R[[x]]\big)$ is quasipolar.

Proof. (1) $\Rightarrow$ (2) It is known that $R[[x]]$ is local. To complete the proof we consider the following cases:

 (i) $A(0) \in GL_2(R)$,

 (ii) $det A(0), tr A(0) \in J(R)$,

 (iii) $det A(0) \in J(R), tr A(0) \in U(R)$ and $\chi\big(A(0)\big)$ is solvable in R.

If $A(0) \in GL_2(R)$, then $A(x) \in GL_2\big(R[[x]]\big)$ and so $A(x) \in M_2\big(R[[x]]\big)$ is quasipolar by [6, Example 2.1]. If $det A(0), tr A(0) \in J(R)$, then $tr A(x), det A(x) \in J\big(R[[x]]\big)$ and so $A(x)$ is quasipolar by [6, Theorem 2.6]. Now suppose that $det A(0) \in J(R), tr A(0) \in U(R)$ and $\chi\big(A(0)\big)$ has two roots $\alpha, \beta \in R$. Then $det A(x) \in J\big(R[[x]]\big)$ and $tr A(x) \in U\big(R[[x]]\big)$. Since $det A(0) \in J(R)$ and $tr A(0) \in U(R)$, either $\alpha \in J(R)$ or $\beta \in J(R)$. Without loss of generality, we assume that $\alpha \in J(R)$ and $\beta \in U(R)$. According to Lemma 4.1, $\chi\big(A(x)\big)$ has a root in $J\big(R[[x]]\big)$ and a root in $U\big(R[[x]]\big)$. Hence $A(x)$ is quasipolar in $M_2\big(R[[x]]\big)$ by [6, Proposition 2.8].
(2) $\Rightarrow$ (1) is similar to the proof of (1) $\Rightarrow$ (2). $\qquad\square$

Example 4.3. Let $R = \mathbb{Z}_4[[x]]$, and let

$$
A(x) = \left[\begin{array}{cc} \bar{0} & - \sum\limits_{n=1}^{\infty} (\bar{1} + \bar{3}^n)x^n \\ \bar{1} & \bar{3} - \sum\limits_{n=1}^{\infty} (\bar{1} + \bar{3}^n)x^n \end{array} \right] \in M_2(R).
$$

Obviously, $\mathbb{Z}_4$ is a commutative local ring. Since $A(0) = \left[\begin{array}{cc} \bar{0} & \bar{0} \\ \bar{1} & \bar{3} \end{array} \right]$, $\chi\big(A(0)\big) = t^2 - tr A(0)t + det A(0) = t^2 - \bar{3}t = t(t - \bar{3})$ is solvable in $\mathbb{Z}_4$. By [6, Proposition 2.8], $A(0) \in M_2(\mathbb{Z}_4)$ is quasipolar. In view of Theorem 4.2, $A(x) \in M_2(R)$ is quasipolar.

Theorem 4.4. Let R be a commutative local ring and for $m \geq 1$ $A(x) \in M_2\big(R[[x]]/(x^m)\big)$. The following are equivalent.

 (1) $A(0) \in M_2(R)$ is quasipolar.

 (2) $A(x) \in M_2\big(R[[x]]/(x^m)\big)$ is quasipolar.

Proof. The proof is similar to that of Theorem 4.2. $\qquad\square$

Example 4.5. Let $R = \mathbb{Z}_4[[x]]/(x^2)$, and let

$$\overline{A}(x) = \left[\begin{array}{cc} \overline{3} + (x^2) & \overline{2} + \overline{2}x + (x^2) \\ \overline{2} + x + (x^2) & \overline{2} + \overline{3}x + (x^2) \end{array} \right] \in M_2(R).$$

Obviously, $\mathbb{Z}_4$ is a commutative local ring. Since $A(0) = \left[\begin{array}{cc} \overline{3} & \overline{2} \\ \overline{2} & \overline{2} \end{array} \right]$, $\chi\big(A(0)\big) = t^2 - trA(0)t + detA(0) = t^2 - t + \overline{2} = (t - \overline{3})(t + \overline{2})$ is solvable in $\mathbb{Z}_4$. By [6, Proposition 2.8], $A(0) \in M_2(\mathbb{Z}_4)$ is quasipolar. In view of Theorem 4.4, $A(x) \in M_2(R)$ is quasipolar.

Lemma 4.6. Let R be a local ring. Then R is uniquely bleached if and only if $R[[x]]$ is uniquely bleached.

Proof. Assume that R is uniquely bleached. Then $l_u - r_j$ is an isomorphism for any $j \in J(R)$ and $u \in U(R)$ and let $f(x) = \sum\limits_{i=1}^{\infty} a_i x^i \in R[[x]]$. Since R is bleached, by [8, Example 2.1.11(6)], $R[[x]]$ is bleached. If, for $j(x) = \sum\limits_{i=1}^{\infty} j_i x^i \in J\big(R[[x]]\big)$ and $u(x) = \sum\limits_{i=1}^{\infty} u_i x^i \in U\big(R[[x]]\big)$, $(l_{j(x)} - r_{u(x)})(f(x)) = 0$, then

$$j_0 a_0 = a_0 u_0 \qquad\qquad\qquad (i_1)$$
$$j_0 a_1 + j_1 a_0 = a_0 u_1 + a_1 u_0 \qquad\qquad (i_2)$$
$$j_0 a_2 + j_1 a_1 + j_2 a_0 = a_0 u_2 + a_1 u_1 + a_2 u_0 \qquad (i_3)$$
$$\vdots \qquad\qquad\qquad\qquad\qquad \vdots$$

By assumption, $l_{j_0} - r_{u_0}$ is an isomorphism and so $a_0 = 0$ by (i_1). As $a_0 = 0$, by (i_2), $j_0 a_1 = a_1 u_0$ and so $a_1 = 0$ by assumption. Since $a_0 = 0 = a_1$, by (i_3), $j_0 a_2 = a_2 u_0$ and so $a_2 = 0$ by assumption. By iteration of this process, we deduce that $f(x) = 0$. Hence $l_{j(x)} - r_{u(x)}$ is an isomorphism and so $R[[x]]$ is uniquely bleached. Conversely, suppose that $R[[x]]$ is uniquely bleached. Then $l_{j(x)} - r_{u(x)}$ is an isomorphism for any $j(x) = \sum\limits_{i=1}^{\infty} j_i x^i \in J\big(R[[x]]\big)$ and $u(x) = \sum\limits_{i=1}^{\infty} u_i x^i \in U\big(R[[x]]\big)$ and let $r \in R$. Let $(l_j - r_u)(r) = 0$ with $j \in J(R)$ and $u \in U(R)$. Since $j \in J\big(R[[x]]\big)$ and $u \in U\big(R[[x]]\big)$, by assumption, $r = 0$ and so $l_j - r_u$ is injective. The remaining proof is to show that $l_j - r_u$ is surjective. Since $l_{j(x)} - r_{u(x)}$ is an isomorphism where $j(0) = j$ and $u(0) = u$, for any $r \in R$, we can find some $f(x) = \sum\limits_{i=1}^{\infty} a_i x^i \in R[[x]]$ such that $j(x)f(x) - f(x)u(x) = r$. Hence $ja_0 - a_0 u = r$ with $a_0 \in R$ and so $l_j - r_u$ is surjective. Thus R is uniquely bleached. $\square$

Proposition 4.7. Let R be a bleached local ring. The following are equivalent.

(1) $\mathcal{T}_3(R)$ is quasipolar.

(2) $\mathcal{T}_3\big(R[[x]]\big)$ is quasipolar.

Proof. (1) $\Rightarrow$ (2) Assume that $\mathcal{T}_3(R)$ is quasipolar. By Theorem 3.1, R is uniquely bleached. Note that if R is local, then so is $R[[x]]$ because $R/J(R) \cong R[[x]]/J\big(R[[x]]\big)$. According to Lemma 4.6, $R[[x]]$ is uniquely bleached. Hence $\mathcal{T}_3\big(R[[x]]\big)$ is quasipolar by Theorem 3.1.

(2) $\Rightarrow$ (1) Suppose that $\mathcal{T}_3\big(R[[x]]\big)$ is quasipolar. Then $R[[x]]$ is uniquely bleached by Theorem 3.1. In view of Lemma 4.6, R is uniquely bleached. Hence $\mathcal{T}_3(R)$ is quasipolar by Theorem 3.1. $\square$

Corollary 4.8. Let R be a bleached local ring. For any positive integer n, the following are equivalent.

(1) $T_n(R)$ is quasipolar.

(2) $T_n\big(R[[x]]\big)$ is quasipolar.

Proof. By [7, Proposition 2.9] and Lemma 4.6, the proof is completed. $\square$

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