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ATTACHMENT OBSERVABILITY OF A ROTATING BODY-BEAM

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Abstract

In this article we analyze the admissibility and the exact observability of a body-beam system when the output is taken in a point of attachment from the beam to the body. The single output case chosen here is the practical measurement of the strength, its velocity or its moment. We prove the exact observability for the moment and the admissibility for the other cases. These results are obtained by the spectral properties of rotating body beam system operator and Ingham's inequalities.

1 Introduction

Understanding complex systems means focusing on their internal structures, functions, behaviour and interactions with other systems. Systems' thinking involves the exploration of interdependencies, dynamics, and feedback loops occurring within the system. This involves asking how the parts function within the whole and moving among the different levels of abstraction to understand global function [11]. Knowledge concerning these systems [3] [7] [16] [20] is important to forecast their behaviour, improve their maintenance and transformation, self-organization, better understand structural tendencies of non-orderly appearing phenomena [18].

The problem of body - beam system' stabilization [8] has been extensively studied in the literature. Bailleul and Levi [2] proved that in the presence of a

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structural friction and in the control' s absence, the system has a finite number of equilibrium states in rotation. Bloch and Titi [4] proved the existence of a inertial variety of the system, in the situation of viscous friction. Considering the effect of viscous amortization, Xu and Bailleul [22] proved that for all constant angular velocity, less than a critical value, the system is exponentially stabilisable only by a feedback of moment of control type:

$$\Gamma_1(t) = 0;\ \Gamma_2(t) = 0;\ \Gamma_3(t) = -\nu\omega(t), \nu \geq 0.$$

Novel and Coron [1] built another law of moment feedback, that globally stabilizes the system:

$$\Gamma_1(t) = 0;\ \Gamma_2(t) = 0;\ \Gamma_3(t) = \omega_t(Id + \int_0^1 u^2\, dx) + 2\omega \int_0^1 uu_t\, dx.$$

The problem of the boundary stabilization at the free extremity of the beam ($x = L$) has also been studied. Laousy et al. [14] proved the exponentially stability of the system (without amortization) for two boundary linear controls at the free extremity of the beam ($\Gamma_1(t) = -\alpha u_{tx}(L,t)$; $\Gamma_2(t) = \beta u_{tx}(L,t)$) and a control of moment type ($\Gamma_3(t) = -\gamma(\omega(t) - w^*)$. Chentouf and Couchouron [6] extended the results of [14] by a class of non linear boundary controls ($\Gamma_1(t) = -f(u_{tx}(L,t))$; $\Gamma_2(t) = g(u_{tx}(L,t)); \Gamma_3(t) = -\gamma(\omega(t) - \omega^*)$).

Since the stabilization of a system is possible for measures and feedback laws at the free extremity of the beam, the problem that naturally arises, and is studied here, is the system stabilization for measures collected at the attachment point. Therefore, we analyse here the existence of admissible and stable states in the bearing edge.

2 Preliminaries

Let consider a distributed, non excited system, described by the equation [13]:

$$(\sum)\begin{cases} \dot{\phi}(t) &= A\phi(t),\ \forall t \geq 0, \\ \phi(0) &= \phi_0. \end{cases}$$

The state of this kind of system can not always be directly measured from physical point of view. But, sometimes it is possible to collect some information on the system and monitor its evolution during an interval of time $[0, \tau_0]$. Thus, the observability problem is the reconstruction of the system' state, using the measures collected on that time interval.

Assume that the state space of the study system is a Banach space, X. If the operator associated to the system generates a C^0 semi-group, then the solution can be written as $\phi(t) = T(t)\phi_0$, and it is enough to determine ϕ at each moment t, knowing the initial condition ϕ_0.

Suppose that we collect q measures on the system, defined by the output function:

$$(S)\left\{\begin{array}{rcl} y(t) & = & (y_1(t), y_2(t), \dots, y_q(t)) \\ & = & C\phi(t), \end{array}\right.$$

where C is an unbounded operator, whose domain $D(C) \subset X$ is invariant with respect to the C^0 semi-group $T(t)_{t\geq 0}$ and $y(.) \in L^2(0,T;\mathbf{R}^q)$. We have: $y(t) = CT(t)\phi_0 = \Psi\phi_0(t)$. The observability is equivalent to the existence of an inverse of the following operator:

$$\Upsilon : L^2(0,T;\mathbf{R}^q) \mapsto X, y \mapsto \Upsilon y = \phi_0.$$

Definition 2.1. The system $(\sum)$ together with (S) is exactly observable if there are constants $\tau_0 > 0$ and $M > 0$ such that:

$$M^{-1} \parallel \phi_0 \parallel_X^2 \leq \int_0^{\tau_0} \parallel CT(t)\phi_0 \parallel_O^2 \, dt \leq M \parallel \phi_0 \parallel_X^2 . \tag{1}$$

Let $(\prod^0)$ be the open loop system given by:

$$(\textstyle\prod^0)\colon \left\{\begin{array}{rcl} \dot{\phi}(t) & = & A\phi(t), \\ y(t) & = & C\phi(t), \\ \phi(0) & = & \phi_0. \end{array}\right.$$

The first equation of $(\prod^0)$ implies that $\phi(t) = T(t)\phi_0$ is a weak solution. Analyzing the second equation of $(\prod^0)$, it results that if $\phi_0 \notin D(A)$, we can have $\phi(t) \notin D(A)$ and, therefore, $y(t)$ is not defined. To get around this difficulty, we work with the following:

Definition 2.2 [19] C is an output admissible operator for the semi-group $(T(t))_{t\geq 0}$ or the couple (A, C) is admissible if there are two positive constants M, τ_0 such that:

$$\int_0^{\tau_0} \parallel CT(t)\phi_0 \parallel_0^2 \, dt \leq M \parallel \phi_0 \parallel_X^2 . \tag{2}$$

Proposition 2.3. An operator A has a compact resolvent in a Hilbert space X, iff the resolvent set, $\rho(A)$, is not void and the injection of $D(A)$ in X is compact, $D(A)$ carrying the graph norm.

3 Results

The simplified model of body-beam system discussed in the following is described by:

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) = \omega_*^2 u(x,t), & t>0, \quad x \in (0,1) \\ u(0,t) = u_x(0,t) = 0, & t>0, \\ u_{xx}(1,t) = u_{xxx}(1,t) = 0, & t>0, \\ u(0,x) = u_0(x), u_t(0,x) = v_0(x), & \qquad\quad x \in (0,1) \\ y(t) = u_{xx}(0,t), & t>0 \end{cases} \tag{3}$$

where ω_* is a positive constant, representing the rotation speed of the disc about its axis, $u(x,t)$ is the transverse displacement of the beam at the abscissa x at the moment t, $y(t)$ is the only measure of the force moment at the clamping point. Let consider

$$H_L^2(0,1) = \{f / f, f_x, f_{xx} \in L^2(0,1), f(0) = f_x(0) = 0\}.$$

The state space of system (3) is the Hilbert space $X = H_L^2(0,1) \times L^2(0,1)$, endowed with the inner product:

$$< f, g >_X = \int_0^1 \{f_{1xx}(x) g_{1xx}(x) + f_2(x) g_2(x) - \omega_*^2 f_1(x) g_1(x)\} \, dx.$$

The observation space is $O = \mathbf{R}$, endowed with its usual inner product.

Denoting by $H^4(0,1) = \left\{ f \in L^2(0,1) : \frac{\partial^\beta f}{\partial x^\beta} \in L^2(0,1), \forall \beta \in \mathbf{N} \right\}$, we define the operators A_0, and A_{ω_*}, respectively by:

$$\begin{cases} D(A_0) &= \{f \in H^4(0,1) / f(0) = f_x(0) = f_{xx}(1) = f_{xxx}(1) = 0\} \\ A_0 f(x) &= f^{(4)}(x), \quad \forall f \in D(A_0). \end{cases} \tag{4}$$

$$D(A_{\omega_*}) = D(A_0) \times H_L^2(0,1), \tag{5}$$

$$A_{\omega_*} \begin{pmatrix} u \\ v \end{pmatrix}(x) = \begin{bmatrix} 0 & I \\ -A_0 + \omega_*^2 & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}(x). \tag{6}$$

In the following we shall denote by A the operator A_{ω_*} for $\omega_* = 0$.

Proposition 3.1. The unbounded operator A_0 is linear and admits a countable infinity of eigenvalues $0 < l_1 < l_2 < \ldots < l_n < \ldots$ such that $\lim_{n\to+\infty} l_n = +\infty$. The corresponding eigenvectors $(e_n)_{n\geq 1}$ form an orthogonal basis of $L^2(0,1)$.

Proof. **I.** A_0 is self-adjoint on $L^2(0,1)$

We prove that A_0 is maximal monotone, symmetric on $L^2(0,1)$ and we apply Proposition VII.6, page 113 from [5].

• A_0 is monotone and positive:

$$\langle A_0 f, f\rangle_{L^2(0,1)} = \int_0^1 (f''(x))^2\,dx \geq 0 \ \ \forall f \in D(A_0).$$

• A_0 is maximal:

We want to prove, by Lax Milgram, that $Im(I + A_0) = L^2(0,1)$.

Let $f \in L^2(0,1)$. We want to determine an element $u \in D(A_0)$ such that

$$(I + A_0)u = f. \tag{7}$$

(7) is equivalent with:

$$\begin{cases} u + u_{xxxx} = f \\ u(0) = u_x(0) = 0 \\ u_{xx}(1) = u_x xx(1) = 0. \end{cases} \tag{8}$$

Let $l : H_L^2(0,1) \to \mathbf{R}, v \mapsto \int_0^1 fv\,dx$. l is a linear and continuous form and :

$$|l(v)| = \left|\int_0^1 fv\,dx\right| \leq \|f\|_{L^2(0,1)}\|v\|_{L^2(0,1)} \tag{9}$$

But

$v(x) = \int_0^x v_\xi(\xi)\,d\xi \ \ \forall v \in H_L^2(0,1)$ and $v_\xi(\xi) = \int_0^\xi v_{yy}(y)\,dy \ \ \forall v \in H_L^2(0,1)$.

By Fubini's theorem it results that:

$$v(x) \ = \ \int_0^x \int_0^\xi v_{yy}(y)\,dyd\xi = \int_0^x v_{yy}(y) \int_y^x \,d\xi\,dy = \int_0^x (x-y)v_{yy}(y)\,dy.$$

Using Hölder' s inequality, we deduce that:

$$|v(x)| \ \leq \ \left[\int_0^x (x-y)^2\,dy\right]^{\frac{1}{2}} \left[\int_0^x |v_{yy}|^2\,dy\right]^{\frac{1}{2}} = \sqrt{\frac{x^3}{3}}\|v_{yy}\|^2_{L^2(0,1)} \Rightarrow$$

$$|v(x)|^2 \leq \frac{x^3}{3}\|v\|^2_{H_L^2(0,1)} \Rightarrow$$

$$\|v\|_{L^2(0,1)} \leq \frac{\sqrt{3}}{6}\|v\|_{H_L^2(0,1)} \quad \forall v \in H_L^2(0,1). \tag{10}$$

From (9) and (10), it results that:

$$|l(v)| \leq \frac{\sqrt{3}}{6}\|f\|_{H_L^2(0,1)}\|v\|_{H_L^2(0,1)}. \tag{11}$$

Let g be the application: $g : H^2_L(0,1) \times H^2_L(0,1) \to \mathbf{R}$,

$$(u,v) \mapsto \int_0^1 (uv + u_{xx}v_{xx})\,dx =< u,v >_{L^2(0,1)} + < u,v >_{H^2_L(0,1)} .$$

- g is a bilinear form,
- g is $H^2_L(0,1)$ - coercive:

$$g(u,u) \quad = \quad \|u\|^2_{L^2(0,1)} + \|u\|^2_{H^2_L(0,1)} \geq \|u\|^2_{H^2_L(0,1)},$$

- g is continuous (from (10)):

$$\begin{aligned} g(u,u) &\leq \|u\|_{L^2(0,1)}\|v\|_{L^2(0,1)} + \|u\|_{H^2_L(0,1)}\|v\|_{H^2_L(0,1)} \\ &\leq \left(1+\frac{1}{12}\right)\|u\|_{H^2_L(0,1)}\|v\|_{H^2_L(0,1)}. \end{aligned}$$

By Lax Milgram (Corollary V.8 page 84 of [5]) we deduce that the equation $g(u,v) = l(v)$ has an unique solution $u \in H^2_L(0,1)$ for all $v \in H^2_L(0,1)$.

By direct calculation, we prove that u satisfies the equation for all $v \in H^2_L(0,1)$,

$$\begin{aligned} &\int_0^1 \left[u_{xx}(x) + \int_1^x \int_1^{\tau_1} u(\tau_2)\,d\tau_2 d\tau_1\right] v_{xx}(x)\,dx - \\ &- \int_0^1 \left[\int_1^x \int_1^{\tau_1} f(\tau_2)\,d\tau_2 d\tau_1\right] v_{xx}(x)\,dx = 0, \end{aligned}$$

for all $v \in H^2_L(0,1)$. This implies that, for all $v \in L^2(0,1)$,

$$\begin{aligned} &\int_0^1 \left[u_{xx}(x) + \int_1^x \int_1^{\tau_1} u(\tau_2)\,d\tau_2 d\tau_1\right] v(x)\,dx - \\ &- \int_0^1 \left[\int_1^x \int_1^{\tau_1} f(\tau_2)\,d\tau_2 d\tau_1\right] v(x)\,dx = 0. \end{aligned}$$

Therefore u is a solution of the equation:

$$u_{xx}(x) + \int_1^x \int_1^{\tau_1} u(\tau_2)\,d\tau_2 d\tau_1 - \int_1^x \int_1^{\tau_1} f(\tau_2)\,d\tau_2 d\tau_1 = 0. \tag{12}$$

For $x = 1$, from (12) we obtain $u_{xx}(1) = 0$.

By differentiation with respect to x, in equation (12), it results that:

$$u_{xxx}(x) + \int_1^x u(\tau)\,d\tau - \int_1^x f(\tau)\,d\tau = 0. \tag{13}$$

The condition $u_{xxx}(1) = 0$ is obtained putting $x = 1$ in (13).

By differentiation with respect to x, in equation (13), the following relation results:

$$u_{xxxx} + u = f, u \in D(A_0)$$

II. A_0 is symmetric.

A double integration per parts prove that:

$$< A_0 f, g >_{L^2(0,1)} = < f, A_0 g >_{L^2(0,1)} = \int_0^1 f_{xx} g_{xx}\, dx \quad \forall f, g \in D(A_0).$$

III. A_0 has a compact resolvent.

A_0 is the operator associated to the form g of the variational triplet $(L^2(0,1), H^2(0,1), b)$. Since g is $H^2(0,1)$ coercive, then A_0 est invertible by Lax Milgram. Therefore $0 \in \rho(A_0)$.

The injection from $H^4(0,1)$ to $L^2(0,1)$ is compact and $D(A_0) \subset H^4(0,1)$, so the injection from $D(A_0)$ to $L^2(0,1)$ is compact.

Applying Proposition 2.3, it results that A_0 has a compact resolvent.

Since A_0 is monotone, positive and maximal, the spectrum is formed by real positive numbers. From III it results that the spectrum of A_0 is formed by a countable infinity of eigenvalues $0 < l_1 < l_2 < \ldots < l_n < \ldots$ such that $\lim_{n\to+\infty} l_n = +\infty$. The corresponding eigenvectors $(e_n)_{n\geq 1}$ form an orthogonal basis of $L^2(0,1)$ [17]. Furthermore, the geometric multiplicity of each eigenvalue of A_0 is equal to its algebraic multiplicity [21]. □

Remark 3.2. Xu and Bailleul proved in [22] that: If $\omega_*^2 < l_1$, then A_{ω_*} generates a C^0 group of unity operators on X and the output operator, C, defined by: $C(\varphi_1, \varphi_2) = (\varphi_1)_{xx}(0)$, belongs to $L(X_1, O)$, where X_1 is the Banach space $D(A_{\omega_*})$ with a graph norm on X.

In the following we consider that ω_* is a very small constant. Since it is possible to estimate asymptotically the size of eigenvectors of the operator A_{ω_*}, we show the exact observability of the couple (A_{ω_*}, C) by spectral analysis and the inequality of Ingham.

Proposition 3.3. For all angular constant speed ω_*, the spectrum of A_{ω_*} can be written as:

$$\sigma(A_{\omega_*}) = \left\{ \nu_{\pm n} | \nu_{\pm n} = \pm\sqrt{\omega_*^2 - l_n}, \forall n = 1, 2, \ldots, m-1 \right\} \cup$$
$$\cup \left\{ \nu_{\pm n} | \nu_{\pm n} = \pm i\sqrt{l_n - \omega_*^2}, \forall n = m, m+1, \ldots \right\},$$

where m is the smallest positive integer such that $\omega_*^2 < l_m$. Moreover, the corresponding generalized eigenvectors can be chosen to form a Riesz basis on X and the eigenvalues ν_n or ν_{-n}, with $n \neq m-1$ are algebraically simple.

Proof. The proof of this proposition is given in [22]. If $\omega_*^2 \neq l_n, \forall n \in \mathbf{N}\backslash\{0\}$, all the eigenvalues of the system are algebraically simples. If $\omega_*^2 = l_{n_0}$ for a certain $n_0 \in \mathbf{N}\backslash\{0\}$, the corresponding eigenvalue $\nu_{\pm n_0}$ has a multiplicity of second order. $\square$

Proposition 3.4.[12] Let $f(t) = \sum_{n=N}^{N'} a_n e^{-i\lambda_n t}$, where λ_n are real numbers, satisfying the separation relation: there is $\gamma > 0$ such that: $\lambda_n - \lambda_{n-1} \geq \gamma > 0, \quad \forall \ N \leq n \leq N'$. Then, for each $\epsilon > 0$ and $T = \pi + \epsilon$, there is a constant $C(\epsilon) > 0$ (independent on N and N') such that:

$$\frac{1}{2TC(\epsilon)} \int_{-T}^{T} |f(t)|^2 dt \leq \sum_{n=N}^{N'} |a_n|^2 \leq \frac{C(\epsilon)}{2T} \int_{-T}^{T} |f(t)|^2 dt. \tag{14}$$

Moreover, the conclusion remains true for the uniformly convergent series on the interval $[-T, T]$.

Theorem 3.5. For any constant angular velocity $\omega_* < \sqrt{l_1}$, the system (3) is exactly observable.

Proof. **Part I**. We find a form for the solution of (3).

$\nu \in \mathbf{C}$ is the eigenvalue of A_{ω_*} iff there is $(u, v) \in D(A_{\omega_*})$, $(u, v) \neq (0, 0)$ that satisfies:

$$\begin{cases} v = \nu u \\ u_{xxxx} - (\omega_*^2 - \nu^2)u = 0 \\ u(0) = u_x(0) = 0 \\ u_{xx}(1) = u_{xxx}(1) = 0. \end{cases} \tag{15}$$

Then $\omega_*^2 - \nu^2$ is the eigenvalue of the operator A_0 defined in (4). For all $n \in \mathbf{N}\backslash\{0\}$, we have $l_n = \omega_*^2 - \nu_n^2$. Choosing ω_* small enough, such that

$$0 \leq \omega_*^2 < l_1 < l_2 < \ldots < l_n \ldots, \tag{16}$$

the spectrum of A_{ω_*} can be written as:

$$\sigma(A_{\omega_*}) = \left\{ \nu_{\pm n} = \pm i\lambda_n | \lambda_n = \sqrt{l_n - \omega_*^2}, n \in \mathbf{N}^* \right\}, \tag{17}$$

taking into account Proposition 3.3. The corresponding eigenvectors are:

$$\phi_n = \begin{bmatrix} e_n \\ i\lambda_n e_n \end{bmatrix}, \ \phi_{-n} = \begin{bmatrix} e_n \\ -i\lambda_n e_n \end{bmatrix} \quad \forall n \in \mathbf{N}\backslash\{0\}.$$

We choose $(e_n)_{n\geq 1}$ such that $(\phi_n)_{n\in\mathbf{Z}^*}$ form an orthonormal basis of X.

For $n \geq 1$, we define: $\alpha_n = <\phi_n, (u_0, v_0)>_X, \alpha_{-n} = \overline{\alpha_n}, \lambda_{-n} = -\lambda_n$.

With these notations, the solution of (3) can be written as:

$$(u,v) = \sum_{n\geq 1} \alpha_n e^{-i\lambda_n t}\phi_n + \sum_{n\geq 1} \overline{\alpha_n e^{-i\lambda_n t}\phi_n} = \sum_{n\in \mathbf{Z}^*} \alpha_n e^{-i\lambda_n t}\phi_n, \tag{18}$$

$$y(t) = \sum_{n\in \mathbf{Z}^*} \alpha_n (e_n)_{xx}(0) e^{-i\lambda_n t}. \tag{19}$$

Part II. We prove that: $(e_{n_0})_{xx}(0) \neq 0$ for all $n \in \mathbf{N}\backslash\{0\}$.

Suppose, by reduction ad absurdum, that there is $n_0 \in \mathbf{N}\backslash\{0\}$ such that $(e_{n_0})_{xx}(0) = 0$. Then e_{n_0} satisfies the ordinary differential equation:

$$\begin{cases} (e_{n_0})_{xxxx}(x) = l_{n_0} e_{n_0}(x), \\ e_{n_0}(0) = (e_{n_0})_x(0) = (e_{n_0})_{xx}(1) = (e_{n_0})_{xxx}(1) = 0. \end{cases} \tag{20}$$

Using $(e_{n_0})_x(x)$ as multiplier and integrating by part with respect to x on $[0,1]$, we obtain from (20) that: $e_{n_0}(1) = 0$. Analogous, using the multiplier $(e_{n_0})_x(x)$, it results that:

$$\int_0^1 \left[3(e''_{n_0})^2(x) + l_{n_0}(e_{n_0})^2(x)\right] dx = 0.$$

Therefore, $e_{n_0}(x) = 0$, $\forall x \in (0,1)$, that argues against the hypothesis that $e_{n_0}(x)$ is an eigenvector of the operator A_0. So, the proof of Part II is complete.

Part III. We prove that:

$$\lim_{n\to+\infty} |(e_n)_{xx}(0)| = \sqrt{2}, |\lambda_{n+1} - \lambda_n| = O(n^2).$$

Denote by:

$$l = \mu^4 = \omega_*^2 - \nu^2. \tag{21}$$

From the second and the third equations of the system (15), it results that:

$$u(x) = 2a\left(\cosh(\mu x) - \cos(\mu x)\right) + 2b\left(\sinh(\mu x) - \sin(\mu x)\right),$$

where a, b are real constants. The last condition of (15) can be written as:

$$\begin{pmatrix} \cosh\mu + \cos\mu & \sinh\mu + \sin\mu \\ \sinh\mu - \sin\mu & \cosh\mu + \cos\mu \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

u is the non-vanishing solution of the system (15) iff:

$$\begin{vmatrix} \cosh\mu + \cos\mu & \sinh\mu + \sin\mu \\ \sinh\mu - \sin\mu & \cosh\mu + \cos\mu \end{vmatrix} = 0 \Leftrightarrow \cosh\mu\cos\mu + 1 = 0. \tag{22}$$

A solution, μ, of (22) has the following properties:
- $\mu \in i\mathbf{R}^* \cup \mathbf{R}$;
- $\mu > 0$ is a solution of (22) $\Leftrightarrow -\mu$ is a solution of (22);
- $\mu > 0$ is the solution of (22) $\Leftrightarrow i\mu$ is a solution of (22).

To avoid the repetition of the eigenvalues of A_{ω_*}, we consider $\mu > 0$. The equation (22) is transcendent, so it has no algebraic solutions. The asymptotic behavior of the solution can be studied, when $n \to +\infty$. Its positive solutions can be estimated by:

$$\mu_n = \mu_{0n} + O(e^{-n}),\ \mu_{0n} = (n - 1/2)\,\pi, \quad \forall\, n \geq 1. \tag{23}$$

where μ_{0n} are the solutions of the equation $\cos\mu = 0$.

The constants a and b satisfy the relation: $a = k_n b$, with

$$k_n = -\left[\frac{\sinh\mu_n + \sin\mu_n}{\cosh\mu_n + \cos\mu_n}\right]$$

and

$$(e_n)_{xx}(0) = 4bk_n\mu_n^2.$$

Normalizing ϕ_n in X:

$$\parallel \phi_n \parallel_X^2 = (2\mu_n^4 - \omega_*^2) \parallel e_n \parallel_{L^2(0,1)}^2 = 1,$$

we obtain:

$$|b| = \sqrt{|2\mu_n^4 - \omega_*^2|}$$

$$\sqrt{\left|\ (k_n+1)^2\frac{e^{2\mu_n}}{2\mu_n} - (k_n-1)^2\frac{e^{-2\mu_n}}{2\mu_n} + 8(k_n^2-1)(-1)^n\frac{\cosh\mu_n}{\mu_n} - \frac{k_n}{\mu_n} + 4k_n^2\right|}.$$

From (23) it results that:

$$|k_n| \sim 1, |b|\mu_n^2 \sim \left|2 - O\left(1/n^4\right)\right|^{-\frac{1}{2}} |4 + O\left(1/n\right)|^{-\frac{1}{2}} \Rightarrow \lim_{n\to+\infty} |(e_n)_{xx})(0)| = \sqrt{2}.$$

From (17), (23) and (21) it results that $|\lambda_{n+1} - \lambda_n| = O(n^2)$, so, the proof of Part III is complete. From (19), Definition 2.1, the results of Part III and Proposition 3.4 it results that the system (3) is exactly observable. This complete the proof of Theorem 3.5. □

Remark. 3.6. The exact observability, within the meaning of the left inequality in (1) is also true if $\omega_* < \sqrt{l_1}$, when the output $y(t) = u_{xx}(0,t)$ is replaced by $y(t) = u_{xxx}(0,t)$ or $y(t) = u_{txx}(0,t)$ in the system (3). Moreover, in the last case, the output operator C is in $L(X_1, O)$. It would be interesting to look to other states spaces to make them admissible and to build observers.

Indeed, to prove the left inequality, the same procedure can be followed, noting that:

$$u_{xxx}(0,t) = \sum_{n\in\mathbf{Z}^*} \alpha_n e^{i\lambda_n t} e_{nxxx}(0), u_{txx}(0,t) = \sum_{n\in\mathbf{Z}^*} \alpha_n e^{i\lambda_n t}(i\lambda_n e_{nxx}(0)),$$

and writing the initial condition: $(u_0, v_0) = \sum_{n\in\mathbf{Z}^*} \alpha_n \phi_n$, with

$\phi_n = [e_n, i\lambda_n e_n]^T,\ \phi_{-n} = \overline{\phi}_n,\ \lambda_n = i\mu_n^2,\ \mu_n = (n-1/2)\pi + O(1/n^4),\ \ n \in \mathbf{N}^*,$

$$e_n(x) = \frac{1}{\sqrt{2}\mu_n^2}\left\{-\frac{\sinh\mu_n + \sin\mu_n}{\cosh\mu_n + cos\mu_n}\left[\cosh\mu_n x - \cos\mu_n x\right] + \sinh\mu_n x - \sin\mu_n x\right\}.$$

From $\lim_{n\to\infty} e_{nxx}(0) = \sqrt{2}$, it results that

$$\lim_{n\to\infty} e_{nxxx}(0)n^{-1} = \sqrt{2}\pi, and \lim_{n\to\infty} |\lambda_n| n^{-2} = \pi^2.$$

By Proposition 3.4. we deduce the left inequality from the exact observability definition.

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