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POINTWISE AND GLOBAL SUMS AND NEGATIVES OF BINARY RELATIONS

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Abstract

For any two relations F and G on one groupoid X to another Y, we define $(F+G)(x) = F(x) + G(x)$ for all $x \in X$ and

$$F \oplus G = \big\{ (x+z,\, y+w) : \quad (x,\, y) \in F, \quad (z,\, w) \in G \big\}.$$

Moreover, if in particular X and Y are groups, then we may also naturally define $(-F)(x) = -F(x)$ for all $x \in X$ and

$$\ominus F = \big\{ (-x,\, -y) : \quad (x,\, y) \in F \big\}.$$

By using these definitions, we prove some basic theorems about the images of subsets of X under the relations $-F$, $\ominus F$, $F+G$ and $F \oplus G$. In particular, we show that

$$(F \oplus G)(x) = \bigcup_{x=u+v} \big(F(u) + G(v) \big)$$

for all $x \in X$. Therefore, in contrast to the intersection convolution [1], the union convolution of relations need not be introduced. Moreover, it is also worth mentioning that the results obtained can, for instance, be applied to translation and additive relations [2].

1 A few basic facts on relations and groupoids

A subset F of a product set $X \times Y$ is called a relation on X to Y. In particular, the relations $\Delta_X = \{ (x,\, x) : \ x \in X \}$ and $X^2 = X \times X$ are called the identity and universal relations on X, respectively.

Namely, if in particular $F \subset X^2$, then we may simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X \cup Y$. Therefore, it is frequently not a severe restriction to assume that $X = Y$.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{ y \in Y : (x, y) \in F \}$ and $F[A] = \bigcup_{x \in A} F(x)$ are called the images of x and A under F, respectively. Whenever $A \in X$ seems unlikely, we may write $F(A)$ in place of $F[A]$.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse F^{-1} of F can, for instance, be defined such that $F^{-1}(y) = \big\{ x \in X : \ y \in F(x) \big\}$ for all $y \in Y$.

If F is a relation on X to Y, then the sets $D_F = F^{-1}(X)$ and $R_F = F(X)$ are called the domain and range of F, respectively. If in particular $X = D_F$ (and $Y = R_F$), then we say that F is a relation of X into (onto) Y.

A relation F on X to Y is called a function if for each $x \in D_F$ there exists $y \in Y$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ in place of $F(x) = \{y\}$.

If X is nonvoid set and $+$ is a function of X^2 into X, then the ordered pair $X(+) = (X, +)$ is called a groupoid. In this case, we may also naturally write $x + y = +(x, y)$ for all $x, y \in X$.

Moreover, if X is a groupoid, then we may also naturally write $A + B = \big\{ x + y : \ x \in A, \ y \in B \big\}$ for all $A, B \subset X$. Thus, the family $\mathcal{P}(X)$ of all subsets of X is also a groupoid.

Note that if X is, in particular, a group, then $\mathcal{P}(X)$ is, in general, only a semigroup with zero element $\{0\}$. However, we can still naturally use the notations $-A = \big\{ -x : \ x \in A \big\}$ and $A - B = A + (-B)$.

2 Pointwise and global sums and negatives of relations

Definition 2.1 *If F and G are relations on a set X to groupoid Y and $F + G$ is the relation on X to Y such that*

$$(F + G)(x) = F(x) + G(x)$$

for all $x \in X$, then $F + G$ is called the pointwise sum of F and G.

While, if F and G are relations on one groupoid X to another Y and

$$F \oplus G = \big\{ (x + z, \, y + w) : \quad (x, y) \in F, \quad (z, w) \in G \big\},$$

then the relation $F \oplus G$ is called the global sum of F and G.

Remark 2.2 *Thus, we have*

$$D_{F+G} = D_F \cap D_G \qquad \textit{and} \qquad D_{F \oplus G} = D_F + D_G .$$

The global sum $F \oplus G$ is, in general, quite different from the pointwise one $F + G$ even if $D_{F+G} = D_{F \oplus G}$.

Example 2.3 *If X is a groupoid, then*

(1) $\Delta_X + \Delta_X = \Delta_X$ *if and only if* $x = x + x$ *for all* $x \in X$;

(2) $\Delta_X \oplus \Delta_X = \Delta_X$ *if and only if for each* $x \in X$ *there exist* $u\,,\, v \in X$ *such that* $x = u + v$.

Therefore, if in particular X *is a group, then* $\Delta_X \oplus \Delta_X = \Delta_X$, *but* $\Delta_X + \Delta_X = \Delta_X$ *if and only if* $X = \{0\}$.

However, in some very particular cases, the global sum of relations may coincide with the pointwise one.

Example 2.4 *Let X be a nonvoid set, and for all $x, y \in X$ define $x + y = x$. Then X is a semigroup such that, for any two relations F and G on X, we have $F \oplus G = F$ whenever $G \neq \emptyset$, and $F + G = F$ whenever $G(x) \neq \emptyset$ for all $x \in D_F$.*

Analogously to Definition 2.1, we may also naturally introduce the following

Definition 2.5 *If F is a relation on a set X to group Y and $-F$ is the relation on X to Y such that*

$$(\,-F\,)(x) = -F\,(x)$$

for all $x \in X$, then $-F$ is called the pointwise negative of F.

While, if F is a relation on one group X to another Y and

$$\ominus F = \big\{ (\,-x\,,\,-y\,) : \quad (x,\,y) \in F \,\big\} ,$$

then the relation $\ominus F$ is called the global negative of F.

Remark 2.6 *Thus, we have*

$$D_{-F} = D_F \qquad \textit{and} \qquad D_{\ominus F} = -D_F .$$

The global negative $\ominus F$ is, in general, also quite different from the pointwise one $-F$ even if $D_{\ominus F} = D_{-F}$.

Example 2.7 *If* X *is a group, then* $\ominus\Delta_X = \Delta_X$ *, but* $-\Delta_X = \Delta_X$ *if and only if* $-x = x$ *for all* $x \in X$*.*

However, in some very particular cases, the global negative of a relation may coincide with the pointwise one.

Example 2.8 *If* X *is a group such that* $-x = x$ *for all* $x \in X$*, then* $-F = F$ *and* $\ominus F = F$ *for any relation* F *on* X*.*

Concerning the images of sets under the relations $-F$, $\ominus F$, $F+G$ and $F \oplus G$, we can easily prove the following theorems.

Theorem 2.9 *If* F *is a relation on a set* X *to a group* Y*, then*

$$(-F)(A) = -F(A)$$

for all $A \subset X$*.*

Theorem 2.10 *If* F *is a relation on one group* X *to another* Y*, then*

$$(\ominus F)(A) = -F(-A)$$

for all $A \subset X$*.*

Proof. If $y \in (\ominus F)(A)$, then there exists $x \in A$ such that $y \in (\ominus F)(x)$, and thus $(x, y) \in \ominus F$. Hence, it follows that $(-x, -y) \in F$, and thus $-y \in F(-x)$. Thus, since $F(-x) \subset F(-A)$, we also have $y \in -F(-A)$. Therefore, $(\ominus F)(A) \subset -F(-A)$.

Now, by writing $\ominus F$ in place of F and $-A$ in place A, we can also see that

$$F(-A) = \big(\ominus(\ominus F)\big)(-A) \subset -(\ominus F)\big(-(-A)\big) = -(\ominus F)(A),$$

and thus $-F(-A) \subset (\ominus F)(A)$ is also true. □

Corollary 2.11 *If* F *is a relation on one group* X *to another* Y*, then*

(1) $\ominus F = F$ *if and only if* $F(-x) = -F(x)$ *for all* $x \in X$*;*

(2) $\ominus F = -F$ *if and only if* $F(-x) = F(x)$ *for all* $x \in X$*.*

Theorem 2.12 *If* F *and* G *are relations on a set* X *to groupoid* Y*, then*

$$(F+G)(A) \subset F(A) + G(A)$$

for all $A \subset X$*.*

Theorem 2.13 *If F and G are relations on one groupoid X to another Y, then*

$$F(A)+G(B) \subset (F \oplus G)(A+B)$$

for all $A, B \subset X$.

Proof If $w \in F(A)+G(B)$, then there exist $y \in F(A)$ and $z \in G(B)$ such that $w = y+z$. Moreover, there exist $a \in A$ and $b \in B$ such that $y \in F(a)$ and $z \in G(b)$, and thus $(a, y) \in F$ and $(b, z) \in G$. Hence, it follows that $(a+b, w) = (a+b, y+z) \in F \oplus G$, and thus $w \in (F \oplus G)(a+b)$. Thus, since $(F \oplus G)(a+b) \subset (F \oplus G)(A+B)$, we also have $w \in (F \oplus G)(A+B)$. $\square$

Corollary 2.14 *If F and G are relations on one groupoid X to another Y, and A is a subgroupoid of X, then*

$$F(A)+G(A) \subset (F \oplus G)(A).$$

3 Some further results on the global sums of relations

Theorem 3.1 *If F and G are relations on one groupoid X to another Y, then*

$$(F \oplus G)(A) = \bigcup_{u+v \in A} \big(F(u)+G(v)\big)$$

for all $A \subset X$.

Proof If $y \in (F \oplus G)(A)$, then there exists $x \in A$ such that $y \in (F \oplus G)(x)$, and hence $(x, y) \in F \oplus G$. Therefore, there exist $(u, z) \in F$ and $(v, w) \in G$ such that $(x, y) = (u+v, z+w)$. Hence, it follows that $z \in F(u)$ and $w \in G(v)$, and moreover $x = u+v$ and $y = z+w$. Therefore, $y \in F(u)+G(v)$, and hence $y \in \bigcup_{x=u+v} \big(F(u)+G(v)\big) \subset \bigcup_{u+v \in A} \big(F(u)+G(v)\big)$.

While, if $y \in \bigcup_{u+v \in A} \big(F(u)+G(v)\big)$, then there exist $u, v \in X$, with $x = u+v \in A$, such that $y \in F(u)+G(v)$. Therefore, there exist $z \in F(u)$ and $w \in G(v)$ such that $y = z+w$. Hence, it is clear that $(u, z) \in F$ and $(v, w) \in G$ such that $(x, y) = (u+v, z+w)$. Therefore, $(x, y) \in F \oplus G$, and hence $y \in (F \oplus G)(x) \subset (F \oplus G)(A)$. $\square$

Remark 3.2 *The $A = \{x\}$ particular case of the above theorem shows that, in contrast to the intersection convolution*

$$(F * G)(x) = \bigcap_{x=u+v} \big(F(u)+G(v)\big),$$

the union convolution of relations not be introduced since it coincides with the global sum.

Now, as a useful consequence of Theorem 3.1, we can also prove

Corollary 3.3 *If* F *and* G *are relations on one group* X *to a groupoid* Y, *then*

$$(F \oplus G)(A) = \bigcup_{v \in X} \big(F (A - v) + G (v) \big)$$

for all $A \subset X$.

Proof If $y \in (F \oplus G)(A)$, then by Theorem 3.1 $y \in \bigcup_{u+v \in A} \big(F (u) + G (v) \big)$. Therefore, there exist $u, v \in X$, with $x = u + v \in A$, such that $y \in F (u) + G (v)$. Hence, it follows that $y \in F (x - v) + G (v) \subset F (A - v) + G (v)$, and thus $y \in \bigcup_{v \in X} \big(F (A - v) + G (v) \big)$.

While, if $y \in \bigcup_{v \in X} \big(F (A - v) + G (v) \big)$, then there exists $v \in X$ such that $y \in F (A - v) + G (v)$. Therefore, there exists $x \in A$ such that $y \in F (x - v) + G (v)$. Hence, by defining $u = x - v$, we can see that $u \in X$ such that $x = u + v$ and $y \in F (u) + G (v) \big)$. Therefore, $y \in \bigcup_{x=u+v} \big(F (u) + G (v) \big) \subset \bigcup_{u+v \in A} \big(F (u) + G (v) \big)$, and hence by Theorem 3.1 $y \in (F \oplus G)(A)$. □

Moreover, as a simple reformulation of the above corollary we can also state

Corollary 3.4 *If* F *and* G *are relations on one group* X *to a groupoid* Y, *then*

$$(F \oplus G) (A) = \bigcup_{u \in X} \big(F (u) + G (- u + A) \big)$$

for all $A \subset X$.

Proof If $y \in (F \oplus G)(A)$, then by Corollary 3.3 $y \in \bigcup_{v \in X} \big(F (A - v) + G (v) \big)$. Therefore, there exists $v \in X$ such that $y \in F (A - v) + G (v)$. Thus, there exists $x \in A$ such that $y \in F (x - v) + G (v)$. Now, by defining $u = x - v$, we can see that $y \in F (u) + G (-u + x) \subset F (u) + G (-u + A)$. Therefore, $y \in \bigcup_{u \in X} \big(F (u) + G (- u + A) \big)$.

While, if $y \in \bigcup_{u \in X} \big(F (u) + G (- u + A) \big)$, then there exists $u \in X$ such that $y \in F (u) + G (- u + A)$. Therefore, there exists $x \in A$ such that $y \in F (u) + G (- u + x)$. Now, by defining $v = -u + x$, we can see that $y \in F (x - v) + G (v) \subset F (A - v) + G (v)$. Therefore, $y \in \bigcup_{v \in X} \big(F (A - v) + G (v) \big)$, and hence by Corollary 3.3 $y \in (F \oplus G)(A)$. □

Remark 3.5 *Now, by using the preceding results, one can also easily establish some properties of the images of sets under the relations*

$$F - G = F + (-G) \qquad \textit{and} \qquad F \ominus G = F \oplus (\ominus G).$$

References

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