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LAGRANGE SPACES WITH INDICATRICES AS CONSTANT MEAN CURVATURE SURFACES OR MINIMAL SURFACES

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Dedicated to Academician Radu Miron on the occasion of 75th birthday

Abstract

Constant mean curvature, particularly minimal, surfaces given by indicatrices of Lagrange and generalized Lagrange spaces are studied.

Introduction

The search of minimal surfaces in $\mathbb{R}^3$ is an old exciting problem([6]) and several methods appear in the study of these surfaces: Lie groups methods([3]), theory of integrable systems via the Weierstrass representation([4]).

Also, very interesting generalizations are fruitful: minimal surfaces in Riemannian manifolds([5]), constant mean curvature(CMC) surfaces([4], [9]).

In this paper we search CMC surfaces, particularly minimal surfaces, provided by indicatrices of Lagrange and generalized Lagrange manifolds of dimension three. Because the indicatrices of these spaces are given in implicit form, in first section the equations of CMC and minimal surfaces in implicit form are derived. In next two sections several equations of CMC and minimal indicatrices are obtained in the Lagrange and generalized Lagrange framework and the last section is devoted to examples.

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1 CMC surfaces equation for surfaces in implicit form

Let in $\mathbb{R}^3$ a surface S given in explicit form $S : u = u(x, y)$. The mean curvature function of S is:

$$H = \frac{u_{xx}\left(1+u_y^2\right) + u_{yy}\left(1+u_x^2\right) - 2u_x u_y u_{xy}}{2\left(1+u_x^2+u_y^2\right)^{\frac{3}{2}}}. \tag{1.1}$$

The equation H=constant is called *CMC surfaces equation* and in particular the equation $H = 0$ is called *minimal surfaces equation.*

If S is given in implicit form $S : F(x, y, z) = 0$ from relation $F(x, y, u(x, y)) = 0$ it results:

$$\begin{cases} u_x = -\frac{F_x}{F_z} \\ u_y = -\frac{F_y}{F_z} \end{cases} \tag{1.2}$$

$$\begin{cases} u_{xx} = \frac{F_x(F_{xz}F_z - F_{zz}F_x) - F_z(F_{xx}F_z - F_{xz}F_x)}{F_z^3} \\ u_{xy} = \frac{F_x(F_{yz}F_z - F_{zz}F_y) - F_z(F_{xy}F_z - F_{xz}F_y)}{F_z^3} \\ u_{yy} = \frac{F_y(F_{yz}F_z - F_{zz}F_y) - F_z(F_{yy}F_z - F_{yz}F_y)}{F_z^3} \end{cases}. \tag{1.3}$$

After a straightforward computation we get the CMC surfaces equation:

$$2\left(F_{xy}F_xF_y + F_{yz}F_yF_z + F_{zx}F_zF_x\right) -$$

$$-\left[F_{xx}\left(F_y^2+F_z^2\right) + F_{yy}\left(F_z^2+F_x^2\right) + F_{zz}\left(F_x^2+F_y^2\right)\right] = 2H\left(F_x^2+F_y^2+F_z^2\right)^{\frac{3}{2}} \tag{1.4}$$

and the minimal surfaces equation:

$$2(F_{xy}F_xF_y + F_{yz}F_yF_z + F_{zx}F_zF_x) = F_{xx}\left(F_y^2+F_z^2\right) +$$

$$F_{yy}\left(F_z^2+F_x^2\right) + F_{zz}\left(F_x^2+F_y^2\right). \tag{1.5}$$

2 CMC indicatrices in Lagrange geometry

Let us denote $T\mathbb{R}^3$ the tangent bundle of $\mathbb{R}^3$ for which we use the coordinates $(x, y) = \left(x^i, y^i\right)_{1\le i\le 3}$ with $x = \left(x^i\right)$ the coordinates in $\mathbb{R}^3$ and $y = \left(y^i\right)$ the coordinates in the fiber $T_x\mathbb{R}^3$. A function $f \in C^\infty\left(T\mathbb{R}^3\right)$ which does not depends of x i.e. $f = f(y)$ is called *Minkowskian function*. A tensor field of (r, s)-type on $T\mathbb{R}^3$ with law of change, at a change of coordinates on $T\mathbb{R}^3$, exactly as a tensor field of (r, s)-type on $\mathbb{R}^3$ is called *d-tensor field* of (r, s)-type.

After [8] a smooth Lagrangian $L : T\mathbb{R}^3 \to \mathbb{R}$ is called *regular* if the matrix $g = (g_{ij})_{1\le i,j\le 3}, g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L$, is of rank 3 i.e. $\det g \neq 0$ where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

The pair $(\mathbb{R}^3, L)$ is called then *Lagrange space* and the d-tensor field $g = (g_{ij})$ is called *the Lagrange metric*.

For every $x \in \mathbb{R}^3$ we have *the indicatrix* of L, $I_x = \{y \in T_x\mathbb{R}^3; L(x,y) = 1\}$ which appears as a surface defined by $F(y) = L(x,y) - 1$, x being fixed. Using the last relations of previous section it results that I_x is CMC surface if:

$$\begin{gathered} 2\left(F_{12}F_1F_2 + F_{23}F_2F_3 + F_{31}F_3F_1\right) - \\ -\left[F_{11}\left(F_2^2 + F_3^2\right) + F_{22}\left(F_3^2 + F_1^2\right) + F_{33}\left(F_1^2 + F_2^2\right)\right] = \\ = 2H_x\left(F_1^2 + F_2^2 + F_3^2\right)^{\frac{3}{2}} \end{gathered} \tag{2.1}$$

and I_x is minimal surface if:

$$\begin{gathered} 2\left(F_{12}F_1F_2 + F_{23}F_2F_3 + F_{31}F_3F_1\right) = \\ = F_{11}\left(F_2^2 + F_3^2\right) + F_{22}\left(F_3^2 + F_1^2\right) + F_{33}\left(F_1^2 + F_2^2\right) \end{gathered} \tag{2.2}$$

where $F_i = \dot{\partial}_i L$ and $F_{ij} = \dot{\partial}_i\dot{\partial}_j L$. From $F_{ij} = 2g_{ij}$ it follows:

Proposition 2.1 (i) *CMC indicatrices are given by*:

$$\begin{gathered} 2\left(g_{12}F_1F_2 + g_{23}F_2F_3 + g_{31}F_3F_1\right) - \\ -\left[g_{11}\left(F_2^2 + F_3^2\right) + g_{22}\left(F_3^2 + F_1^2\right) + g_{33}\left(F_1^2 + F_2^2\right)\right] = \\ = H_x\left(F_1^2 + F_2^2 + F_3^2\right)^{\frac{3}{2}} \end{gathered} \tag{2.3}$$

(ii) *minimal indicatrices are given by*:

$$\begin{gathered} 2\left(g_{12}F_1F_2 + g_{23}F_2F_3 + g_{31}F_3F_1\right) = \\ = g_{11}\left(F_2^2 + F_3^3\right) + g_{22}\left(F_3^2 + F_1^2\right) + g_{33}\left(F_1^2 + F_2^2\right). \end{gathered} \tag{2.4}$$

Let us remark that for a Minkowski Lagrangian if there exists a CMC (minimal) indicatrix then all indicatrices are CMC (minimal) surfaces.

A particular important case is that of a r-homogeneous Lagrangian i.e. $L(x, \lambda y) = \lambda^r L(x,y)$ for every $\lambda \in \mathbb{R}$.

Proposition 2.2 *If L is r-homogeneous with $r \neq 1$ then*:
(i) *CMC indicatrices are given by*:

$$\begin{gathered} 2\left(g_{12}g_{1a}g_{2b} + g_{23}g_{2a}g_{3b} + g_{31}g_{3a}g_{1b}\right)y^ay^b - \{g_{11}\left[\left(g_{2a}y^a\right)^2 + \left(g_{3a}y^a\right)^2\right] + \\ +g_{22}\left[\left(g_{3a}y^a\right)^2 + \left(g_{1a}y^a\right)^2\right] + g_{33}\left[\left(g_{1a}y^a\right)^2 + \left(g_{2a}y^a\right)^2\right]\} = \end{gathered}$$

$$= \frac{2H_x}{r-1}\left[(g_{1a}y^a)^2 + (g_{2a}y^a)^2 + (g_{3a}y^a)^2\right]^{\frac{3}{2}} \tag{2.5}$$

(ii) *minimal indicatrices are given by*:

$$2\left(g_{12}g_{1a}g_{2b} + g_{23}g_{2a}g_{3b} + g_{31}g_{3a}g_{1b}\right)y^a y^b =$$

$$= g_{11}\left[(g_{2a}y^a)^2 + (g_{3a}y^a)^2\right] + g_{22}\left[(g_{3a}y^a)^2 + (g_{1a}y^a)^2\right] +$$

$$+ g_{33}\left[(g_{1a}y^a)^2 + (g_{2a}y^a)^2\right]. \tag{2.6}$$

Proof From Euler relation $\dot{\partial}_i \, Ly^i = rL$ applying $\dot{\partial}_j$ we have $2g_{ij}y^i + F_j = rF_j$ which means that:

$$F_j = \frac{2}{r-1} g_{ia} y^a \tag{2.7}$$

and substituting this relation in (2.3) and (2.4) we get (2.5) and (2.6). □

The most important case is $r = 2$ for:

(i) *Riemann spaces* when $g = (g_{ij}(x))$ is a Riemannian metric and L is the kinetic energy of g i.e. $L = g_{ij}y^i y^j$

(ii) *Finsler spaces* when $g = (g_{ij}(x, y))$ is a Finsler metric([8]) and $L = g_{ij}y^i y^j$.

Proposition 2.3 *For Finsler, particularly Riemann, spaces*:

(i) *the CMC indicatrices are given by*:

$$2\left(g_{12}g_{1a}g_{2b} + g_{23}g_{2a}g_{3b} + g_{31}g_{3a}g_{1b}\right)y^a y^b - -\{g_{11}\left[(g_{2a}y^a)^2 + (g_{3a}y^a)^2\right] +$$

$$+ g_{22}\left[(g_{3a}y^a)^2 + (g_{1a}y^a)^2\right] + g_{33}\left[(g_{1a}y^a)^2 + (g_{2a}y^a)^2\right]\} =$$

$$= 2H_x\left[(g_{1a}y^a)^2 + (g_{2a}y^a)^2 + (g_{3a}y^a)^2\right]^{\frac{3}{2}} \tag{2.8}$$

(ii) *the minimal indicatrices are given by* (2.6).

Returning to the general case of proposition 2.1 because the matrix $g = (g_{ij})$ is symmetric let us suppose that this matrix is diagonal: $g_{12} = g_{32} = g_{31} = 0$. Let us call *diagonal Lagrange space* this type of Lagrange spaces.

Proposition 2.4 A) *In a diagonal Lagrange space*:

(i) *the CMC indicatrices are given by*:

$$g_{11}\left(F_2^2 + F_3^2\right) + g_{22}\left(F_3^2 + F_1^2\right) + g_{33}\left(F_1^2 + F_2^2\right) = -H_x\left(F_1^2 + F_2^2 + F_3^2\right)^{\frac{3}{2}} \tag{2.9}$$

(ii) *the minimal indicatrices are given by*:

$$g_{11}\left(F_2^2 + F_3^2\right) + g_{22}\left(F_3^2 + F_1^2\right) + g_{33}\left(F_1^2 + F_2^2\right) = 0. \tag{2.10}$$

If the diagonal Lagrange metric is positive definite i.e. $g_{ii} > 0, 1 \leq i \leq 3$, *it results that there are no minimal indicatrices.*
B) *In a diagonal* r*-homogeneous Lagrange space*:
(i) *the CMC indicatrices are given by*:

$$g_{11}\left[\left(g_{22}y^2\right)^2+\left(g_{33}y^3\right)^2\right]+g_{22}\left[\left(g_{33}y^3\right)^2+\left(g_{1a}y^1\right)^2\right]+$$

$$+g_{33}\left[\left(g_{11}y^1\right)^2+\left(g_{22}y^2\right)^2\right]=\frac{2H_x}{1-r}\left[\left(g_{11}y^1\right)^2+\left(g_{22}y^2\right)^2+\left(g_{33}y^3\right)^2\right]^{\frac{3}{2}} \tag{2.11}$$

(ii) *the minimal indicatrices are given by*:

$$g_{11}\left[\left(g_{22}y^2\right)^2+\left(g_{33}y^3\right)^2\right]+g_{22}\left[\left(g_{33}y^3\right)^2+\left(g_{11}y^1\right)^2\right]+$$

$$+g_{33}\left[\left(g_{11}y^1\right)^2+\left(g_{22}y^2\right)^2\right]=0 \tag{2.12}$$

C) *In a diagonal Finsler, particularly Riemann, space*:
(i) *the CMC indicatrices are given by*:

$$g_{11}\left[\left(g_{22}y^2\right)^2+\left(g_{33}y^3\right)^2\right]+g_{22}\left[\left(g_{33}y^3\right)^2+\left(g_{11}y^1\right)^2\right]+$$

$$+g_{33}\left[\left(g_{11}y^1\right)^2+\left(g_{22}y^2\right)^2\right]=-2H_x\left[\left(g_{11}y^1\right)^2+\left(g_{22}y^2\right)^2+\left(g_{33}y^3\right)^2\right]^{\frac{3}{2}} \tag{2.13}$$

(ii) *the minimal indicatrices are given by* (2.12).

Example 2.5(**The Euclidean case**) Let $g_{ij} = \delta_{ij}$ be the usual Euclidean metric of $\mathbb{R}^3$ which is a diagonal Riemann metric. The relation (2.13) becomes:

$$2\left[\left(y^1\right)^2+\left(y^2\right)^2+\left(y^3\right)^2\right]=-2H_x\left[\left(y^1\right)^2+\left(y^2\right)^2+\left(y^3\right)^2\right]^{\frac{3}{2}}.$$

In this case $L = \left(y^1\right)^2 + \left(y^2\right)^2 + \left(y^3\right)^2$ and thus the equation of I_x is $\left(y^1\right)^2 + \left(y^2\right)^2 + \left(y^3\right)^2 = 1$ and then for every $x \in \mathbb{R}^3$:
(i) the only CMC indicatrix is the unit sphere S^2 with $H_x = -1$,
(ii) there are not minimal indicatrices.

3 CMC indicatrices in generalized Lagrange spaces

A d-tensor field of $(0, 2)$-type on $T\mathbb{R}^3$, denoted $g = (g_{ij}(x, y))$, is called *generalized Lagrange metric* (*GL-metric*, on short) if the following properties hold([8]):

(i) symmetry, $g_{ij} = g_{ji}$
(ii) nondegeneracy: $\det(g_{ij}) \neq 0$
(iii) the signature of quadratic form $g(\xi) = g_{ij}\xi^i\xi^j, \xi = (\xi^i) \in \mathbb{R}^3$, is constant.

The function $\mathcal{E}(g) = g_{ij}y^iy^j$ is called *the absolute energy* of the given GL-metric.

Definition 3.1([8]) The GL-metric is called *weak regular* if $\mathcal{E}(g)$ is a regular Lagrangian.

It follows that for a weak regular GL-metric the d-tensor field of $(0,2)$-type:

$$g^*_{ij} = \frac{1}{2}\,\dot{\partial}_i\dot{\partial}_j\,\mathcal{E}(g) \tag{3.1}$$

is a Lagrange metric and then we can associate the indicatrix:

$$I_x = \{(x,y) \in T\mathbb{R}^3; \mathcal{E}(g)(x,y) = 1\}.$$

Applying proposition 2.1 we get:

Proposition 3.2 *For a weak regular GL-metric*:
(i) *the CMC indicatrices are given by*:

$$\begin{aligned}
&2\left[g^*_{12}\,\dot{\partial}_1\,\mathcal{E}(g)\,\dot{\partial}_2\,\mathcal{E}(g) + g^*_{23}\,\dot{\partial}_2\,\mathcal{E}(g)\,\dot{\partial}_3\,\mathcal{E}(g) + g^*_{31}\,\dot{\partial}_3\,\mathcal{E}(g)\,\dot{\partial}_1\,\mathcal{E}(g)\right] - \\
&-\{g^*_{11}\left[\left(\dot{\partial}_2\,\mathcal{E}(g)\right)^2 + \left(\dot{\partial}_3\,\mathcal{E}(g)\right)^2\right] + g^*_{22}\left[\left(\dot{\partial}_3\,\mathcal{E}(g)\right)^2 + \left(\dot{\partial}_1\,\mathcal{E}(g)\right)^2\right] + \\
&+g^*_{33}\left[\left(\dot{\partial}_1\,\mathcal{E}(g)\right)^2 + \left(\dot{\partial}_2\,\mathcal{E}(g)\right)^2\right]\} = \\
&= H_x\left[\left(\dot{\partial}_1\,\mathcal{E}(g)\right)^2 + \left(\dot{\partial}_2\,\mathcal{E}(g)\right)^2 + \left(\dot{\partial}_3\,\mathcal{E}(g)\right)^2\right]^{\frac{3}{2}}
\end{aligned} \tag{3.2}$$

(ii) *the minimal indicatrices are given by*:

$$\begin{aligned}
&2\left[g^*_{12}\,\dot{\partial}_1\,\mathcal{E}(g)\,\dot{\partial}_2\,\mathcal{E}(g) + g^*_{23}\,\dot{\partial}_2\,\mathcal{E}(g)\,\dot{\partial}_3\,\mathcal{E}(g) + g^*_{31}\,\dot{\partial}_3\,\mathcal{E}(g)\,\dot{\partial}_1\,\mathcal{E}(g)\right] = \\
&= g^*_{11}\left[\left(\dot{\partial}_2\,\mathcal{E}(g)\right)^2 + \left(\dot{\partial}_3\,\mathcal{E}(g)\right)^2\right] + g^*_{22}\left[\left(\dot{\partial}_3\,\mathcal{E}(g)\right)^2 + \left(\dot{\partial}_1\,\mathcal{E}(g)\right)^2\right] + \\
&+g^*_{33}\left[\left(\dot{\partial}_1\,\mathcal{E}(g)\right)^2 + \left(\dot{\partial}_2\,\mathcal{E}(g)\right)^2\right].
\end{aligned} \tag{3.3}$$

A straightforward computation gives:

$$\begin{cases}
g^*_{ij} = g_{ij} + \left(\dot{\partial}_i\dot{\partial}_j\,g_{ab}\right)y^ay^b + \left(\dot{\partial}_i\,g_{ja} + \dot{\partial}_j\,g_{ia}\right)y^a \\
\dot{\partial}_i\,\mathcal{E}(g) = \left(\dot{\partial}_i\,g_{ab}\right)y^ay^b + 2g_{ia}y^a
\end{cases}. \tag{3.4}$$

The above formulae become more simple in the following case:

Definition 3.3([8]) A weak regular GL-metric is called *regular* if:

$$\dot{\partial}_i \, \mathcal{E}(g) = 2g_{ij}y^j. \tag{3.5}$$

It results([8]):

$$g^*_{ij} = g_{ij} + \left(\dot{\partial}_j \, g_{ik}\right) y^k \tag{3.6}$$

and then:

Proposition 3.4 *For a regular GL-metric*:
(i) *the CMC indicatrices are given by*:

$$\begin{gathered}
2\{\left[g_{12} + \left(\dot{\partial}_2 \, g_{1k}\right) y^k\right] g_{1a}g_{2b} + \left[g_{23} + \left(\dot{\partial}_3 \, g_{2k}\right) y^k\right] g_{2a}g_{3b} \\
+ \left[g_{31} + \left(\dot{\partial}_1 \, g_{3k}\right) y^k\right] g_{3a}g_{1b}\}y^a y^b - \{\left[g_{11} + \left(\dot{\partial}_1 \, g_{1k}\right) y^k\right] \left[(g_{2a}y^a)^2 + (g_{3a}y^a)^2\right] + \\
+ \left[g_{22} + \left(\dot{\partial}_2 \, g_{2k}\right) y^k\right] \left[(g_{3a}y^a)^2 + (g_{1a}y^a)^2\right] + \\
+ \left[g_{33} + \left(\dot{\partial}_3 \, g_{3k}\right) y^k\right] \left[(g_{1a}y^a)^2 + (g_{2a}y^a)^2\right]\} = \\
2H_x \left[(g_{1a}y^a)^2 + (g_{2a}y^a)^2 + (g_{3a}y^a)^2\right]^{\frac{3}{2}}
\end{gathered} \tag{3.7}$$

(ii) *the minimal indicatrices are given by*:

$$\begin{gathered}
2\{\left[g_{12} + \left(\dot{\partial}_2 \, g_{1k}\right) y^k\right] g_{1a}g_{2b} + \left[g_{23} + \left(\dot{\partial}_3 \, g_{2k}\right) y^k\right] g_{2a}g_{3b} + \\
\left[g_{31} + \left(\dot{\partial}_1 \, g_{3k}\right) y^k\right] g_{3a}g_{1b}\}y^a y^b = \left[g_{11} + \left(\dot{\partial}_1 \, g_{1k}\right) y^k\right] \left[(g_{2a}y^a)^2 + (g_{3a}y^a)^2\right] + \\
+ \left[g_{22} + \left(\dot{\partial}_2 \, g_{2k}\right) y^k\right] \left[(g_{3a}y^a)^2 + (g_{1a}y^a)^2\right] + \\
+ \left[g_{33} + \left(\dot{\partial}_3 \, g_{3k}\right) y^k\right] \left[(g_{1a}y^a)^2 + (g_{2a}y^a)^2\right].
\end{gathered} \tag{3.8}$$

Another approach in the regular case is provided by homogeneity. By multiplication of (3.5) with y^i we have:

$$\dot{\partial}_i \, \mathcal{E}(g) \, y^i = 2g_{ij}y^i y^j = 2\mathcal{E}(g) \tag{3.9}$$

which means that $\mathcal{E}(g)$ is 2-homogeneous i.e. $\mathcal{E}(g)$ is a Finslerian function. Then we apply proposition 2.3:

Proposition 3.5 *For a regular GL-metric*:

(i) *the CMC indicatrices are given by*:

$$2\left(g^*_{12}g^*_{1a}g^*_{2b}+g^*_{23}g^*_{2a}g^*_{3b}+g^*_{31}g^*_{3a}g^*_{1b}\right)y^a y^b-$$
$$-\{g^*_{11}\left[\left(g^*_{2a}y^a\right)^2+\left(g^*_{3a}y^a\right)^2\right]+g^*_{22}\left[\left(g^*_{3a}y^a\right)^2+\left(g^*_{1a}y^a\right)^2\right]+$$
$$+g^*_{33}\left[\left(g^*_{1a}y^a\right)^2+\left(g^*_{2a}y^a\right)^2\right]\}=$$
$$=2H_x\left[\left(g^*_{1a}y^a\right)^2+\left(g^*_{2a}y^a\right)^2+\left(g^*_{3a}y^a\right)^2\right]^{\frac{3}{2}} \tag{3.10}$$

(ii) *the minimal indicatrices are given by*:

$$2\left(g^*_{12}g^*_{2a}g^*_{3b}+g^*_{23}g^*_{2a}g^*_{3b}+g^*_{31}g^*_{3a}g^*_{1b}\right)y^a y^b=$$
$$=g^*_{11}\left[\left(g^*_{2a}y^a\right)^2+\left(g^*_{3a}y^a\right)^2\right]+g^*_{22}\left[\left(g^*_{3a}y^a\right)^2+\left(g^*_{1a}y^a\right)^2\right]+$$
$$+g^*_{33}\left[\left(g^*_{1a}y^a\right)^2+\left(g^*_{2a}y^a\right)^2\right] \tag{3.11}$$

where for g^*_{ij} we use the relation (3.6).

4 Beil metrics as examples

Let $\widetilde{g}=\left(\widetilde{g}_{ij}\left(x,y\right)\right)$ be a Finsler metric and $B=B^i\left(x,y\right)\dot{\partial}_i$ a d-vector field for which we denote $B_i=\widetilde{g}_{ij}B^j$ and $B_0=B_iy^i$. Let also $a,b\in C^\infty\left(T\mathbb{R}^3\right)$. In [1] and [2] the following GL-metric is studied:

$$g_{ij}=a\widetilde{g}_{ij}+bB_iB_j. \tag{4.1}$$

These GL-metrics, called *Beil metrics*, are not Lagrange metrics. From:

$$\mathcal{E}\left(g\right)=a\mathcal{E}\left(\widetilde{g}\right)+b\left(B_0\right)^2 \tag{4.2}$$

we get:

$$\dot{\partial}_i\,\mathcal{E}\left(g\right)=\left(\dot{\partial}_i\,a\right)\mathcal{E}\left(\widetilde{g}\right)+a\left(\dot{\partial}_i\,\mathcal{E}\left(\widetilde{g}\right)\right)+\left(\dot{\partial}_i\,b\right)\left(B_0\right)^2+2bB_0\left(\dot{\partial}_i\,B_0\right) \tag{4.3}$$

$$2g^*_{ij}=2a\widetilde{g}_{ij}+\dot{\partial}_i\dot{\partial}_j\,a\mathcal{E}\left(\widetilde{g}\right)+\dot{\partial}_i\,a\,\dot{\partial}_j\,\mathcal{E}\left(\widetilde{g}\right)+\dot{\partial}_j\,a\,\dot{\partial}_i\,\mathcal{E}\left(\widetilde{g}\right)+\dot{\partial}_i\dot{\partial}_j\,b\left(B_0\right)^2+$$
$$2B_0\left(\dot{\partial}_i\,b\,\dot{\partial}_j\,B_0+\dot{\partial}_j\,b\,\dot{\partial}_i\,B_0+b\,\dot{\partial}_i\dot{\partial}_j\,B_0\right)+2b\,\dot{\partial}_i\,B_0\,\dot{\partial}_j\,B_0. \tag{4.4}$$

I) On $T_0\mathbb{R}^3=T\mathbb{R}^3\backslash\{\text{null section}\}$ let:

$$a=\frac{1}{2},b=\frac{1}{2\|y\|^2_F} \tag{4.5}$$

where $\|,\|_F$ is the norm induced by the Finsler metric $\widetilde{g}$ i.e. $\|y\|_F^2 = E(\widetilde{g}) = \widetilde{g}_{ij}y^i y^j$. Let $B = y^i \dot{\partial}_i$ be *the Liouville vector field*, it results $B_i = \widetilde{g}_{ij}y^j \overset{denoted}{=} \widetilde{y}_i$. The associated Beil metric is:

$$g = \frac{1}{2\|y\|_F^2}\begin{pmatrix} (\widetilde{y}_1)^2 + \widetilde{g}_{11}\|y\|_F^2 & \widetilde{y}_1\widetilde{y}_2 + \widetilde{g}_{12}\|y\|_F^2 & \widetilde{y}_1\widetilde{y}_3 + \widetilde{g}_{13}\|y\|_F^2 \\ \widetilde{y}_1\widetilde{y}_2 + \widetilde{g}_{12}\|y\|_F^2 & (\widetilde{y}_2)^2 + \widetilde{g}_{22}\|y\|_F^2 & \widetilde{y}_2\widetilde{y}_3 + \widetilde{g}_{23}\|y\|_F^2 \\ \widetilde{y}_1\widetilde{y}_3 + \widetilde{g}_{13}\|y\|_F^2 & \widetilde{y}_2\widetilde{y}_3 + \widetilde{g}_{23}\|y\|_F^2 & (\widetilde{y}_3)^2 + \widetilde{g}_{33}\|y\|_F^2 \end{pmatrix}. \tag{4.6}$$

Thus:

$$\mathcal{E}(g) = \|y\|_F^2 = \mathcal{E}(\widetilde{g}) \tag{4.7}$$

which is 2-homogeneous and then a Finsler function. It results that the Beil metric is regular GL-metric with $g^*_{ij} = \widetilde{g}_{ij}$ and then the CMC(minimal) indicatrices of Beil metric are exactly the CMC(minimal) indicatrices of Finsler metric $\widetilde{g}$.

II) (**Miron-Tavakol metrics**) For $a = \exp(2\sigma), b = 0$ with $\sigma \in C^\infty(T\mathbb{R}^3)$ and $\widetilde{g} = \widetilde{g}(x)$ a Riemannian metric we have the so-called *Miron-Tavakol metrics*([7]):

$$g_{ij}(x,y) = e^{2\sigma(x,y)}\widetilde{g}_{ij}(x) \tag{4.8}$$

for which:

$$\dot{\partial}_i\, \mathcal{E}(g) = 2\left(g_{ia}y^a + \left(\dot{\partial}_i\, \sigma\right) g_{ab}y^a y^b\right) \tag{4.9}$$

$$g^*_{ij} = g_{ij} + \left(\dot{\partial}_i\dot{\partial}_j\, \sigma + 2\,\dot{\partial}_i\, \sigma\, \dot{\partial}_j\, \sigma\right) g_{ab}y^a y^b + 2\left(g_{ja}\,\dot{\partial}_i\, \sigma + g_{ia}\,\dot{\partial}_j\, \sigma\right). \tag{4.10}$$

Particular cases:

II1. ([7, p. 219]) $\sigma = \frac{1}{2}\mathcal{E}(\widetilde{g}) = \frac{1}{2}\widetilde{g}_{ij}y^i y^j$

$$\dot{\partial}_i\, \mathcal{E}(g) = 2e^{\mathcal{E}(\widetilde{g})}\left(1 + \mathcal{E}(\widetilde{g})\right)\widetilde{g}_{ia}y^a \tag{4.11}$$

$$g^*_{ij} = g_{ij} + \left(\widetilde{g}_{ij} + 2\widetilde{g}_{ia}\widetilde{g}_{jb}y^a y^b\right) g_{uv}y^u y^v + 2\left(g_{ja}\widetilde{g}_{iu}y^u + g_{ia}\widetilde{g}_{ia}\right) y^a. \tag{4.12}$$

II2. $\sigma = \gamma_i(x)\, y^i$ with $\gamma_i \in C^\infty(\mathbb{R}^3), 1 \leq i \leq 3$

$$\dot{\partial}_i\, \mathcal{E}(g) = 2\left(g_{ia}y^a + \gamma_i\mathcal{E}(g)\right) \tag{4.13}$$

$$g^*_{ij} = g_{ij} + 2\gamma_i\gamma_j\mathcal{E}(g) + 2\left(\gamma_i g_{ja} + \gamma_j g_{ia}\right) y^a. \tag{4.14}$$

References

[1] Anastasiei, M., Shimada, H., *Beil metrics associated to a Finsler space*, Balkan J. Geom. Appl., 3(1998), no. 2, 1-16.

[2] Anastasiei, M., Shimada, H., *Deformations of Finsler metrics*, in Antonelli, P. L.(Ed.)-*Finslerian geometries. A meeting of minds*, Kluwer Academic Publishers, FTPH no. 109, 2000, 53-65.

[3] Bîlă, N., *Lie groups applications to minimal surfaces PDE*, Differential Geometry-Dynamical Systems, 1(1999), no. 1, 1-9.

[4] Hélein, F., *Constant mean curvature surfaces, harmonic maps and integrable systems* (Notes taken by Roger Moser), Birkhäuser, Boston, 2001.

[5] Ji, M., Wang, G. Y., *Minimal surfaces in Riemannian manifolds*, Memoirs AMS, 104(1993), no. 495.

[6] Lucas, J., Barbosa, M., Gervasio Colares, A., *Minimal surfaces in* $\mathbb{R}^3$, Springer, Berlin, 1986.

[7] Miron, R., Tavakol, R. K., Bălan, V., Roxburgh, I., *Geometry of space-time and generalized Lagrange gauge theory*, Publ. Math. Debrecen, 42(1993), no. 3-4, 215-224.

[8] Miron, R., Anastasiei, M., *The geometry of Lagrange spaces: theory and applications*, Kluwer Academic Publishers, FTPH no. 59, 1994.

[9] Wente, H. C., *Constant mean curvature immersions of Enneper type*, Memoirs AMS, 100(1992), no. 478.

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