An. Şt. Univ. Ovidius Constanţa Vol. 10(1), 2002, 49–62

SYZYGIETIC PROPERTIES OF A MODULE AND TORSION FREENESS OF ITS SYMMETRIC POWERS

Vittoria Bonanzinga and Gaetana Restuccia

Abstract

Let E be a finitely generated R-module of finite projective dimension. We establish necessary and sufficient conditions for the q-torsion freeness of the symmetric powers $Sym_t(E)$, $(t \geq 1)$. In projective dimension > 1, we study the connection between the acyclicity of the complex $\mathcal{Z}(E)$ of a module E and the condition $\mathfrak{F}_0$ on E.

Introduction

Let R be a commutative noetherian ring with unit and let E be a finitely generated R-module. Let $Sym_R(E) = S(E) = \bigoplus_{t \geq 0} Sym_t(E)$ be the symmetric algebra of E. It is well-known that if R is an integral domain, $Sym_R(E)$ is hardly ever an integral domain itself.

It is so if and only if each of the symmetric powers $Sym_t(E)$ is a torsion free R-module [6]. If $E = I$, an ideal of R, and (R, m) is local then $Sym_R(m)$ is an integral domain if and only if R is regular ([6]).

We say that an ideal I is of linear type ([13]) if the canonical epimorphism $Sym_R(I) \to \mathfrak{R}(I) \to 0$, where $\mathfrak{R}(I) = \bigoplus_{t \geq 0} I^t$ is the Rees algebra of I, is an isomorphism, and $Sym_R(I)$ is an integral domain if and only if R is an integral domain.

If E is a module of finite presentation $R^m \longrightarrow R^n \longrightarrow E \longrightarrow 0$ then the torsion freeness of the symmetric powers of E is connected with some conditions of finiteness for the depth of the Fitting ideals $F_k(E)$ of E, $e+1 \leq k \leq n$

Key Words: Approximation complex, Cohen-Macaulay ring, symmetric algebra
Mathematical Reviews subject classification: 13D25, 13H10

when the module E admits rank $e > 0$. Consequently, we can deduce theoretic properties of the symmetric algebra $Sym_R(E)$ by the syzygetic properties of the module E.

It is interesting to investigate the q-torsion freeness of the symmetric powers of E, by using acyclicity criteria for canonical complexes associated to the symmetric algebra of E.

In projective dimension 1, the basic result of Avramov [2] solves the problem completely, in the sense, that, for each t, $Sym_t(E)$ is q-torsion free if and only if E is $\mathfrak{F}_q{:=}\quad depth(F_k(E)) \geqslant k - e + q,\ e+1 \leq k \leq n$.

In section 1, we consider modules of finite projective dimension and we establish necessary and sufficient conditions for the q-torsion freeness of the symmetric powers $Sym_t(E)$ $(t \geqslant 1)$.

In section 2, we examine the relation between the acyclicity of the $\mathcal{Z}(E)$-complex of a module E and the condition $\mathfrak{F}_0$ on E, when the $\mathcal{Z}(E)$-complex coincides with the Koszul complex of the immersion $0 \longrightarrow L \longrightarrow R^n$, with $0 \longrightarrow L \longrightarrow R^n \longrightarrow E$ a finite presentation of E and L not necessarily free.

1

We consider a module E of finite projective dimension with the following free resolution

$$\mathbf{F}.:\qquad 0 \longrightarrow F_s \xrightarrow{f_s} F_{s-1} \xrightarrow{f_{s-1}} \dots \xrightarrow{f_1} F_0 \longrightarrow E \longrightarrow 0 \tag{1}$$

with F_i, free R-modules.

By theorem 2.1 [11] we can associate to E a canonical complex $\mathcal{S}_i(\mathbf{F}.)$. The goal of this section is to use this canonical complex in order to study when the symmetric powers of E are q-torsion free $(q \geqslant 1)$.

We recall that a module E is called q-torsion free if every R-regular sequence of lenght q is also E-regular.

Proposition 1 *Let E be a module of finite projective dimension over a noetherian ring R and let q be an integer. The following are equivalent:*

1. *E is q-torsion free.*
2. *For every prime ideal $\wp$ of R, $depth(E_\wp) \geq \min(q, depth(R_\wp))$*
3. *E is a q-th syzygy.*

Proof. See [1], [2]. ■

Remark 1 *In general, for an arbitrary module E, we have only the implications $3) \Rightarrow 2) \Rightarrow 1)$. We have also the equivalence for arbitrary modules if R is a normal domain and $q \leq 2$ ([10], prop. 1). In this case and in finite projective dimension, the 2-torsion free modules are the reflexive ones.*

Let E be a module generated by n elements. For the next theorem we switch from determinantal ideals to Fitting invariants $F_k(E) = I_{n-k+1}(f_1)$, where $I_{n-k+1}(f_1)$ denotes the ideal generated by the $n-k+1$-sized minors of f_1.

We say that a module E has rank e if $E \otimes_R Q(R)$ is a free $Q(R)$-module of rank e, where $Q(R)$ the total quotient ring of R.

Theorem 2 *Let E be a module of projective dimension 2 over a noetherian ring R, of rank r and with resolution*

$$\mathbf{F}. : \qquad 0 \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \longrightarrow E \longrightarrow 0 \tag{2}$$

where F_2, F_1, F_0 are free R-modules of rank p, m, and n, respectively. Suppose that E is $2(i-1)+q$-torsion free, ($q \geq 1$), $i \geq 2$; $i!$ is invertible in R. Then we have:

1. *$Sym_i(E)$ is q-torsion free*
2. *depth $F_k(E) \geq k - r + q$, $r+1 \leq k \leq n$.*

Proof. It suffices to observe that the complex $\mathcal{S}_i(\mathbf{F}.)$ associated to E, of theorem 2.1 [11], is acyclic, and by Corollary [14] $Sym_i(E)$ is q-torsion free, hence 1).

Since E is $2(i-1)+q-$torsion free, there exists an exact sequence

$$0 \longrightarrow E \longrightarrow G_1 \longrightarrow \cdots \longrightarrow G_{2(i-1)+q}$$

with G_j-free of finite type.

Then the sequence (2) is extendable to the right by an exact sequence in this way:

$$0 \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \longrightarrow E \longrightarrow G_1 \longrightarrow \cdots \longrightarrow G_{2(i-1)+q}.$$

By Buchsbaum-Eisenbud criterion we must necessarily have $depth(I_{r_1}(f_1)) \geq 2(i-1)+q+1$, where $r_1 = rank\,\mathrm{Im}\,f_1 = m-p$. But $I_k(f_1) \supseteq I_{r_1}(f_1)$, $\forall k \leq r_1$, then $depth(I_k(f_1)) \geq depth(I_{r_1}(f_1))$, $\forall k \leq r_1$.

In particular $depth(I_{r_1-i+1}(f_1)) \geq 2i+q-1 > i+q-1 \geq i+q$; if we put $r_1 - i + 1 = k$, then

$$depth(I_k(f_1)) > r_1 - k + q + 1$$

hence 2). ■

Theorem 3 *Let E be a module of projective dimension 2 over a noetherian ring R, of rank r and with resolution (2), with depth $I_p(f_2) \geq m-p-1$. Let q be an integer, $q \geq 1$ and let i be an integer ≥ 2; such that $2i < m-p$; $i!$ is invertible in R. We suppose that:*

$$depth\ F_k(E) \geq k - r + q, \qquad r+1 \leq k \leq n.$$

Then $Sym_i(E)$ is $q-torsion\ free$.

Proof. It follows by [1.9, 11] that

$$pd_R(Sym_i(E)) \leq 2i \qquad \forall i > 0.$$

Then, if $\wp$ is a prime ideal such that $depth(R_\wp) \geq 2i+q$, we have:

$$\begin{aligned} depth(Sym_i(E)_\wp) &= depth(R_\wp) - pd_{R_\wp}(Sym_i(E)_\wp) \geq \\ &\geq 2i + q - pd_{R_\wp}(Sym_i(E)_\wp) \geq 2i + q - 2i = q. \end{aligned}$$

Now, we consider a prime ideal $\wp$ such that

$$depth(R_\wp) < 2i + q.$$

By hypothesis, $depth(I_1(f_1)) \geq (m-p)+q > 2i+q$ and we have that $I_1(f_1) \nsubseteq \wp$. It follows that the module $E_\wp$ admits a free resolution over $R_\wp$

$$0 \longrightarrow F_2' \longrightarrow F_1' \xrightarrow{f_1'} F_0' \longrightarrow E_\wp \longrightarrow 0$$

with $F_0' = R_\wp^n = R_\wp^{n-1} \oplus R_\wp$, $F_1' = R_\wp^m = R_\wp^{m-1} \oplus R_\wp$ Moreover, since $I_{k-1}(f_1') = I_k(f_1)_\wp$, we have the inequalities

$$depth(I_{k-1}(f_1')) \geq (m-1)-p-(k-1)+1+q, \qquad \text{for } 1 \leq k-1 \leq (m-1)-p$$

We can then apply the induction on m. In fact, if $m = 0$, E is free and the result is trivial. By induction, then we can conclude that

$$depth\ (Sym_i(E)_\wp) \geq \min(q, depth(R_\wp)).$$

■

Corollary 4 *Let E be a module of projective dimension 2 over a noetherian ring R, $2(i-1)+q$-torsion free of rank r and with resolution (2). Let q be an integer ≥ 1 and let i be an integer ≥ 2; such that $2i < m-p$; $i!$ is invertible in R. Then the following conditions are equivalent:*

1. *depth* $(F_k(E)) \geq k-r+q, \qquad r+1 \leq k \leq n$
2. $Sym_i(E)$ *is q-torsion free.*

We say that a module E satisfies $\mathfrak{F}_q$ if $ht\, I_t(f_1) \geq rank\ f_1 - t + 1 + q,\ \ 1 \leq t \leq rank\ f_1$, where $q \geq 0$ is an integer and $rank\ f_1 = \sup\{t/I_t(f_1) \neq 0\}$

Theorem 5 *Let E be a module of finite type over R, of rank e, generated by n elements, of finite projective dimension. Let q be an integer, $q \geq 1$. We suppose that:*

1. $pd_R\,(Sym_R(E)) \leq n-e$
2. *E is $\mathfrak{F}_q$.*

Then $Sym_t(E)$ is $q-torsion\ free$, $\forall\ t > 0$.

Proof. We consider a presentation of the module E

$$R^m \overset{f_1=(a_{ji})}{\longrightarrow} R^n \longrightarrow E \longrightarrow 0 \tag{3}$$

and let

$$0 \longrightarrow L \longrightarrow R^n \longrightarrow E \longrightarrow 0.$$

We have $pd_R\,(Sym_i(E)) \leq n-e = \ell = rank\ L \ \ \forall\ i > 0$.

Let $\wp \in Spec(R)$ such that $depth\ (R_\wp) \geq \ell + q$. Then

$depth\ (Sym_i(E)_\wp) = depth\ (R_\wp) - pd_{R_\wp}(Sym_i(E)_\wp \geq \ell + q - \ell = q = \min(q, depth(R_\wp))$.

Let $\wp \in Spec(R)$ such that $depth\ (R_\wp) < \ell + q$.

By $\mathfrak{F}_q$, $depth(I_1(f_1)) \geq \ell + q$, $I_1(f_1) \nsubseteq \wp$ and localizing (3) at the prime ideal $\wp$, we have:

$$R_\wp^{m-1} \oplus R_\wp \overset{f_1' \oplus id}{\longrightarrow} R_\wp^{n-1} \oplus R_\wp \longrightarrow E_\wp \longrightarrow 0$$

and $E_\wp$ has the presentation:

$$0 \longrightarrow L_\wp^{'} \overset{f_1'}{\longrightarrow} R_\wp^{n-1} \longrightarrow E_\wp \longrightarrow 0$$

$L_\wp = L_\wp^{'} \oplus R_\wp$, $rank\ (L_\wp^{'}) = rank(L_\wp) - 1$ and $I_{k-1}(f_1^{'}) = I_k(f_1)_\wp \quad \forall k$.

We proceed by induction on ℓ.
If $\ell = 0$, E is free and the equivalences are trivial.
By induction hypothesis,

$$depth\ I_{k-1}(f_1^{'}) \geq \ell - k + q + 1 = (\ell - 1) - (k-1) + q + 1$$

where $1 \leq k-1 \leq \ell - 1$ and it follows
$depth\ (Sym_i(E)_{\wp}) \geq q \geq \min(q, depth(R_{\wp}))$.
■

Remark 2 *If R is a Cohen-Macaulay ring and E is a module of finite type over R of finite projective dimension 1, which is $\mathfrak{F}_0$(or, equivalently, if $Sym_t(E)$ is torsion free, $\forall\ t > 0$), the condition $pd_R(Sym_R(E)) \leq n - e$ is always verified. In fact, in this case $Sym_R(E)$ has a free finite resolution ([5], Prop. 4.1) of lenght $n - e$.*

Theorem 6 *Let E be an R-module of finite type, being R a Cohen-Macaulay ring, of rank e, of finite projective dimension. Let $q \geq 1$ be an integer and we suppose that:*

1. *For all prime ideal $\wp$ of R such that depth $(\wp R_{\wp}) > \ell$, depth $I_k(f_1)_{\wp} \geq \ell - k + q + 1$, $1 \leq k \leq \ell$;*
2. *$Sym_t(E)$ is q-torsion free, $\forall\ t > 0$.*

Then E is $\mathfrak{F}_q$.

Proof. We proceed by induction on $\ell = rank\ L$, $L = \ker(R^m \to E)$.
If $\ell = 0$, the assertion is trivial.
If $rank\ L = \ell$, by theorem 3.1 [9], we have:

$$depth\ I_1(f_1) \geq \ell.$$

Let $\wp \in Spec(R)$ such that $depth\ \wp R\wp \leq \ell$. Then $I_1(f_1) \nsubseteq \wp$ and, localizing (3) at the prime ideal $\wp$, we have:

$$R_{\wp}^{m-1} \oplus R_{\wp} \xrightarrow{f_1' \oplus id} R_{\wp}^{n-1} \oplus R_{\wp} \longrightarrow E_{\wp} \longrightarrow 0$$

and the presentation:

$$0 \longrightarrow L_{\wp}^{'} \xrightarrow{f_1'} R_{\wp}^{n-1} \longrightarrow E_{\wp} \longrightarrow 0$$

$rank\ (L_{\wp}^{'}) = rank(L_{\wp}) - 1$ and $I_{k-1}(f_1^{'}) = I_k(f_1)_{\wp}$. By induction hypothesis:

$$depth\ I_{k-1}(f_1^{'}) \geq (\ell - 1) - (k-1) + q + 1 = \ell - k + q + 1.$$

Since for every prime ideal $\wp \in Spec(R)$, $depth\ I_k(f_1)_\wp \geq \ell - k + q + 1$, then we can suppose that R is local and it results that $depth\ I_k(f_1) \geq \ell - k + q + 1$, $1 \leq k \leq \ell$, i. e., E is $\mathfrak{F}_q$. ■

2

Let E be a module of finite presentation:

$$R^m \overset{f_1=(a_{ji})}{\longrightarrow} R^n \overset{\varphi}{\longrightarrow} E \longrightarrow 0$$

and let

$$0 \longrightarrow L \longrightarrow R^n \longrightarrow E \longrightarrow 0.$$

If E has rank e, then we have

$$rank\ L = \ell = n - e.$$

We introduce the Z-complex, $\mathcal{Z}(E)$, of the module E that is a complex of graded $S = S(R^n)$-modules:

$\mathcal{Z}(E) := \quad 0 \longrightarrow Z_{n-e} \otimes S[-\ell] \longrightarrow \ldots \longrightarrow Z_1 \bigotimes S[-1] \longrightarrow S \longrightarrow Sym_R(E) \longrightarrow 0,$

where $Z_i = Z_i(E) = \ker(\overset{i}{\bigwedge} R^n \overset{\partial}{\longrightarrow} \overset{i-1}{\bigwedge} R^n \otimes E)$, $\partial(a_1 \wedge \ldots \wedge a_i) = \sum (-1)^j (a_1 \wedge \ldots \wedge \widehat{a_j} \wedge \ldots \wedge a_i) \otimes \varphi(a_j)$, $S[-j]_r = S_{r-j}$.

We consider the case when the complex $\mathcal{Z}(E)$ coincides with the Koszul complex of immersion $0 \longrightarrow L \longrightarrow R^n$

$S(L.) := \quad 0 \longrightarrow \overset{\ell}{\bigwedge} L \otimes S[-\ell] \longrightarrow \overset{\ell-1}{\bigwedge} L \otimes S[-\ell+1] \longrightarrow \ldots \longrightarrow L \bigotimes S[-1] \longrightarrow S \longrightarrow Sym_R(E) \longrightarrow 0.$

We need some preparatory lemmas

Lemma 7 *Let F be a module of finite type over R, not necessarily free. Let*

$$S_R(F) = \bigoplus_{i \geq 0} Sym_i(F) = \bigoplus_{i \geq 0} S_i(F)$$

be the symmetric algebra of F and $\bigwedge F = \bigoplus_{i \geq 0} \overset{i}{\bigwedge} F$ the exterior algebra of F. Then we have:

1. $S_i(F \bigoplus R) \cong \bigoplus_{j=0}^{i} S_j(F)$;

2. $\bigwedge^{i}(F\bigoplus R)\cong\bigwedge^{i}F\bigoplus\bigwedge^{i-1}F$

Proof. For F free see lemma 3 [12], lemma 2 [14].

1. If F is not free, we consider a presentation of F :

$$0\longrightarrow L\longrightarrow R^n\xrightarrow{f}F\longrightarrow 0$$

and the induced exact sequence:

$$0\longrightarrow J\longrightarrow S(R^n)\longrightarrow S(F)\longrightarrow 0\quad .$$

We have

$$0\longrightarrow L\longrightarrow R^n\oplus R\xrightarrow{f\oplus id}F\oplus R\longrightarrow 0,$$

and the induced exact sequence

$$0\longrightarrow J\longrightarrow S(R^n\oplus R)\longrightarrow S(F\oplus R)\longrightarrow 0$$

$$0\longrightarrow J_i\longrightarrow S_i(R^n\oplus R)\longrightarrow S_i(F\oplus R)\longrightarrow 0$$

where $J_i=J\cap S_i(R^n\oplus R)$. Since $S_i(R^n\oplus R)=\bigoplus_{j=0}^{i}S_j(R^n)$, $J_i=J\cap\left(\bigoplus_{j=0}^{i}S_j(R^n)\right)=\bigoplus_{j=0}^{i}J_j$. Hence $S_i(F\oplus R)\cong S_i(R^n\oplus R)/J_i\cong\bigoplus_{j=0}^{i}S_j(R^n)/J_j\cong\bigoplus_{j=0}^{i}S_j(F)$.

2. We consider the presentation $0\longrightarrow L\longrightarrow R^n\xrightarrow{f}F\longrightarrow 0$ and by $0\longrightarrow L\longrightarrow R^n\oplus R\xrightarrow{f\oplus id}F\oplus R\longrightarrow 0$ the induced exact sequence:

$$0\longrightarrow B\longrightarrow\bigwedge(R^n\oplus R)\longrightarrow\bigwedge(F\oplus R)\longrightarrow 0$$

$$0\longrightarrow B_i\longrightarrow\bigwedge^{i}(R^n\oplus R)\longrightarrow\bigwedge^{i}(F\oplus R)\longrightarrow 0$$

where $B_i=B\cap\bigwedge^i(R^n\oplus R)=B\cap(\bigwedge^i R^n\oplus\bigwedge^{i-1}R^n)$.

Hence $\bigwedge^i(F\oplus R)\cong\bigwedge^i(R^n\oplus R)/B_i\cong\bigwedge^i R^n\oplus\bigwedge^{i-1}R^n/B_i\cong\bigwedge^{i}F\bigoplus\bigwedge^{i-1}F$.
■

Lemma 8 *Let E be a module of finite type on R and let $K. := \ 0 \longrightarrow L \xrightarrow{f_1} F_0 \longrightarrow E \longrightarrow 0$; $\ K'. := \ 0 \longrightarrow L \oplus R \xrightarrow{f_1 \oplus id_R} F_0 \oplus R \longrightarrow E \longrightarrow 0$ be two presentations of E, L not necessarily free, F_0 free on R. Then the Koszul complexes $S(L.)$ and $S(L.')$ of immersion $0 \longrightarrow L \xrightarrow{f_1} F_0$, $0 \longrightarrow L \oplus R \xrightarrow{f_1 \oplus id_R} F_0 \oplus R \longrightarrow E$ have the same homology.*

Proof. If we call $S(L.)$ and $S(L.')$ the two Koszul complexes of immersions $0 \longrightarrow L \longrightarrow F_0$, $0 \longrightarrow L' \longrightarrow F_0 \oplus R$, $L' = L \oplus R$, in the component of degree $t > 0$, we have:

$$S_t(L.) := \qquad ... \longrightarrow \bigwedge^{i} L \otimes S_{t-i}(F_0) \longrightarrow \bigwedge^{i-1} L \otimes S_{t-i+1}(F_0) \longrightarrow ...$$

$$S_t(L.') := \qquad ... \longrightarrow \bigwedge^{i} L' \otimes S_{t-i}(F_0 \oplus R) \longrightarrow \bigwedge^{i-1} L' \otimes S_{t-i+1}(F_0 \oplus R) \longrightarrow ...$$

Let:

$$\left(S_t(L.)\right)_i = \bigwedge^{i} L \otimes S_{t-i}(F_0) \text{ and } \left(S_t(L.')\right)_i = \bigwedge^{i} L' \otimes S_{t-i}(F_0 \oplus R).$$

From lemma 7, we have:

$$\begin{aligned}
&\left(S_t(L'.)\right)_i = \bigwedge^{i}(L \oplus R) \otimes S_{t-i}(F_0 \oplus R) \cong \\
&\cong \left(\bigwedge^{i} L \oplus \bigwedge^{i-1} L\right) \otimes \left(S_{t-i}(F_0) \oplus S_{t-i-1}(F_0) \oplus ... \oplus F_0 \oplus R\right) = \\
&= \left(S_t(L.)_i \oplus S_{t-1}(L.)_i \oplus ... \oplus S_{i+1}(L.)_i \oplus S_i(L.)_i\right) \oplus \\
&\oplus \left(S_{t-1}(L.)_{i-1} \oplus S_{t-2}(L.)_{i-1} \oplus ... \oplus S_i(L.)_{i-1} \oplus S_{i-1}(L.)_{i-1}\right).
\end{aligned}$$

We proceed in a similar way to that contained in [12] or [14], prop. 3, and we can conclude that $S(L.)$ and $S(L.')$ have the same homology. ■

Theorem 9 *Let E be a torsion free module of finite type on R, Cohen-Macaulay ring of finite projective dimension, of rank e and with resolution:*

$$0 \longrightarrow R^p \longrightarrow ... \longrightarrow R^m \xrightarrow{f_1} R^n \longrightarrow E \longrightarrow 0. \tag{4}$$

We suppose that:

1. *E is $\mathfrak{F}_0$;*
2. *If $0 \longrightarrow L \longrightarrow R^n \longrightarrow E \longrightarrow 0$, $\ell = rank\ L$, the complex $S(L.)$ is exact $\Longleftrightarrow S(L.) \otimes R_\wp$ is exact, for all $\wp \in Spec(R)$ such that $depth(\wp R_\wp) < \ell$;*
3. *$\bigwedge^{r} L = (\bigwedge^{r} L)^{**}$ for $r < rank\ L$.*

Then the complex $\mathcal{Z}(E)$ is acyclic.

Proof. Since E is a torsion free module of finite projective dimension, then $E_\wp$ is a free $R_\wp$-module for every $\wp \in Spec(R)$ such that *depth* $(R_\wp) \leq 1$. By prop. 4.1, [5], we have $(\bigwedge^r L)^{**} \cong Z_r(E)$, $\forall r < rank E$.

Being L of finite projective dimension, $\left(\bigwedge^\ell L\right)^{**} = \det L = R$ and the $\mathcal{Z}(E)$-complex of E is the following:

$\mathcal{Z}(E) := \quad 0 \to R \otimes S_{t-\ell}(R^n) \to \bigwedge^{\ell-1} L \otimes S_{t-\ell+1}(R^n) \to ... \to$
$\to L \otimes S_{t-1}(R^n) \to S_t(R^n) \to S_t(E) \to 0$

We show, by induction on ℓ, that $\mathcal{Z}(E)$ is acyclic. If $\ell = 0$, E is free and $\mathcal{Z}(E)$ is acyclic.([8]).

We suppose that $\ell > 0$. Let $\wp \in Spec(R)$. Since $\mathfrak{F}_0$ implies *depth* $I_1(f_1) \geq \ell$, $I_1(f_1) \nsubseteq \wp$.

Localizing (4) at the prime ideal $\wp$,we have:

$$R_\wp^{m-1} \oplus R_\wp \xrightarrow{f_1' \oplus id} R_\wp^{n-1} \oplus R_\wp \longrightarrow E_\wp \longrightarrow 0$$

with the presentation:

$$0 \longrightarrow L'_\wp \xrightarrow{f_1'} R_\wp^{n-1} \longrightarrow E_\wp \longrightarrow 0$$

$L_\wp = L'_\wp \oplus R_\wp$, $rank(L'_\wp) = rank(L_\wp) - 1$ and $I_{k-1}(f_1') = I_k(f_1)_\wp$, where *depth* $I_k(f_1') \geq (\ell-1) - (k-1) + q + 1 = \ell - k + q + 1$, $1 \leq k-1 \leq \ell - 1$, the module $E_\wp$ is $\mathcal{F}_0$. We can suppose then R is local and we can conclude by lemma 8 and by the induction hypothesis. ■

Theorem 10 *Let E be a module of finite type over R, Cohen-Macaulay ring of finite projective dimension, of rank e and with resolution (4). We suppose that:*

1. *the complex $S(L.)$ is exact $\Longleftrightarrow$ $S(L.) \otimes R_\wp$ is exact for all $\wp \in Spec(R)$ such that depth $(\wp R_\wp) < \ell$;*
2. *E is free on the prime ideals $\wp$ such that depth $(\wp R_\wp) \leq 1$;*
3. *$\bigwedge^r L = \left(\bigwedge^r L\right)^{**}$, $\forall\ r < rank\ L$;*
4. *depth $I_1(f_1) \geq \ell$ and $\bigwedge^{\ell-1} L \cong R^\ell$;*
5. *$\mathcal{Z}(E)$ is acyclic.*

Then the module E is $\mathfrak{F}_0$.

Proof. We have the maps:

$$\bigwedge^r L \longrightarrow Z_r(E)$$

where the modules are reflexive.

Localizing at the prime ideals $\wp \in Spec(R)$ such that $depth\ (\wp R_\wp) \leq 1$, the modules $\bigwedge^r L$ and $Z_r(E)$ coincide. Hence :

$\mathcal{Z}(E) := 0 \to R \otimes S_{t-\ell}(R^n) \xrightarrow{d_\ell} R^\ell \otimes S_{t-\ell+1}(R^n) \xrightarrow{d_{\ell-1}} \bigwedge^{\ell-2} L \otimes S_{t-\ell+2}(R^n) \to$ ... $\to L \otimes S_{t-1}(R^n) \xrightarrow{d_1} S_t(R^n) \to Sym(E) \to 0$

Since the complex $\mathcal{Z}(E)$ is exact, $depth\ I_{\binom{\ell}{\ell}}(d_\ell) = depth\ I_1(f_1) \geq \ell$.

Let $\wp \in Spec(R)$ such that $depth\ (\wp R_\wp) < \ell$. Then $I_1(f_1) \nsubseteq \wp$. Localizing (4) at the prime ideal $\wp$, we obtain:

$$R_\wp^{m-1} \oplus R_\wp \xrightarrow{f_1 \oplus id_R} R_\wp^{n-1} \oplus R_\wp \longrightarrow E_\wp \longrightarrow 0$$

with the presentation:

$$0 \longrightarrow L'_\wp \xrightarrow{f_1} R_\wp^{n-1} \longrightarrow E_\wp \longrightarrow 0$$

$L_\wp = L'_\wp \oplus R_\wp$, $rank(L'_\wp) = rank(L_\wp) - 1$ and $I_{k-1}(f'_1) = I_k(f_1)_\wp$.

By lemma 8 , the complex $\mathcal{Z}(E) \otimes R_\wp$ is acyclic and

$$depth\ I_k(f_1)_\wp = depth\ I_{k-1}(f'_1) \geq (\ell - 1) - (k - 1) + 1 = \ell - k + 1,$$

$1 \leq k \leq \ell - 1$, by the induction hypothesis. Then we can suppose that R is local and we can conclude that E is $\mathfrak{F}_0$.

Example 1 *We exibit an example of a module that is high torsion free, containing a field and that fulfills the hypothesis in theorem 10. More precisely, let E be a q-torsion free module, where $q = (t-1)(\ell-1) + \ell$, $\ell = rankL$, $pdE = t$, $t > 1$, $\ell \geq 3$, and $d > t(\ell - 1) + 2$, where $d = depthR$. We verify all points of theorem 10.*

1. L is $(q+1)$-torsion free, then $\bigwedge^i L$ is $(\ell-1)$-torsion free, $\forall\ i \leq \ell - 1$.

If $\wp \in Spec(R)$ such that $depthR_\wp \geq \ell$, $depth\ (\bigwedge^i L)_\wp \geq \min\{\ell - 1, \ell\} = \ell - 1$. Then the complex $(S(L.))_\wp = S(L.) \otimes R_\wp$ is exact, by the criterion of Peskine-Szpiro, [14, theorem B].

Then we have to verify only that $(S(L.))_{\wp}$ is exact for every $\wp$, $depthR_{\wp} < \ell$.

2. Since E is torsion free and of finite projective dimension, then hypothesis 2 is verified.

3. L is of projective dimension $t-1$ and L is $q-1 = (t-1)(\ell-1)+(\ell-1)$-torsion free. Moreover L is $(t-1)i+(\ell-1)-$ torsion free, for every $i \leq \ell-1$, because $(t-1)i+(\ell-1) \leq (t-1)(\ell-1)+(\ell-1)$. By Corollary [14], this implies $\bigwedge^i L$ is $(\ell-1)$-torsion free, for every $i \leq \ell-1$ and since $\ell \geq 3$, $\bigwedge^i L$ is 2-torsion free, for every $i \leq \ell-1$. By Corollary [14] and since $\bigwedge^i L$ has a finite projective dimension, $\bigwedge^i L$ is a reflexive module for every i (Remark 1).

4. Since E is $(t-1)(\ell-1)+\ell$-torsion free, by the exact sequence

$$0 \longrightarrow F_t \longrightarrow \cdots \to F_1 \xrightarrow{f_1} F_0 \longrightarrow G_1 \longrightarrow \cdots \longrightarrow G_{(t-1)(\ell-1)+\ell},$$

$depthI_{r_1}(f_1) \geq (t-1)(\ell-1)+\ell+1$, $depthI_{r_1}(f_1) \geq t(\ell-1)+2 \geq \ell$, since $t > 1$. Since $I_{r_1}(f_1) \subset I_1(f_1)$, $depthI_1(f_1) \geq depthI_{r_1}(f_1) \geq \ell$.

We show that $\bigwedge^{\ell-1} L$ is a free module. Since $rank \bigwedge^{\ell-1} L = \ell$, we have to prove that $\bigwedge^{\ell-1} L \cong R^{\ell}$.

$L_{\ell-1}F$ is a resolution of $\bigwedge^{\ell-1} L$ of length $(t-1)(\ell-1)$, $depth \bigwedge^{\ell-1} L = depthR-pd \bigwedge^{\ell-1} L = d-(t-1)(\ell-1) \geq t(\ell-1)+2-(t-1)(\ell-1) = 2+\ell-1 = \ell+1$, by [4], syzygy theorem, $\bigwedge^{\ell-1} L$ is free and is isomorphic to R^{ℓ}.

5. $\mathcal{Z}(E)$ is acyclic. For every $i \leq \ell-1$, $depth \bigwedge^i L = depthR-(t-1)i \geq d-(t-1)(\ell-1) \geq$

$\geq d-(\ell-1)t > t(\ell-1)+2-(\ell-1)(t-1) =$

$= t(\ell-1)+2-t(\ell-1)+\ell = 2+\ell > i$ and the complex is acyclic by the acyclicity of Peskine-Szpiro, [3, lemma 3], [7, lemma 1.8].

References

[1] Auslander M., Bridger M., *Stable module theory*, Mem. Amer. Math. Soc. **94**, Providence, R. I. (1969).

[2] Avramov L., *Complete Intersections and Symmetric Algebras*, J. Algebra **72**, (1981), 248-263.

[3] D. A. Buchsbaum, D. Eisenbud, *What makes a Complex exact?*, J. Algebra, **25**, (1973), 259-268.

[4] Evans E. G., Griffith, *The syzygy problem*, Ann. of Math. (2), 114, (1981), 323-333.

[5] Herzog J. , Simis A., Vasconcelos W. V., *On the arithmetic and homology of algebras of linear type*, Trans. Amer. Math. Soc., **283**, N. 2, (1984), 661-683.

[6] Micali A., *Algèbres intégres et sans torsion*, Bull. Soc. Math. France, **94**, (1966), 5-14.

[7] C. Peskine, L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. I. H. E. S., **42**, (1972), 47-119.

[8] Restuccia G., *Formes linéaires et algèbres symétriques*, Bull. Sc. Math., **110** (1986), 391-410.

[9] Restuccia G., Ionescu C., *q-torsion freeness of symmetric powers,* Rend. Circolo Mat. di Palermo, serie II, XLVI (1997), 329-346.

[10] Samuel P., *Anneaux gradués factoriels et modules réfléxifs*, Bull. Soc. Math. France, **92** (1964) 234-249.

[11] Tchernev A. B., *Acyclicity of symmetric and exterior powers of complexes*, J. of Algebra **184**, (1996), 1113-1135.

[12] Utano R., *Moduli di dimensione proiettiva 1 e algebre simmetriche*, Università di Messina, (1987).

[13] Valla G., *On the symmetric and Rees algebras of an ideal*, Manuscripta Math. **30** (1980), 235-255.

[14] Weyman J., *Resolution of exterior and symmetric powers of a module*, J. of Algebra **58** (1979), 333-341.

Università di Reggio Calabria,
DIMET, Facoltà di Ingegneria,
via Graziella (Feo di Vito) 89100,
REGGIO CALABRIA
Italia
e-mail: bonanzin@ns.ing.unirc.it

Università di Messina,
Dipartimento di Matematica,
Contrada Papardo (salita Sperone), 98166
MESSINA
Italia
e-mail: grest@dipmat.unime.it