



# Necessary and sufficient conditions for the boundedness of the anisotropic Riesz potential in anisotropic modified Morrey spaces

Malik S. Dzhabrailov and Sevinc Z. Khaligova

## Abstract

We prove that the anisotropic fractional maximal operator  $M_{\alpha,\sigma}$  and the anisotropic Riesz potential operator  $I_{\alpha,\sigma}$ ,  $0 < \alpha < |\sigma|$  are bounded from the anisotropic modified Morrey space  $\tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$  to the weak anisotropic modified Morrey space  $W\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$  if and only if,  $\alpha/|\sigma| \leq 1 - 1/q \leq \alpha/(|\sigma|(1-b))$  and from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$  if and only if,  $\alpha/|\sigma| \leq 1/p - 1/q \leq \alpha/((1-b)|\sigma|)$ . In the limiting case  $\frac{|\sigma|(1-b)}{\alpha} \leq p \leq \frac{|\sigma|}{\alpha}$  we prove that the operator  $M_{\alpha,\sigma}$  is bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $L_\infty(\mathbb{R}^n)$  and the modified anisotropic Riesz potential operator  $\tilde{I}_{\alpha,\sigma}$  is bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $BMO_\sigma(\mathbb{R}^n)$ .

## 1 Introduction

For  $x \in \mathbb{R}^n$  and  $t > 0$ , let  $B(x, t)$  denote the open ball centered at  $x$  of radius  $t$  and  ${}^c B(x, t) = \mathbb{R}^n \setminus B(x, t)$ . Let  $0 \leq b \leq 1$ ,  $\sigma = (\sigma_1, \dots, \sigma_n)$  with  $\sigma_i > 0$  for  $i = 1, \dots, n$ ,  $|\sigma| = \sigma_1 + \dots + \sigma_n$  and  $t^\sigma x \equiv (t^{\sigma_1} x_1, \dots, t^{\sigma_n} x_n)$  for  $t > 0$ . For  $x \in \mathbb{R}^n$  and  $t > 0$ , let  $E_\sigma(x, t) = \prod_{i=1}^n (x_i - t^{\sigma_i}, x_i + t^{\sigma_i})$  denote the open parallelepiped centered at  $x$  of side length  $2t^{\sigma_i}$  for  $i = 1, \dots, n$ .

Key Words: anisotropic Riesz potential, anisotropic fractional maximal function, anisotropic modified Morrey space, anisotropic BMO space

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By [3, 10], the function  $F(x, \rho) = \sum_{i=1}^n x_i^2 \rho^{-2\sigma_i}$ , considered for any fixed  $x \in \mathbb{R}^n$ , is a decreasing one with respect to  $\rho > 0$  and the equation  $F(x, \rho) = 1$  is uniquely solvable. This unique solution will be denoted by  $\rho(x)$ . Define  $\rho(x) = \rho$  and  $\rho(0) = 0$ . It is a simple matter to check that  $\rho(x - y)$  defines a distance between any two points  $x, y \in \mathbb{R}^n$ . Thus  $\mathbb{R}^n$ , endowed with the metric  $\rho$ , defines a homogeneous metric space ([3, 5, 10]). Note that  $\rho(x)$  is equivalent to  $|x|_\sigma = \max_{1 \leq i \leq n} |x_i|^{\frac{1}{\sigma_i}}$ .

One of the most important variants of the anisotropic maximal function is the so-called anisotropic fractional maximal function defined by the formula

$$M_{\alpha, \sigma} f(x) = \sup_{t > 0} |E_\sigma(x, t)|^{-1 + \alpha/|\sigma|} \int_{E_\sigma(x, t)} |f(y)| dy, \quad 0 \leq \alpha < |\sigma|,$$

where  $|E_\sigma(x, t)| = 2^{nt} t^{|\sigma|}$  is the Lebesgue measure of the parallelepiped  $E_\sigma(x, t)$ .

It coincides with the anisotropic maximal function  $M_\sigma f \equiv M_{0, \sigma} f$  and is intimately related to the anisotropic Riesz potential operator

$$I_{\alpha, \sigma} f(x) = \int_{\mathbb{R}^n} \frac{f(y) dy}{|x - y|_\sigma^{|\sigma| - \alpha}}, \quad 0 < \alpha < |\sigma|.$$

If  $\sigma = \mathbf{1}$ , then  $M_\alpha \equiv M_{\alpha, \mathbf{1}}$  and  $I_\alpha \equiv I_{\alpha, \mathbf{1}}$  is the fractional maximal operator and Riesz potential, respectively. The operators  $M_\alpha$ ,  $M_{\alpha, \sigma}$ ,  $I_\alpha$  and  $I_{\alpha, \sigma}$  play important role in real and harmonic analysis (see, for example [4] and [35]).

**Definition 1.1.** Let  $0 \leq b \leq 1$ ,  $1 \leq p < \infty$  and  $[t]_1 = \min\{1, t\}$ . We denote by  $L_{p, b, \sigma}(\mathbb{R}^n)$  anisotropic Morrey space, and by  $\tilde{L}_{p, b, \sigma}(\mathbb{R}^n)$  the modified anisotropic Morrey space, the set of locally integrable functions  $f(x)$ ,  $x \in \mathbb{R}^n$ , with the finite norms

$$\|f\|_{L_{p, b, \sigma}} = \sup_{x \in \mathbb{R}^n, t > 0} \left( t^{-b|\sigma|} \int_{E_\sigma(x, t)} |f(y)|^p dy \right)^{1/p},$$

$$\|f\|_{\tilde{L}_{p, b, \sigma}} = \sup_{x \in \mathbb{R}^n, t > 0} \left( [t]_1^{-b|\sigma|} \int_{E_\sigma(x, t)} |f(y)|^p dy \right)^{1/p}$$

respectively.

**Remark 1.1.** Note that  $L_{p, 0, \sigma} = L_p(\mathbb{R}^n)$  and  $L_{p, 1, \sigma} = L_\infty(\mathbb{R}^n)$ . If  $b < 0$  or  $b > 1$ , then  $L_{p, b, \sigma} = \Theta$ , where  $\Theta$  is the set of all functions equivalent to 0 on  $\mathbb{R}^n$ . In the case  $\sigma \equiv \mathbf{1} = (1, \dots, 1)$  and  $b = \frac{\lambda}{n}$  we get the classical Morrey space  $L_{p, \lambda}(\mathbb{R}^n) = L_{p, \frac{\lambda}{n}, \mathbf{1}}(\mathbb{R}^n)$ ,  $0 \leq \lambda \leq n$ .

In the theory of partial differential equations, together with weighted  $L_{p,w}(\mathbb{R}^n)$  spaces, Morrey spaces  $L_{p,\lambda}(\mathbb{R}^n)$  play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [25]). Later, Morrey spaces found important applications to Navier-Stokes ([22], [36]) and Schrödinger ([26], [28], [29], [31], [32]) equations, elliptic problems with discontinuous coefficients ([8], [11]), and potential theory ([1], [2]). An exposition of the Morrey spaces can be found in the book [20].

The modified Morrey space  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  firstly was defined and investigated by [19] (see also [4]).

Note that

$$\begin{aligned} \tilde{L}_{p,0,\sigma}(\mathbb{R}^n) &= L_{p,0,\sigma}(\mathbb{R}^n) = L_p(\mathbb{R}^n), \\ \tilde{L}_{p,b,\sigma}(\mathbb{R}^n) &\subset_{\supset} L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \quad \text{and} \\ \max\{\|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p}\} &\leq \|f\|_{\tilde{L}_{p,b,\sigma}} \end{aligned} \tag{1.1}$$

and if  $b < 0$  or  $b > 1$ , then  $L_{p,b,\sigma}(\mathbb{R}^n) = \tilde{L}_{p,b,\sigma}(\mathbb{R}^n) = \Theta$ .

**Definition 1.2.** [6] Let  $1 \leq p < \infty$ ,  $0 \leq b \leq 1$ . We denote by  $WL_{p,b,\sigma}(\mathbb{R}^n)$  the weak anisotropic Morrey space and by  $W\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  the weak modified anisotropic Morrey space as the set of locally integrable functions  $f(x)$ ,  $x \in \mathbb{R}^n$  with finite norms

$$\|f\|_{WL_{p,b,\sigma}} = \sup_{r>0} r \sup_{x \in \mathbb{R}^n, t>0} \left( t^{-b|\sigma|} |\{y \in E_\sigma(x, t) : |f(y)| > r\}| \right)^{1/p},$$

$$\|f\|_{W\tilde{L}_{p,b,\sigma}} = \sup_{r>0} r \sup_{x \in \mathbb{R}^n, t>0} \left( [t]_1^{-b|\sigma|} |\{y \in E_\sigma(x, t) : |f(y)| > r\}| \right)^{1/p}$$

respectively.

Note that

$$\begin{aligned} WL_p(\mathbb{R}^n) &= WL_{p,0,\sigma}(\mathbb{R}^n) = W\tilde{L}_{p,0,\sigma}(\mathbb{R}^n), \\ L_{p,b,\sigma}(\mathbb{R}^n) &\subset WL_{p,b,\sigma}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{WL_{p,b,\sigma}} \leq \|f\|_{L_{p,b,\sigma}}, \\ \tilde{L}_{p,b,\sigma}(\mathbb{R}^n) &\subset W\tilde{L}_{p,b,\sigma}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{W\tilde{L}_{p,b,\sigma}} \leq \|f\|_{\tilde{L}_{p,b,\sigma}}. \end{aligned}$$

The anisotropic result by Hardy-Littlewood-Sobolev states that if  $1 < p < q < \infty$ , then  $I_{\alpha,\sigma}$  is bounded from  $L_p(\mathbb{R}^n)$  to  $L_q(\mathbb{R}^n)$  if and only if  $\alpha = |\sigma| \left( \frac{1}{p} - \frac{1}{q} \right)$  and for  $p = 1 < q < \infty$ ,  $I_{\alpha,\sigma}$  is bounded from  $L_1(\mathbb{R}^n)$  to  $WL_q(\mathbb{R}^n)$  if and only if  $\alpha = |\sigma| \left( 1 - \frac{1}{q} \right)$ . Spanne (see [33]) and Adams

[1] studied boundedness of the Riesz potential  $I_\alpha$  for  $0 < \alpha < n$  in Morrey spaces  $L_{p,\lambda}$ . Later on Chiarenza and Frasca [9] was reproved boundedness of the Riesz potential  $I_\alpha$  in these spaces. By more general results of Guliyev [13] (see also [14, 17]) one can obtain the following generalization of the results in [1, 9, 33] to the anisotropic case.

**Theorem A.** *Let  $0 < \alpha < |\sigma|$  and  $0 \leq b < 1$ ,  $1 \leq p < \frac{(1-b)|\sigma|}{\alpha}$ .*

1) *If  $1 < p < \frac{(1-b)|\sigma|}{\alpha}$ , then condition  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-b)|\sigma|}$  is necessary and sufficient for the boundedness of the operator  $I_{\alpha,\sigma}$  from  $L_{p,b,\sigma}(\mathbb{R}^n)$  to  $L_{q,b,\sigma}(\mathbb{R}^n)$ .*

2) *If  $p = 1$ , then condition  $1 - \frac{1}{q} = \frac{\alpha}{(1-b)|\sigma|}$  is necessary and sufficient for the boundedness of the operator  $I_{\alpha,\sigma}$  from  $L_{1,b,\sigma}(\mathbb{R}^n)$  to  $WL_{q,b,\sigma}(\mathbb{R}^n)$ .*

If  $\alpha = (1-b)|\sigma|(\frac{1}{p} - \frac{1}{q})$ , then  $b = 0$  and the statement of Theorem A reduces to the aforementioned anisotropic result by Hardy-Littlewood-Sobolev.

Recall that, for  $0 < \alpha < |\sigma|$ ,

$$M_{\alpha,\sigma}f(x) \leq 2^{n(\frac{\alpha}{|\sigma|}-1)} I_{\alpha,\sigma}(|f|)(x), \quad (1.2)$$

hence Theorem A also implies the boundedness of the fractional maximal operator  $M_{\alpha,\sigma}$ . It is known that the anisotropic maximal operator  $M_\sigma$  is also bounded from  $L_{p,b,\sigma}$  to  $L_{p,b,\sigma}$  for all  $1 < p < \infty$  and  $0 < b < 1$ , which isotropic case proved by F. Chiarenza and M. Frasca [9].

In this paper we study the fractional maximal integral and the Riesz potential in the modified Morrey space. In the case  $p = 1$  we prove that the operators  $M_{\alpha,\sigma}$  and  $I_{\alpha,\sigma}$  are bounded from  $\tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$  to  $W\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$  if and only if,  $\alpha/|\sigma| \leq 1 - 1/q \leq \alpha/((1-b)|\sigma|)$ . In the case  $1 < p < \frac{(1-b)|\sigma|}{\alpha}$  we prove that the operators  $M_{\alpha,\sigma}$  and  $I_{\alpha,\sigma}$  are bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$  if and only if,  $\alpha/|\sigma| \leq 1/p - 1/q \leq \alpha/((1-b)|\sigma|)$ . In the limiting case  $\frac{|\sigma|(1-b)}{\alpha} \leq p \leq \frac{|\sigma|}{\alpha}$  we prove that the operator  $M_{\alpha,\sigma}$  is bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $L_\infty(\mathbb{R}^n)$  and the modified anisotropic Riesz potential operator  $\tilde{I}_{\alpha,\sigma}$  is bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $BMO_\sigma(\mathbb{R}^n)$ .

The structure of the paper is as follows. In section 2 the boundedness of the anisotropic maximal operator in anisotropic modified Morrey space  $\tilde{L}_{p,b,\sigma}$  is proved. The main result of the paper is the Hardy-Littlewood-Sobolev inequality in anisotropic modified Morrey space for the anisotropic Riesz potential, established in section 3. In section 4 we prove that the operator  $I_{\alpha,\sigma}$  is bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $BMO_\sigma(\mathbb{R}^n)$  for  $\frac{|\sigma|(1-b)}{\alpha} \leq p \leq \frac{|\sigma|}{\alpha}$ .

By  $A \lesssim B$  we mean that  $A \leq CB$  with some positive constant  $C$  independent of appropriate quantities. If  $A \lesssim B$  and  $B \lesssim A$ , we write  $A \approx B$  and say that  $A$  and  $B$  are equivalent.

## 2 $\tilde{L}_{p,b,\sigma}$ -boundedness of the maximal operator

Define  $f_{t^\sigma}(x) =: f(t^\sigma x)$  and  $[t]_{1,+} = \max\{1, t\}$ . Then

$$\begin{aligned} \|f_{t^\sigma}\|_{L_p} &= t^{-\frac{|\sigma|}{p}} \|f\|_{L_p}, \\ \|f_{t^\sigma}\|_{L_{p,b,\sigma}} &= t^{-\frac{|\sigma|}{p}} \sup_{x \in \mathbb{R}^n, r > 0} \left( r^{-b|\sigma|} \int_{E_\sigma(t^\sigma x, tr)} |f(y)|^p dy \right)^{1/p} \\ &= t^{\frac{(b-1)|\sigma|}{p}} \|f\|_{L_{p,b,\sigma}}, \end{aligned}$$

and

$$\begin{aligned} \|f_{t^\sigma}\|_{\tilde{L}_{p,b,\sigma}} &= \sup_{x \in \mathbb{R}^n, r > 0} \left( [r]_1^{-b|\sigma|} \int_{E_\sigma(x,r)} |f_{t^\sigma}(y)|^p dy \right)^{1/p} \\ &= t^{-\frac{|\sigma|}{p}} \sup_{x \in \mathbb{R}^n, r > 0} \left( [r]_1^{-b|\sigma|} \int_{E_\sigma(t^\sigma x, tr)} |f(y)|^p dy \right)^{1/p} \\ &= t^{-\frac{|\sigma|}{p}} \sup_{r > 0} \left( \frac{[tr]_1}{[r]_1} \right)^{b|\sigma|/p} \sup_{x \in \mathbb{R}^n, r > 0} \left( [tr]_1^{-b|\sigma|} \int_{E_\sigma(t^\sigma x, tr)} |f(y)|^p dy \right)^{1/p} \\ &= t^{-\frac{|\sigma|}{p}} [t]_{1,+}^{\frac{b|\sigma|}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}}. \end{aligned} \tag{2.1}$$

In this section we study the  $\tilde{L}_{p,b,\sigma}$ -boundedness of the maximal operator  $M_\sigma$ .

**Lemma 2.1.** *Let  $1 \leq p < \infty$ ,  $0 \leq b \leq 1$ . Then*

$$\tilde{L}_{p,b,\sigma}(\mathbb{R}^n) = L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$$

and

$$\|f\|_{\tilde{L}_{p,b,\sigma}} = \max \left\{ \|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p} \right\}.$$

*Proof.* Let  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ . Then from (1.1) we have that  $f \in L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$  and  $\max \left\{ \|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p} \right\} \leq \|f\|_{\tilde{L}_{p,b,\sigma}}$ .

Let now  $f \in L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ . Then

$$\begin{aligned} \|f\|_{\tilde{L}_{p,b,\sigma}} &= \sup_{x \in \mathbb{R}^n, t > 0} \left( [t]_1^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p} \\ &= \max \left\{ \sup_{x \in \mathbb{R}^n, 0 < t \leq 1} \left( t^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p}, \sup_{x \in \mathbb{R}^n, t > 1} \left( \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p} \right\} \\ &\leq \max \left\{ \|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p} \right\}. \end{aligned}$$

Therefore,  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  and the embedding

$$L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \subset_{\gamma} \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$$

is valid.

Thus

$$\begin{aligned} \tilde{L}_{p,b,\sigma}(\mathbb{R}^n) &= L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \text{ and} \\ \|f\|_{\tilde{L}_{p,b,\sigma}} &= \max \left\{ \|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p} \right\}. \end{aligned}$$

□

Analogously proved the following statement.

**Lemma 2.2.** *Let  $1 \leq p < \infty$ ,  $0 \leq b \leq 1$ . Then*

$$W\tilde{L}_{p,b,\sigma}(\mathbb{R}^n) = WL_{p,b,\sigma}(\mathbb{R}^n) \cap WL_p(\mathbb{R}^n)$$

and

$$\|f\|_{W\tilde{L}_{p,b,\sigma}} = \max \left\{ \|f\|_{WL_{p,b,\sigma}}, \|f\|_{WL_p} \right\}.$$

To prove our main result in this section we need the following statement.

**Theorem 2.1.** [23] 1. *If  $f \in L_{1,b,\sigma}(\mathbb{R}^n)$ ,  $0 \leq b < 1$ , then  $M_\sigma f \in WL_{1,b,\sigma}(\mathbb{R}^n)$  and*

$$\|M_\sigma f\|_{WL_{1,b,\sigma}} \leq C_{b,\sigma} \|f\|_{L_{1,b,\sigma}},$$

where  $C_{b,\sigma}$  depends only on  $n$ ,  $b$  and  $\sigma$ .

2. *If  $f \in L_{p,b,\sigma}(\mathbb{R}^n)$ ,  $1 < p < \infty$ ,  $0 \leq b < 1$ , then  $M_\sigma f \in L_{p,b,\sigma}(\mathbb{R}^n)$  and*

$$\|M_\sigma f\|_{L_{p,b,\sigma}} \leq C_{p,b,\sigma} \|f\|_{L_{p,b,\sigma}},$$

where  $C_{p,b,\sigma}$  depends only on  $n$ ,  $p$ ,  $b$  and  $\sigma$ .

Our main theorem in this section is the following statement:

**Theorem 2.2.** 1. If  $f \in \tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ ,  $0 \leq b < 1$ , then  $M_\sigma f \in W\tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$  and

$$\|M_\sigma f\|_{W\tilde{L}_{1,b,\sigma}} \leq \bar{C}_{1,b,\sigma} \|f\|_{\tilde{L}_{1,b,\sigma}},$$

where  $C_{1,b,\sigma}$  depends only on  $b$  and  $\sigma$ .

2. If  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ ,  $1 < p < \infty$ ,  $0 \leq b < 1$ , then  $M_\sigma f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  and

$$\|M_\sigma f\|_{\tilde{L}_{p,b,\sigma}} \leq \bar{C}_{p,b,\sigma} \|f\|_{\tilde{L}_{p,b,\sigma}},$$

where  $C_{p,b,\sigma}$  depends only on  $p$ ,  $b$  and  $\sigma$ .

*Proof.* It is obvious that (see Lemmas 2.1 and 2.2)

$$\|M_\sigma f\|_{\tilde{L}_{p,b,\sigma}} = \max \left\{ \|M_\sigma f\|_{L_{p,b,\sigma}}, \|M_\sigma f\|_{L_p} \right\}$$

for  $1 < p < \infty$  and

$$\|M_\sigma f\|_{W\tilde{L}_{1,b,\sigma}} = \max \left\{ \|M_\sigma f\|_{WL_{1,b,\sigma}}, \|M_\sigma f\|_{WL_1} \right\}$$

for  $p = 1$ .

Let  $1 < p < \infty$ . By the boundedness of  $M_\sigma$  on  $L_p(\mathbb{R}^n)$  and from Theorem 2.1 we have

$$\|M_\sigma f\|_{\tilde{L}_{p,b,\sigma}} \leq \max \{C_{p,\sigma}, C_{p,b,\sigma}\} \|f\|_{\tilde{L}_{p,b,\sigma}}.$$

Let  $p = 1$ . By the boundedness of  $M_\sigma$  from  $L_1(\mathbb{R}^n)$  to  $WL_1(\mathbb{R}^n)$  and from Theorem 2.1 we have

$$\|M_\sigma f\|_{W\tilde{L}_{1,b,\sigma}} \leq \max \{C_{1,\sigma}, C_{1,b,\sigma}\} \|f\|_{\tilde{L}_{1,b,\sigma}}.$$

□

### 3 Hardy-Littlewood-Sobolev inequality in modified Morrey spaces

The following Hardy-Littlewood-Sobolev inequality in modified Morrey spaces is valid.

**Theorem 3.3.** Let  $0 < \alpha < |\sigma|$ ,  $0 \leq b < 1 - \frac{\alpha}{|\sigma|}$  and  $1 \leq p < \frac{(1-b)|\sigma|}{\alpha}$ .

1) If  $1 < p < \frac{(1-b)|\sigma|}{\alpha}$ , then condition  $\frac{\alpha}{|\sigma|} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{(1-b)|\sigma|}$  is necessary and sufficient for the boundedness of the operator  $I_{\alpha,\sigma}$  from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ .

2) If  $p = 1 < \frac{(1-b)|\sigma|}{\alpha}$ , then condition  $\frac{\alpha}{|\sigma|} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{(1-b)|\sigma|}$  is necessary and sufficient for the boundedness of the operator  $I_{\alpha,\sigma}$  from  $\tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$  to  $W\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ .

*Proof.* 1) *Sufficiency.* Let  $0 < \alpha < |\sigma|$ ,  $0 < b < 1 - \frac{\alpha}{|\sigma|}$ ,  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  and  $1 < p < \frac{(1-b)|\sigma|}{\alpha}$ . Then

$$I_{\alpha,\sigma}f(x) = \left( \int_{E_\sigma(x,t)} + \int_{\mathfrak{c}_{E_\sigma(x,t)}} \right) f(y)|x-y|_{\sigma}^{\alpha-|\sigma|} dy \equiv A(x,t) + C(x,t).$$

For  $A(x,t)$  we have

$$\begin{aligned} |A(x,t)| &\leq \int_{E_\sigma(x,t)} |x-y|_{\sigma}^{\alpha-|\sigma|} |f(y)| dy \\ &\leq \sum_{j=1}^{\infty} (2^{-j}t)^{\alpha-|\sigma|} \int_{E_\sigma(x,2^{-j+1}t) \setminus E_\sigma(x,2^{-j}t)} |f(y)| dy. \end{aligned}$$

Hence

$$|A(x,t)| \lesssim t^\alpha Mf(x). \quad (3.1)$$

In the second integral by the Hölder's inequality we have

$$\begin{aligned} |C(x,t)| &\leq \left( \int_{\mathfrak{c}_{E_\sigma(x,t)}} |x-y|_{\sigma}^{-\beta} |f(y)|^p dy \right)^{1/p} \\ &\quad \times \left( \int_{\mathfrak{c}_{E_\sigma(x,t)}} |x-y|_{\sigma}^{(\frac{\beta}{p} + \alpha - |\sigma|)p'} dy \right)^{1/p'} = J_1 \cdot J_2. \end{aligned}$$

For  $J_2$  we obtain

$$J_2 \lesssim \left( \int_t^\infty r^{|\sigma|-1 + (\frac{\beta}{p} + \alpha - |\sigma|)p'} dr \right)^{\frac{1}{p'}} \approx t^{\frac{\beta}{p} + \alpha - \frac{|\sigma|}{p}}, \quad (3.2)$$

where  $\beta < |\sigma| - \alpha p$ .



Let  $b|\sigma| < \beta < |\sigma| - \alpha p$ . For  $J_1$  we get

$$\begin{aligned}
 J_1 &= \left( \sum_{j=0}^{\infty} \int_{E_{\sigma}(x, 2^{j+1}t) \setminus E_{\sigma}(x, 2^j t)} |x-y|^{-\beta} |f(y)|^p dy \right)^{1/p} \\
 &\leq t^{-\frac{\beta}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}} \left( \sum_{j=0}^{\infty} 2^{-\beta j} [2^{j+1}t]_1^{b|\sigma|} \right)^{1/p} \\
 &= t^{-\frac{\beta}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}} \begin{cases} \left( 2^{b|\sigma|} t^{b|\sigma|} \sum_{j=0}^{\lfloor \log_2 \frac{1}{2t} \rfloor} 2^{(b|\sigma|-\beta)j} + \sum_{j=\lfloor \log_2 \frac{1}{2t} \rfloor + 1}^{\infty} 2^{-\beta j} \right)^{1/p}, & 0 < t < \frac{1}{2}, \\ \left( \sum_{j=0}^{\infty} 2^{-\beta j} \right)^{1/p}, & t \geq \frac{1}{2} \end{cases} \\
 &\approx t^{-\frac{\beta}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}} \begin{cases} \left( t^{b|\sigma|} + t^{\beta} \right)^{1/p}, & 0 < t < \frac{1}{2}, \\ 1, & t \geq \frac{1}{2} \end{cases} \\
 &\approx \|f\|_{\tilde{L}_{p,b,\sigma}} \begin{cases} t^{\frac{b|\sigma|-\beta}{p}}, & 0 < t < \frac{1}{2}, \\ t^{-\frac{\beta}{p}}, & t \geq \frac{1}{2} \end{cases} \\
 &= [2t]_1^{\frac{b|\sigma|}{p}} t^{-\frac{\beta}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}}. \tag{3.3}
 \end{aligned}$$

From (3.2) and (3.3) we have

$$|C(x, t)| \lesssim [t]_1^{\frac{b|\sigma|}{p}} t^{\alpha - \frac{|\sigma|}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}}. \tag{3.4}$$

Thus for all  $t > 0$  we get

$$\begin{aligned}
 |I_{\alpha,\sigma} f(x)| &\lesssim t^{\alpha} M_{\sigma} f(x) + [t]_1^{\frac{b|\sigma|}{p}} t^{\alpha - \frac{|\sigma|}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}} \\
 &\leq \min \left\{ t^{\alpha} M_{\sigma} f(x) + t^{\alpha - \frac{|\sigma|}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}}, t^{\alpha} M_{\sigma} f(x) + t^{\alpha - \frac{(1-b)|\sigma|}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}} \right\}.
 \end{aligned}$$

Minimizing with respect to  $t$ , at

$$t = \left[ (M_{\sigma} f(x))^{-1} \|f\|_{\tilde{L}_{p,b,\sigma}} \right]^{p/((1-b)|\sigma|)}$$

and

$$t = \left[ (M_{\sigma} f(x))^{-1} \|f\|_{\tilde{L}_{p,b,\sigma}} \right]^{p/|\sigma|}$$

we get

$$|I_{\alpha,\sigma} f(x)| \lesssim \min \left\{ \left( \frac{M_{\sigma} f(x)}{\|f\|_{\tilde{L}_{p,b,\sigma}}} \right)^{1 - \frac{p\alpha}{(1-b)|\sigma|}}, \left( \frac{M_{\sigma} f(x)}{\|f\|_{\tilde{L}_{p,b,\sigma}}} \right)^{1 - \frac{p\alpha}{|\sigma|}} \right\} \|f\|_{\tilde{L}_{p,b,\sigma}}.$$

Then

$$|I_{\alpha,\sigma}f(x)| \lesssim (M_\sigma f(x))^{p/q} \|f\|_{\tilde{L}_{p,b,\sigma}}^{1-p/q}.$$

Hence, by Theorem 2.2, we have

$$\begin{aligned} \int_{E_\sigma(x,t)} |I_{\alpha,\sigma}f(y)|^q dy &\lesssim \|f\|_{\tilde{L}_{p,b,\sigma}}^{q-p} \int_{E_\sigma(x,t)} (M_\sigma f(y))^p dy \\ &\lesssim [t]_1^{b|\sigma|} \|f\|_{\tilde{L}_{p,b,\sigma}}^q, \end{aligned}$$

which implies that  $I_{\alpha,\sigma}$  is bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ .

*Necessity.* Let  $1 < p < \frac{(1-b)|\sigma|}{\alpha}$ ,  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  and  $I_{\alpha,\sigma}$  bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ . Then from (2.1) we have

$$\|f_{t^\sigma}\|_{\tilde{L}_{p,b,\sigma}} = t^{-\frac{|\sigma|}{p}} [t]_{1,+}^{\frac{b|\sigma|}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}},$$

and

$$I_{\alpha,\sigma}f_{t^\sigma}(x) = t^{-\alpha} I_{\alpha,\sigma}f(t^\sigma x), \quad (3.5)$$

$$\begin{aligned} \|I_{\alpha,\sigma}f_{t^\sigma}\|_{\tilde{L}_{q,b,\sigma}} &= t^{-\alpha} \sup_{x \in \mathbb{R}^n, r > 0} \left( [r]_1^{-b|\sigma|} \int_{E_\sigma(x,r)} |I_{\alpha,\sigma}f(t^\sigma y)|^q dy \right)^{1/q} \\ &= t^{-\alpha - \frac{|\sigma|}{q}} \sup_{r > 0} \left( \frac{[tr]_1}{[r]_1} \right)^{b|\sigma|/q} \sup_{x \in \mathbb{R}^n, r > 0} \left( [tr]_1^{-b|\sigma|} \int_{E_\sigma(t^\sigma x, tr)} |I_{\alpha,\sigma}f(y)|^q dy \right)^{1/q} \\ &= t^{-\alpha - \frac{|\sigma|}{q}} [t]_{1,+}^{\frac{b|\sigma|}{q}} \|I_{\alpha,\sigma}f\|_{\tilde{L}_{q,b,\sigma}}. \end{aligned}$$

By the boundedness of  $I_{\alpha,\sigma}$  from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$

$$\begin{aligned} \|I_{\alpha,\sigma}f\|_{\tilde{L}_{q,b,\sigma}} &= t^{\alpha + \frac{|\sigma|}{q}} [t]_{1,+}^{-\frac{b|\sigma|}{q}} \|I_{\alpha,\sigma}f_{t^\sigma}\|_{\tilde{L}_{q,b,\sigma}} \\ &\leq t^{\alpha + \frac{|\sigma|}{q}} [t]_{1,+}^{-\frac{b|\sigma|}{q}} \|f_{t^\sigma}\|_{\tilde{L}_{p,b,\sigma}} \\ &= t^{\alpha + \frac{|\sigma|}{q} - \frac{|\sigma|}{p} - \frac{b|\sigma|}{q}} [t]_{1,+}^{\frac{b|\sigma|}{p} - \frac{b|\sigma|}{q}} \|f\|_{\tilde{L}_{p,b,\sigma}}, \end{aligned}$$

where  $C_{p,q,b,\sigma}$  depends only on  $p$ ,  $q$ ,  $b$  and  $\sigma$ .

If  $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{|\sigma|}$ , then in the case  $t \rightarrow 0$  we have  $\|I_{\alpha,\sigma}f\|_{\tilde{L}_{q,b,\sigma}} = 0$  for all  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ .

As well as if  $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{(1-b)|\sigma|}$ , then at  $t \rightarrow \infty$  we obtain  $\|I_{\alpha,\sigma}f\|_{\tilde{L}_{q,b,\sigma}} = 0$  for all  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ .

Therefore  $\frac{\alpha}{|\sigma|} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{(1-b)|\sigma|}$ .

2) *Sufficiency.* Let  $f \in \tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ . We have

$$\begin{aligned} & |\{y \in E_\sigma(x, t) : |I_{\alpha,\sigma} f(y)| > 2\beta\}| \\ & \leq |\{y \in E_\sigma(x, t) : |A(y, t)| > \beta\}| \\ & + |\{y \in E_\sigma(x, t) : |C(y, t)| > \beta\}|. \end{aligned}$$

Then

$$\begin{aligned} C(y, t) &= \sum_{j=0}^{\infty} \int_{E_\sigma(y, 2^{j+1}t) \setminus E_\sigma(y, 2^j t)} |f(z)| |y - z|^{\alpha - |\sigma|} dz \\ &\leq t^{\alpha - |\sigma|} \|f\|_{\tilde{L}_{1,b,\sigma}} \sum_{j=0}^{\infty} 2^{-(|\sigma| - \alpha)j} [2^{j+1}t]_1^{b|\sigma|} = t^{\alpha - |\sigma|} \|f\|_{\tilde{L}_{1,b,\sigma}} \\ &\times \begin{cases} 2^{b|\sigma|} t^{b|\sigma|} \sum_{j=0}^{\lfloor \log_2 \frac{1}{2t} \rfloor} 2^{(b|\sigma| - |\sigma| + \alpha)j} + \sum_{j=\lfloor \log_2 \frac{1}{2t} \rfloor + 1}^{\infty} 2^{-(|\sigma| - \alpha)j}, & 0 < t < \frac{1}{2}, \\ \sum_{j=0}^{\infty} 2^{-(|\sigma| - \alpha)j}, & t \geq \frac{1}{2} \end{cases} \\ &\approx t^{\alpha - |\sigma|} \|f\|_{\tilde{L}_{1,b,\sigma}} \begin{cases} t^{b|\sigma|} + t^{|\sigma| - \alpha}, & 0 < t < \frac{1}{2}, \\ 1, & t \geq \frac{1}{2} \end{cases} \\ &\approx \|f\|_{\tilde{L}_{1,b,\sigma}} \begin{cases} t^{b|\sigma| + \alpha - |\sigma|}, & 0 < t < \frac{1}{2}, \\ t^{\alpha - |\sigma|}, & t \geq \frac{1}{2} \end{cases} \\ &= [2t]_1^{b|\sigma|} t^{\alpha - |\sigma|} \|f\|_{\tilde{L}_{1,b,\sigma}}. \end{aligned}$$

Taking into account inequality (3.1) and Theorem 2.2, we have

$$\begin{aligned} & |\{y \in E_\sigma(x, t) : |A(y, t)| > \beta\}| \\ & \leq \left| \left\{ y \in E_\sigma(x, t) : Mf(y) > \frac{\beta}{C_1 t^\alpha} \right\} \right| \\ & \leq \frac{C_2 t^\alpha}{\beta} \cdot [t]_1^{b|\sigma|} \|f\|_{\tilde{L}_{1,b,\sigma}}, \end{aligned}$$

where  $C_2 = C_1 \cdot C_{1,b,\sigma}$  and thus if  $C_2 [2t]_1^{b|\sigma|} t^{\alpha - |\sigma|} \|f\|_{\tilde{L}_{1,b,\sigma}} = \beta$ , then  $|C(y, t)| \leq \beta$  and consequently,  $|\{y \in E_\sigma(x, t) : |C(y, t)| > \beta\}| = 0$ .

Then

$$\begin{aligned} |\{y \in E_\sigma(x, t) : |I_{\alpha,\sigma} f(y)| > 2\beta\}| &\lesssim \frac{1}{\beta} [t]_1^{b|\sigma|} t^\alpha \|f\|_{\tilde{L}_{1,b,\sigma}} \\ &\lesssim [t]_1^{b|\sigma|} \left( \frac{\|f\|_{\tilde{L}_{1,b,\sigma}}}{\beta} \right)^{\frac{(1-b)|\sigma|}{(1-b)|\sigma| - \alpha}}, \text{ if } 2t < 1 \end{aligned}$$

and

$$\begin{aligned} |\{y \in E_\sigma(x, t) : |I_{\alpha, \sigma} f(y)| > 2\beta\}| &\lesssim \frac{1}{\beta} [t]_1^{b|\sigma|} t^\alpha \|f\|_{\tilde{L}_{1, b, \sigma}} \\ &\lesssim [t]_1^{b|\sigma|} \left( \frac{\|f\|_{\tilde{L}_{1, b, \sigma}}}{\beta} \right)^{\frac{|\sigma|}{|\sigma| - \alpha}}, \text{ if } 2t \geq 1. \end{aligned}$$

Finally we have

$$\begin{aligned} &|\{y \in E_\sigma(x, t) : |I_{\alpha, \sigma} f(y)| > 2\beta\}| \\ &\lesssim [t]_1^{b|\sigma|} \min \left\{ \left( \frac{\|f\|_{\tilde{L}_{1, b, \sigma}}}{\beta} \right)^{\frac{(1-b)|\sigma|}{(1-b)|\sigma| - \alpha}}, \left( \frac{\|f\|_{\tilde{L}_{1, b, \sigma}}}{\beta} \right)^{\frac{|\sigma|}{|\sigma| - \alpha}} \right\} \\ &\leq [t]_1^{b|\sigma|} \left( \frac{1}{\beta} \|f\|_{\tilde{L}_{1, b, \sigma}} \right)^q. \end{aligned}$$

*Necessity.* Let  $I_{\alpha, \sigma}$  is bounded from  $\tilde{L}_{1, b, \sigma}(\mathbb{R}^n)$  to  $W\tilde{L}_{q, b, \sigma}(\mathbb{R}^n)$ . From (3.5) we have

$$\begin{aligned} \|I_{\alpha, \sigma} f t^\sigma\|_{W\tilde{L}_{q, b, \sigma}} &= \sup_{r>0} r \sup_{x \in \mathbb{R}^n, \tau > 0} \left( [t]_1^{-b|\sigma|} \int_{\{y \in E_\sigma(x, \tau) : |I_{\alpha, \sigma} f t^\sigma(y)| > r\}} dy \right)^{1/q} \\ &= \sup_{r>0} r \sup_{x \in \mathbb{R}^n, \tau > 0} \left( [t]_1^{-b|\sigma|} \int_{\{y \in E_\sigma(x, \tau) : |I_{\alpha, \sigma} f(t^\sigma y)| > r t^\alpha\}} dy \right)^{1/q} \\ &= t^{-\alpha - \frac{|\sigma|}{q}} \sup_{\tau > 0} \left( \frac{[t\tau]_1}{[\tau]_1} \right)^{b|\sigma|/q} \sup_{r>0} r t^\alpha \\ &\times \sup_{x \in \mathbb{R}^n, \tau > 0} \left( [t\tau]_1^{-b|\sigma|} \int_{\{y \in E_\sigma(t^\sigma x, t\tau) : |I_{\alpha, \sigma} f(y)| > r t^\alpha\}} dy \right)^{1/q} \\ &= t^{-\alpha - \frac{|\sigma|}{q}} [t]_{1,+}^{\frac{b|\sigma|}{q}} \|I_{\alpha, \sigma} f\|_{W\tilde{L}_{q, b, \sigma}}. \end{aligned}$$

By the boundedness of  $I_{\alpha, \sigma}$  from  $\tilde{L}_{1, b, \sigma}(\mathbb{R}^n)$  to  $W\tilde{L}_{q, b, \sigma}(\mathbb{R}^n)$  and from (2.1) we get

$$\begin{aligned} \|I_{\alpha, \sigma} f\|_{W\tilde{L}_{q, b, \sigma}} &= t^{\alpha + \frac{|\sigma|}{q}} [t]_{1,+}^{-\frac{b|\sigma|}{q}} \|I_{\alpha, \sigma} f t^\sigma\|_{W\tilde{L}_{q, b, \sigma}} \\ &\lesssim t^{\alpha + \frac{|\sigma|}{q}} [t]_{1,+}^{-\frac{b|\sigma|}{q}} \|f t^\sigma\|_{\tilde{L}_{1, b, \sigma}} \\ &\lesssim t^{\alpha + \frac{|\sigma|}{q} - |\sigma|} [t]_{1,+}^{b|\sigma| - \frac{b|\sigma|}{q}} \|f\|_{\tilde{L}_{1, b, \sigma}}. \end{aligned}$$

If  $1 < \frac{1}{q} + \frac{\alpha}{|\sigma|}$ , then in the case  $t \rightarrow 0$  we have  $\|I_{\alpha,\sigma}f\|_{W\tilde{L}_{q,b,\sigma}} = 0$  for all  $f \in \tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ .

Similarly, if  $1 > \frac{1}{q} + \frac{\alpha}{(1-b)|\sigma|}$ , then for  $t \rightarrow \infty$  we obtain  $\|I_{\alpha,\sigma}f\|_{W\tilde{L}_{q,b,\sigma}} = 0$  for all  $f \in \tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ .

Therefore  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-b)|\sigma|}$ .  $\square$

**Corollary 3.1.** *Let  $0 < \alpha < |\sigma|$ ,  $0 \leq b < 1 - \frac{\alpha}{|\sigma|}$  and  $1 \leq p \leq \frac{|\sigma|}{\alpha}$ .*

1) *If  $1 < p < \frac{(1-b)|\sigma|}{\alpha}$ , then condition  $\frac{\alpha}{|\sigma|} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{(1-b)|\sigma|}$  is necessary and sufficient for the boundedness of the operator  $M_{\alpha,\sigma}$  from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ .*

2) *If  $p = 1 < \frac{(1-b)|\sigma|}{\alpha}$ , then condition  $\frac{\alpha}{|\sigma|} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{(1-b)|\sigma|}$  is necessary and sufficient for the boundedness of the operator  $M_{\alpha,\sigma}$  from  $\tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$  to  $W\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ .*

3) *If  $\frac{(1-b)|\sigma|}{\alpha} \leq p \leq \frac{|\sigma|}{\alpha}$ , then the operator  $M_{\alpha,\sigma}$  is bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $L_\infty(\mathbb{R}^n)$ .*

*Proof.* Sufficiency of Corollary 3.1 follows from Theorem 3.3 and inequality (1.2).

*Necessity.* (1) Let  $M_{\alpha,\sigma}$  be bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$  for  $1 < p < \frac{(1-b)|\sigma|}{\alpha}$ . Then we have

$$M_{\alpha,\sigma}f_{t^\sigma}(x) = t^{-\alpha}M_{\alpha,\sigma}f(t^\sigma x),$$

and

$$\|M_{\alpha,\sigma}f_{t^\sigma}\|_{\tilde{L}_{q,b,\sigma}} = t^{-\alpha - \frac{|\sigma|}{q}} [t]_{1,+}^{\frac{b|\sigma|}{q}} \|M_{\alpha,\sigma}f\|_{\tilde{L}_{q,b,\sigma}}.$$

By the same argument in Theorem 3.3 we obtain  $\frac{\alpha}{|\sigma|} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{(1-b)|\sigma|}$ .

(2) Let  $M_{\alpha,\sigma}$  be bounded from  $\tilde{L}_{1,b,\sigma}(\mathbb{R}^n)$  to  $W\tilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ . Then

$$\|M_{\alpha,\sigma}f_{t^\sigma}\|_{W\tilde{L}_{q,b,\sigma}} = t^{-\alpha - \frac{|\sigma|}{q}} [t]_{1,+}^{\frac{b|\sigma|}{q}} \|M_{\alpha,\sigma}f\|_{W\tilde{L}_{q,b,\sigma}}.$$

Hence we obtain  $\frac{\alpha}{|\sigma|} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{(1-b)|\sigma|}$ .

(3) Let  $\frac{(1-b)|\sigma|}{\alpha} \leq p \leq \frac{|\sigma|}{\alpha}$ . Then by the Hölder's inequality we have

$$\begin{aligned}
\|M_{\alpha,\sigma}f\|_{L^\infty} &= 2^{-n} \sup_{x \in \mathbb{R}^n, t > 0} t^{\alpha-|\sigma|} \int_{E_\sigma(x,t)} |f(y)| dy \\
&\leq 2^{-\frac{n}{p}} \sup_{x \in \mathbb{R}^n, t > 0} t^{\alpha-\frac{|\sigma|}{p}} [t]_1^{\frac{b|\sigma|}{p}} \left( [t]_1^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p} \\
&\leq 2^{-\frac{n}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}} \sup_{t > 0} t^{\alpha-\frac{|\sigma|}{p}} [t]_1^{\frac{b|\sigma|}{p}} \\
&= 2^{-\frac{n}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}} \max \left\{ \sup_{0 < t \leq 1} t^{\alpha-\frac{|\sigma|(1-b)}{p}}, \sup_{t > 1} t^{\alpha-\frac{|\sigma|}{p}} \right\} \\
&= 2^{-\frac{n}{p}} \|f\|_{\tilde{L}_{p,b,\sigma}}.
\end{aligned}$$

□

#### 4 The modified anisotropic Riesz potential in the spaces $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$

The examples show that the anisotropic Riesz potential  $I_{\alpha,\sigma}$  are not defined for all functions  $f \in L_{p,b,\sigma}(\mathbb{R}^n)$ ,  $0 \leq b < 1 - \frac{\alpha}{|\sigma|}$ , if  $p \geq \frac{|\sigma|(1-b)}{\alpha}$ , and  $I_{\alpha,\sigma}$  are not defined for all functions  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ ,  $0 \leq b < 1 - \frac{\alpha}{|\sigma|}$ , if  $p \geq \frac{|\sigma|(1-b)}{\alpha}$ .

We consider the modified Riesz potential

$$\tilde{I}_{\alpha,\sigma}f(x) = \int_{\mathbb{R}^n} \left( |x-y|_\sigma^{\alpha-|\sigma|} - |y|_\sigma^{\alpha-|\sigma|} \chi_{\mathbb{C}_{E_\sigma(0,1)}}(y) \right) f(y) dy.$$

Note that in the limiting case  $\frac{|\sigma|(1-b)}{\alpha} \leq p \leq \frac{|\sigma|}{\alpha}$  statement 1) in Theorem A does not hold. Moreover, there exists  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  such that  $I_{\alpha,\sigma}f(x) = \infty$  for all  $x \in \mathbb{R}^n$ . However, as will be proved, statement 1) holds for the modified anisotropic Riesz potential  $\tilde{I}_{\alpha,\sigma}$  if the space  $L_\infty(\mathbb{R}^n)$  is replaced by a wider space  $BMO_\sigma(\mathbb{R}^n)$ .

The following theorem is our main result in which we obtain conditions ensuring that the modified anisotropic Riesz potential  $\tilde{I}_{\alpha,\sigma}$  is bounded from the space  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $BMO_\sigma(\mathbb{R}^n)$ .

**Theorem 4.4.** *Let  $0 < \alpha < |\sigma|$ ,  $0 \leq b < 1 - \frac{\alpha}{|\sigma|}$ , and  $\frac{|\sigma|(1-b)}{\alpha} \leq p \leq \frac{|\sigma|}{\alpha}$ , then the operator  $\tilde{I}_{\alpha,\sigma}$  is bounded from  $\tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$  to  $BMO_\sigma(\mathbb{R}^n)$ .*

Moreover, if the integral  $I_{\alpha,\sigma}f$  exists almost everywhere for  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ ,  $\frac{|\sigma|(1-b)}{\alpha} \leq p \leq \frac{|\sigma|}{\alpha}$ , then  $I_{\alpha,\sigma}f \in BMO_{\sigma}(\mathbb{R}^n)$  and the following inequality is valid

$$\|I_{\alpha,\sigma}f\|_{BMO_{\sigma}} \leq C\|f\|_{\tilde{L}_{p,b,\sigma}},$$

where  $C > 0$  is independent of  $f$ .

*Proof.* For given  $t > 0$  we denote

$$f_1(x) = f(x)\chi_{E_{\sigma}(0,2t)}(y), \quad f_2(x) = f(x) - f_1(x), \quad (4.1)$$

where  $\chi_{E_{\sigma}(0,2t)}$  is the characteristic function of the set  $E_{\sigma}(0,2t)$ . Then

$$\tilde{I}_{\alpha,\sigma}f(x) = \tilde{I}_{\alpha,\sigma}f_1(x) + \tilde{I}_{\alpha,\sigma}f_2(x) = F_1(x) + F_2(x), \quad (4.2)$$

where

$$F_1(x) = \int_{E_{\sigma}(0,2t)} \left( |x-y|_{\sigma}^{\alpha-|\sigma|} - |y|_{\sigma}^{\alpha-|\sigma|} \chi_{\mathfrak{e}_{E_{\sigma}(0,1)}}(y) \right) f(y) dy,$$

$$F_2(x) = \int_{\mathfrak{e}_{E_{\sigma}(0,2t)}} \left( |x-y|_{\sigma}^{\alpha-|\sigma|} - |y|_{\sigma}^{\alpha-|\sigma|} \chi_{\mathfrak{e}_{E_{\sigma}(0,1)}}(y) \right) f(y) dy.$$

Note that the function  $f_1$  has compact (bounded) support and thus

$$a_1 = - \int_{E_{\sigma}(0,2t) \setminus E_{\sigma}(0, \min\{1,2t\})} |y|_{\sigma}^{\alpha-|\sigma|} f(y) dy$$

is finite.

Note also that

$$\begin{aligned} F_1(x) - a_1 &= \int_{E_{\sigma}(0,2t)} |x-y|_{\sigma}^{\alpha-|\sigma|} f(y) dy \\ &\quad - \int_{E_{\sigma}(0,2t) \setminus E_{\sigma}(0, \min\{1,2t\})} |y|_{\sigma}^{\alpha-|\sigma|} f(y) dy \\ &\quad + \int_{E_{\sigma}(0,2t) \setminus E_{\sigma}(0, \min\{1,2t\})} |y|_{\sigma}^{\alpha-|\sigma|} f(y) dy \\ &= \int_{\mathbb{R}^n} |x-y|_{\sigma}^{\alpha-|\sigma|} f_1(y) dy = I_{\alpha,\sigma}f_1(x). \end{aligned}$$

Therefore

$$|F_1(x) - a_1| \leq \int_{\mathbb{R}^n} |y|_{\sigma}^{\alpha-|\sigma|} |f_1(x-y)| dy = \int_{E_{\sigma}(0,2t)} |y|_{\sigma}^{\alpha-|\sigma|} |f(x-y)| dy.$$

Then

$$\begin{aligned}
& |E_\sigma(x, t)|^{-1} \int_{E_\sigma(x, t)} |F_1(y) - a_1| dy \\
&= |E_\sigma(0, t)|^{-1} \int_{E_\sigma(0, t)} |F_1(x - y) - a_1| dy \\
&\leq |E_\sigma(0, t)|^{-1} \int_{E_\sigma(0, t)} \left( \int_{E_\sigma(0, 2t)} |z|_\sigma^{\alpha-|\sigma|} |f(x - y - z)| dz \right) dy \\
&= |E_\sigma(0, t)|^{-1} \int_{E_\sigma(0, 2t)} \left( \int_{E_\sigma(0, t)} |f(x - y - z)| dy \right) |z|_\sigma^{\alpha-|\sigma|} dz \\
&\lesssim t^{-|\sigma|} t^{|\sigma|-\alpha} \|f\|_{L_{1, 1-\frac{\alpha}{|\sigma|}, \sigma}} \int_{E_\sigma(0, 2t)} |z|_\sigma^{\alpha-|\sigma|} dz \\
&\approx \|f\|_{L_{1, 1-\frac{\alpha}{|\sigma|}, \sigma}}. \tag{4.3}
\end{aligned}$$

Denote

$$a_2 = \int_{E_\sigma(0, \max\{1, 2t\}) \setminus E_\sigma(0, 2t)} |y|_\sigma^{\alpha-|\sigma|} f(y) dy.$$

If  $2|x|_\sigma \leq |y|_\sigma$ , then

$$\|x - y|_\sigma^{\alpha-|\sigma|} - |y|_\sigma^{\alpha-|\sigma|}\| \leq C|x|_\sigma |y|_\sigma^{\alpha-|\sigma|-1}.$$

By the Hölder's inequality we have

$$\begin{aligned}
|F_2(x) - a_2| &\leq C|x|_\sigma \int_{E_\sigma(0, 2t)} |y|_\sigma^{\alpha-|\sigma|-1} |f(y)| dy \\
&\lesssim |x|_\sigma \sum_{j=0}^{\infty} \int_{2^{j+1}t \leq |y|_\sigma \leq 2^{j+2}t} |y|_\sigma^{\alpha-|\sigma|-1} |f(y)| dy \\
&\lesssim |x|_\sigma \sum_{j=0}^{\infty} (2^{j+1}t)^{\alpha-|\sigma|-1} \int_{|y|_\sigma \leq 2^{j+2}t} |f(y)| dy \\
&\lesssim |x|_\sigma \|f\|_{L_{1, 1-\frac{\alpha}{|\sigma|}, \sigma}} \sum_{j=0}^{\infty} (2^{j+2}t)^{\alpha-|\sigma|-1} (2^{j+2}t)^{|\sigma|-\alpha} \\
&\approx |x|_\sigma t^{-1} \|f\|_{L_{1, 1-\frac{\alpha}{|\sigma|}, \sigma}}. \tag{4.4}
\end{aligned}$$

Therefore, from (4.3) and (4.4) we have

$$\sup_{x, t} \frac{1}{|E_\sigma(0, t)|} \int_{E_\sigma(0, t)} \left| \tilde{I}_{\alpha, \sigma} f(x - y) - a_f \right| dy \lesssim \|f\|_{L_{1, 1-\frac{\alpha}{|\sigma|}, \sigma}}. \tag{4.5}$$



By the Hölder's inequality we have

$$\begin{aligned}
 \|f\|_{L_{1,1-\frac{\alpha}{|\sigma|},\sigma}} &= \sup_{x \in \mathbb{R}^n, t > 0} t^{\alpha-|\sigma|} \int_{E_\sigma(x,t)} |f(y)| dy \\
 &\leq 2^{\frac{n}{p'}} \sup_{x \in \mathbb{R}^n, t > 0} t^{\alpha-\frac{|\sigma|}{p}} [t]_1^{\frac{b|\sigma|}{p}} \left( [t]_1^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p} \\
 &\leq 2^{\frac{n}{p'}} \|f\|_{\tilde{L}_{p,b,\sigma}} \sup_{t > 0} t^{\alpha-\frac{|\sigma|}{p}} [t]_1^{\frac{b|\sigma|}{p}} \tag{4.6} \\
 &= 2^{\frac{n}{p'}} \|f\|_{\tilde{L}_{p,b,\sigma}} \max \left\{ \sup_{0 < t \leq 1} t^{\alpha-\frac{|\sigma|(1-b)}{p}}, \sup_{t > 1} t^{\alpha-\frac{|\sigma|}{p}} \right\} \\
 &= 2^{\frac{n}{p'}} \|f\|_{\tilde{L}_{p,b,\sigma}}.
 \end{aligned}$$

Finally let  $\frac{|\sigma|(1-b)}{\alpha} \leq p \leq \frac{|\sigma|}{\alpha}$  and  $f \in \tilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ , then from (4.5) and (4.6) we get

$$\left\| \tilde{I}_{\alpha,\sigma} f \right\|_{BMO_\sigma} \leq 2 \sup_{x,t} \frac{1}{|E_\sigma(0,t)|} \int_{E_\sigma(0,t)} \left| \tilde{I}_{\alpha,\sigma} f(x-y) - a_f \right| dy \lesssim \|f\|_{\tilde{L}_{p,b,\sigma}}.$$

The Theorem 4.4 is proved.  $\square$

## References

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Malik S. Dzhabrailov  
Department of Mathematics  
Azerbaijan State Pedagogical University, Baku, Azerbaijan  
E-mail: vagif@guliyev.com

Sevinc Z. Khaligova  
Department of Mathematics  
Azerbaijan State Pedagogical University, Baku, Azerbaijan  
Email: seva.xaligova@hotmail.com