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$3-$ *Fold Local* index theorem

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Abstract

Although this is a slightly modified version of the paper [23], it has to be seen as preliminary work.

$3-$*Fold Local* Index Theorem means $Local(Local(Local\ Index\ Theorem)))$. $Local\ Index\ Theorem$ is the Connes-Moscovici local index theorem [4], [5]. The second "Local" refers to the cyclic homology localised to a certain separable subring of the ground algebra, while the last one refers to Alexander-Spanier type cyclic homology. Localised cyclic homology had already appeared in the literature, see Connes [3], Karoubi [9] [10], Loday [12].

The Connes-Moscovici work is based on the operator $R(A) = \mathbf{P} - \mathbf{e}$ associated to the elliptic pseudo-differential operator A on the smooth manifold M, where $\mathbf{P}$, $\mathbf{e}$ are idempotents, see [4], Pg. 353.

The operator $R(A)$ has two main merits: it is a smoothing operator and its distributional kernel is situated in an arbitrarily small neighbourhood of the diagonal in $M \times M$.

The operator $R(A)$ has also two setbacks: -i) it is not an idempotent and therefore it does not have a genuine Connes-Karoubi-Chern character in the absolute cyclic homology of the algebra of smoothing operators, see Connes [2], [3], Karoubi [9] [10] ; -ii) even if it were an idempotent, its Connes-Karoubi-Chern character would belong to the cyclic homology of the algebra of smoothing operators with *arbitrary* supports, which is *trivial*.

This paper presents a new solution to the difficulties raised by the two setbacks.

Key Words: *Local* index theorem, Local index theorem, *Local* Hochschild and Cyclic homology of Banach algebras, *Local* K-theory, *Local* Chern character.

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For which concerns -i), we show that although $R(A)$ is not an idempotent, it satisfies the identity $(\mathbf{R}(A))^2 = \mathbf{R}(A) - [\mathbf{R}(A).e + e.\mathbf{R}(A)]$. We show that the operator $R(A)$ has a genuine Chern character provided the cyclic homology complex of the algebra of smoothing operators is *localised* to the separable sub-algebra $\Lambda = \mathbb{C} + \mathbb{C}.e$, see Section 7.1.

For which concerns -ii), we introduce the notion of *local* cyclic homology; this is constructed on the foot-steps of the Alexander-Spanier homology, i.e. by filtering the chains of the cyclic homology complex of the algebra of smoothing operators by their distributional support, see Section 6.

Using these new instruments, we give a reformulation of the Connes-Moscovici local Index Theorem, see Theorem 8.1, Section 8. As a corollary of this theorem, we show that the *local* cyclic homology of the algebra of smoothing operators is at least as big as the Alexander-Spanier homology of the base manifold.

The present reformulation of Connes-Moscovici local index theorem opens the way to new investigations, see Section 9.

1 Introduction

Using the language of non-commutative geometry, Connes and Moscovici [4], [5] build an algebraic bridge connecting in a natural way the analytical index and the topological index of elliptic pseudo-differential operators on smooth manifolds. Their construction extends also to topological manifolds with quasi-conformal structure, see Connes-Sullivan-Teleman [6].

Given an elliptic pseudo-differential operator A on the smooth manifold M, Connes and Moscovici associate its index class $Ind(A) \in H_*^{AS}(M)$, where $H_*^{AS}(M)$ denotes Alexander-Spanier homology. The index class is obtained as the result of the composition of two constructions

$$K^1(C(S^*M)) \xrightarrow{\mathbf{R}} K^0(\mathcal{K}_M) \xrightarrow{\tau} H_{ev}^{AS}(M), \tag{1.1}$$

where $K^1(C(S^*M))$ contains the symbol of the elliptic operator A; $K^0(\mathcal{K}_M)$ consists of differences of stably homotopy classes of smooth idempotents with *arbitrary* supports.

The first homomorphism R applied upon the operator A is given by an algebraic construction involving the operator A and one of its parametrices B; as a result one obtains an operator $\mathbf{R}(A)$ who has the following basic properties:

-1) it has small support about the diagonal in $M \times M$

-2) it is a smoothing operator on M

-3) $\mathbf{R}(A) = \mathbf{P} - \mathbf{e}$, where $\mathbf{P}$ and $\mathbf{e}$ are idempotents, with small support about the diagonal. The idempotent $\mathbf{e}$ is a constant operator; it does not contain homological information.

The smoothing operator $\mathbf{R}(A)$ is obtained by implementing the connecting homomorphism $\delta_1 : K^1(C(S^*M)) \longrightarrow K^0(\mathcal{K}_M)$ associated to the short exact sequence of Banach algebras

$$0 \to \mathcal{K}_M \to \mathcal{L}_M \to C(S^*M) \to 0.$$

The operator $\mathbf{R}(A)$ has two inconveniences:

-i) it is not an idempotent

-ii) even if $\mathbf{R}(A)$ were an idempotent, its Chern character, belonging to the cyclic homology of the algebra of smoothing operators with *arbitrary* supports, would be trivial because this cyclic homology is trivial.

Connes-Moscovici [4], Pg. 352 state clearly that the connecting homomorphism ∂_1 takes values in $K^0(\mathcal{K}_M) \simeq \mathbb{Z}$ and that the information carried by it is solely the *index* of the operator. However, [4] states also that by pairing the residue operator $\mathbf{R}(A)$ with the Alexander-Spanier cohomology (which is *local*) one recovers the whole co-homological information carried by its symbol, (see the Connes-Moscovici local Index Theorem 3.1 of Section 3).

The Connes-Moscovici index class $Ind(A) := \tau(\ \mathbf{R}(A)\ )$ is a well defined Alexander-Spanier *homology class* on M. Its correctness depends upon two important ingredients:

-a) the realisation of the Alexander-Spanier co-homology by means of skew-symmetric co-chains; this allows one to get rid of the idempotent $\mathbf{e}$ from the expression of $\mathbf{R}(A)$ and ultimately to treat $\mathbf{R}(A)$ as it were an idempotent, see -3); call it virtual idempotent.

-b) at this point, $Ind(A)$ uses the formal pairing of the Chern character of the virtual idempotent -a) with the cyclic homology of the algebra $C^\infty(M)$. This operation requires to produce trace class operators.

In this paper we address the same problem, i.e. to define an algebraic bridge between the analytical and topological index of elliptic operators. However, we propose here a different way to overcome the difficulties described above. This will be done by introducing two main ideas.

The first idea of the paper is based on the remark that the residue operator $\mathbf{R}(A)$ satisfies the identity

$$(\mathbf{R}(A))^2 \ = \ \mathbf{R}(A) - [\mathbf{R}(A).e + e.\mathbf{R}(A)].$$

We show in Section 7.1 that, based on this identity, the operator $\mathbf{R}(A)$ has a genuine Chern character provided the ordinary cyclic homology complex of the algebra of smoothing operators is localised with respect to the separable ring $\Lambda := \mathbb{C} + \mathbb{C}e$. This replacement does not modify the cyclic homology.

The second idea of the paper consists of replacing the ordinary cyclic homology of the algebra of smoothing operators by the *local* cyclic homology of the algebra. This is done by filtering the cyclic complex of the algebra based on the supports of the chains. The *local* cyclic homology of the algebra of smoothing operators is then defined in the same way as the Alexander-Spanier co-homology is defined.

The combination of these two ideas allows one to reformulate the Connes-Moscovici local index theorem, see Theorem 8.1.

Notice that our considerations do not require necessarily to deal with trace class operators; this opens the way to new applications.

As a corollary of Theorem 8.1 we obtain that the *local* cyclic homology of the algebra of smoothing operators is at least as big as the Alexander-Spanier homology of the base space, see Proposition 8.2.

We stress also that our *local* cyclic homology of the Banach algebra of smoothing operators is independent of Connes' notions of *entire* or *asymptotic* cyclic homology, see [2], [3] and differs from Puschnigg's [15] construction.

Our methods lead to interesting questions, new scenarios and connections with previously known results; these will be addressed elsewhere, see Section 9. This preliminary version of the work does not state completeness of references.

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2 Recall of K-theory groups of Banach algebras.

For the benefit of the reader we recall here the basic definitions regarding the K-theory of Banach algebras.

To begin with, as motivation, suppose X is a compact connected topological space and $C(X)$ denote the C^* algebra of continuous complex valued functions on X. A continuous complex vector bundle over X may be described either as a finite projective module over $C(X)$, or as an idempotent of the matrix algebra $\mathcal{M}_n(C(X))$. Passing to isomorphism classes of bundles allows one to identify bundles over X with continuous homotopy classes of idempotents of the algebra $\mathcal{M}_n(C(X))$, with n sufficiently large. Denote by *Vect*(X) the set of such homotopic classes of idempotents.

The direct sum of finite projective $C(X)$ modules passes to *Vect*(X) so that it becomes a commutative semigroup. Taking the Grothendieck completion of this semigroup, one obtains the $K-$theory group $K^0(X)$. Any element of the group $K^0(X)$ may be represented as $\xi = [p] - [q]$, where $[p]$ and $[q]$ are the homotopy classes of two idempotents $p, q \in \mathcal{M}_m(C(X))$. The idempotent q may be chosen to be a unit matrix. The element ξ is called a virtual vector bundle of rank $m - n$.

The subgroup of $K^0(X)$ consisting of virtual bundles of rank zero is denoted by $\tilde{K}^0(X)$.

An idempotent $p_n \in \mathcal{M}_n(C(X))$ may be used to produce the idempotent $p_{n+1} \in \mathcal{M}_{n+1}(C(X))$ by stabilization

$$p_{n+1} := \begin{pmatrix} p_n & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

Then any element of $\tilde{K}^0(X)$ may be thought of as the stably homotopy class of an idempotent in $\mathcal{M}_m(C(X))$.

As for any general homology functor, one defines $K^1(X) = \tilde{K}^0(\Sigma X)$, where Σ denotes suspension. Any vector bundle over ΣX may be described by a clutching function $f : X \to GL(m, C)$. Alternatively, any such function may be thought of as an element of the subset $GL_m(C(X)) \subset \mathcal{M}_m(C(X))$ consisting precisely of all invertible elements of the algebra $M_m(C(X))$.

As the stabilization formula above may be used not only for idempotents but also for invertibles, one gets an equivalent definition of $K^1(X)$

$$K^1(X) = \textit{stably homotopy classes of invertibles in } \mathcal{M}_m(C(X)), \tag{2.2}$$

or

$$K^1(X) = \pi_0 \; (\; Lim_{m\to\infty} GL_m((C(X)) \;). \tag{2.3}$$

The higher order K theory groups are defined by

$$K^i(X) = \tilde{K}(\Sigma^i X), \quad 1 \leq i. \tag{2.4}$$

For any closed subspace $Y \subset X$, one define the relative K groups

$$K^i(X, Y) := \tilde{K}^i(X/Y), \tag{2.5}$$

where X/Y denotes the quotient space.

The Bott periodicity theorem implies the periodicity of the K-theory groups, which leads to the 6-term exact sequence

$$\begin{array}{ccccc} K^0(X/Y) & \to & K^0(X) & \to & K^0(Y) \\ \uparrow & & & & \downarrow \\ K^1(Y) & \leftarrow & K^1(X) & \leftarrow & K^1(X/Y) \end{array} . \tag{2.6}$$

If we replace in the above constructions the algebra $C(X)$ by an arbitrary unital Banach algebra $\mathcal{A}$, the K theory groups of the algebra $\mathcal{A}$ are defined (with [ ] meaning homotopy class)

$$K^0(\mathcal{A}) := \{ \; [p] - [q] \mid p^2 = p \in \mathcal{M}_m(\mathcal{A}), \;\; q^2 = q \in \mathcal{M}_n(\mathcal{A}), \;\; m, n \in N \} \tag{2.7}$$

$$K^1(\mathcal{A}) := \pi_0 \ (\ Lim_{m\to\infty} GL_m(\mathcal{A}) \ ). \tag{2.8}$$

To the $(X,\ Y)$ pair of compact non-empty topological spaces there corresponds the exact sequence of algebras of continuous functions

$$0 \to C(X,\ Y) \to C(X) \to C(Y) \to 0, \tag{2.9}$$

where

$$C(X,\ Y) \ = \ \{ f \ \in \ C(X) \mid f(Y) = 0 \ \}. \tag{2.10}$$

More generaly, one may consider an arbitrary exact sequence of Banach algebras

$$0 \to J \to A \to B \to 0. \tag{2.11}$$

Let $\tilde{J} := J \oplus C.1$ be the algebra J with the unit $1 \in C$ adjoined and let $\epsilon : \tilde{J} \to C$ be the augmentation mapping. By definition, for $i = 0, 1$

$$K^i(J) = Ker \ \epsilon_i \tag{2.12}$$

where $\epsilon_i : K^i(\tilde{J}) \to K^i(C)$.

By construction, $K^i(C(X)) = K^i(X)$, $i = 0, 1$.

The analogue of Bott periodicity holds for K-theory groups, see Wood [25]; hence, for any short exact sequence of Banach algebras and continuous mappings as above, the 6-terms K exact sequence of K-theory groups holds

$$\begin{array}{ccccc} K^1(J) & \to & K^1(A) & \to & K^1(B) \\ \uparrow & & & & \downarrow \\ K^0(B) & \leftarrow & K^0(A) & \leftarrow & K^0(J) \end{array} . \tag{2.13}$$

3 Connes-Moscovici Local Index Theorem.

In this section we summarize the Connes-Moscovici [4] construction of the *local index class* for an elliptic operator. All constructions and notations in this section are those of [4].

To fix the notation, let $A : L_2(E) \to L_2(F)$ be an elliptic pseudo-differential operator of order zero from the vector bundle E to the vector bundle F on the compact smooth manifold M. Let $\sigma_{pr}(A) = a$ be its principal symbol, seen as a continuous isomorphism from the bundle π^*E to the bundle π^*F over the unit co-sphere bundle $S(T^*M)$ (π is the co-tangent bundle projection). Let B be a pseudo-differential parametrix for the operator A. The parametrix B, having principal symbol $\sigma_{pr}(B) = a^{-1}$, may be chosen so that the operators $S_0 = 1 - BA$ and $S_1 = 1 - AB$ be *s*moothing operators. Additionally, supposing that the distributional support of the operator A is sufficiently small about the diagonal, the operators B, S_0, S_1 may be supposed to have also small supports about the diagonal.

With the operators A, B, S_0, S_1 one manufactures the invertible operator

$$\mathbf{L} = \begin{pmatrix} S_0 & -(1+S_0)B \\ A & S_1 \end{pmatrix} : L_2(E) \oplus L_2(F) \to L_2(E) \oplus L_2(F) \tag{3.1}$$

with inverse

$$\mathbf{L}^{-1} = \begin{pmatrix} S_0 & (1+S_0)B \\ -A & S_1 \end{pmatrix} \tag{3.2}$$

The operator $\mathbf{L}$ is used to produce the idempotent $\mathbf{P}$

$$\mathbf{P} = \mathbf{L} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{L}^{-1}. \tag{3.3}$$

Let $\mathbf{P}_1$, resp. $\mathbf{P}_2$ be the projection onto the direct summand $L_2(E)$, resp. $L_2(F)$.

A direct computation shows that

$$\mathbf{R} := \mathbf{P} - \mathbf{P}_2 = \begin{pmatrix} S_0^2 & S_0(1+S_0)B \\ S_1 A & -S_1^2 \end{pmatrix}. \tag{3.4}$$

This shows that the *residue* operator $\mathbf{R}$ is a smoothing operator on $L_2(E) \oplus L_2(F)$ with small support about the diagonal.

The motivation for the consideration of the operators L, P, R comes from the implementation of the connecting homomorphism

$$\partial_1 : K^1(C(S^*M)) \to K^0(\mathcal{K}_M) \cong \mathsf{Z} \tag{3.5}$$

in the 6-terms K-theory groups exact sequence associated to the short exact sequence of C^* algebras

$$0 \to \mathcal{K}_M \to \mathcal{L}_M \to C(S^*M) \to 0; \tag{3.6}$$

here $\mathcal{K}_M$ is the algebra of compact operators on $L_2(M)$, $\mathcal{L}_M$ is the norm closure in the algebra of bounded operators of the algebra of pseudo-differential operators of order zero and $C(S^*M)$ is the algebra of continuous functions on the unit co-sphere bundle to M.

In fact, if $\mathbf{a}$ denotes the symbol of the elliptic operator A, then after embedding the bundles E and F into the trivial bundle of rank N, the operators $L, \mathbf{P}, \mathbf{P}_2$, resp. R, may be seen as elements of the matrix algebras $\mathcal{M}_N(\mathcal{L}_M)$, resp. $\mathcal{M}_N(\mathcal{K}_M)$, and hence

$$\partial_1([\mathbf{a}]) = [\mathbf{P}] - [\mathbf{P}_2] \in K^0(\mathcal{K}_M). \tag{3.7}$$

Let $C^q(M)$ denote the space of Alexander-Spanier cochains of degree q on M consisting of all smooth, *anti-symmetric* real valued functions ϕ defined on M^{q+1}, which have support on a sufficiently small tubular neighbourhood of the diagonal.

Then, for any $[\mathbf{a}] \in K^1(C(S^*M))$, and for any even number q one considers the linear functional

$$\tau^q_{\mathbf{a}} : C^q(M) \longrightarrow \mathbb{C} \tag{3.8}$$

given by the formula

$$\tau^q_{\mathbf{a}}(\phi) = \int_{M^{q+1}} \mathbf{R}(x_0, x_1)\mathbf{R}(x_1, x_2)...\mathbf{R}(x_q, x_0)\phi(x_0, x_1, ..., x_q), \tag{3.9}$$

where $\mathbf{R}(x_0, x_1)$ is the kernel of the smoothing operator $\mathbf{R}$ defined above.

Using the above construction, Connes and Moscovici [4] produce the *index class homomorphism*

$$Ind : K^1(C(S^*M)) \otimes_{\mathbb{C}} H^{ev}_{AS}(M) \longrightarrow \mathbb{C}, \tag{3.10}$$

where $H^{ev}_{AS}(M)$ denotes Alexander-Spanier cohomology. On the Alexander-Spanier co-chains ϕ it is defined by

$$Ind(\mathbf{a} \otimes_{\mathbb{C}} \phi) := \tau^q_{\mathbf{a}}(\phi) \tag{3.11}$$

The functional $\tau^q_{\mathbf{a}}$ is an Alexander-Spanier cycle of degree q over M; it defines a homology class $[\tau^q_{\mathbf{a}}] \in H_q(M, R)$.

Theorem 3.1. *(Connes-Moscovici [4], Theorem 3.9) Let A be an elliptic pseudo-differential operator on M and let $[\phi] \in H^{2q}_{comp}(M)$. Then*

$$Ind_{[\phi]}A \;=\; \frac{1}{(2\pi i)^q}\frac{q!}{(2q)!}(-1)^{dim M} < Ch\sigma(A)\tau(M)[\phi],\ [T^*M] > \tag{3.12}$$

where $\tau(M) = Todd(TM) \otimes C$ and $H^*(T^*M)$ is seen as a module over $H^*_{comp}(M)$.

4 K-Theory *Local* Symbol Index Class.

The content of this section presents an interest by itself although it is not going to be used in this paper.

In this section we are going to show that by replacing in the above constructions the Hilbert spaces by corresponding bundles and operators by their symbols, one gets a quasi-local *residue bundle* $\mathbf{r}$ on the total space of the cotangent bundle $\pi : T^*(M) \longrightarrow M$. The Connes-Moscovici *residue* operator $\mathbf{R}$ appears to be the quantification of the bundle $\mathbf{r}$.

All considerations here are made on the total space of the co-tangent bundle.

Let $\xi = \pi^*(E)$ and $\eta = \pi^*(F)$ the pullbacks of the bundles E and F. Let $\lambda : T(M) \longrightarrow [0,1]$ be a smooth function which is identically zero on a small neighbourhood U of the zero section and identically 1 on the complement of $2U$.

Let $A : L_2(E) \to L_2(F)$ be the elliptic pseudo-differential operator considered above and let B be the pseudo-differential parametrix for the operator A. The principal symbols of the operators A and B

$$\sigma_{pr}(A) = \mathbf{a} : \xi \longrightarrow \eta \tag{4.1}$$

$$\mathbf{b} = \sigma_{pr}(B) = \mathbf{a}^{-1} : \eta \longrightarrow \xi \tag{4.2}$$

are isomorphisms away from the zero section. The symbol $\mathbf{a}$ defines the K^0-theory triple $(\ \xi,\ \eta,\ \mathbf{a})$ with compact support on $T^*(M)$.

We regularize the bundle homomorphisms $\mathbf{a}$, and $\mathbf{a}^{-1}$ by multiplying them by the function λ; let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be the obtained bundle homomorphisms.

Let $s_0 = 1 - \tilde{\mathbf{b}}\,\tilde{\mathbf{a}}$ and $s_1 = 1 - \tilde{\mathbf{a}}\,\tilde{\mathbf{b}}$. They are bundle homomorphisms with supports in the neighbourhood $2U$ of the zero section.

With the bundle homomorphisms $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, s_0, s_1$ one manufactures the smooth bundle isomorphism

$$l = \begin{pmatrix} s_0 & -(1+s_0)\tilde{\mathbf{b}} \\ \tilde{\mathbf{a}} & s_1 \end{pmatrix} : \xi \oplus \eta \to \xi \oplus \eta \tag{4.3}$$

with inverse

$$l^{-1} = \begin{pmatrix} s_0 & (1+s_0)\tilde{\mathbf{b}} \\ -\tilde{\mathbf{a}} & s_1 \end{pmatrix} \tag{4.4}$$

Let $\mathbf{p}_1$, resp. $\mathbf{p}_2$, be the direct sum projection of the bundle $\xi \oplus \eta$ onto the first, resp. the second, summand.

The isomorphism l is used to produce the idempotent $\mathbf{p}$

$$\mathbf{p} = l \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} l^{-1} \ = \ l\, \mathbf{p}_1\, l^{-1}. \tag{4.5}$$

Therefore, $\tilde{\xi} \ := \ Image(\mathbf{p})$ is a smooth sub-bundle of the bundle $\xi \oplus \eta$.

Proposition 4.1. *-i) $l : \ \xi \longrightarrow \tilde{\xi}$ is a bundle isomorphism*

-ii) away from the neighbourhood $2U$, the bundles $\tilde{\xi}$, η coincide

-iii) the triples $(\ \tilde{\xi},\ \eta,\ \mathbf{I}_\eta\ )$, $\ (\ \xi,\ \eta,\ a)$ are isomorphic and hence they define the same element of $K^0(T^*(M))$.

Proof. -i) Obviously,

$$l^{-1}(\ \tilde{\xi}\ ) = l^{-1}(\ Image(\ l\,\mathbf{p}_1\ l^{-1}\ )\ ) = l^{-1}(\ Image(\ l\,\mathbf{p}_1\ )\ ) =\ Image(\ \mathbf{p}_1\ ) = \xi. \tag{4.6}$$

-ii) Let the subscript $_\infty$ denote the behaviour on the complement of $2U$. Then

$$l_\infty = \begin{pmatrix} 0 & -\mathbf{a}^{-1} \\ a & 0 \end{pmatrix}, \qquad l_\infty^{-1} = \begin{pmatrix} 0 & \mathbf{a}^{-1} \\ -\mathbf{a} & 0 \end{pmatrix} \tag{4.7}$$

and therefore

$$\tilde{\xi}_\infty = Image(\ l_\infty\ \mathbf{p}_1\ l_\infty^{-1}\ ) \tag{4.8}$$

$$= Image \begin{pmatrix} 0 & -\mathbf{a}^{-1} \\ \mathbf{a} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{a}^{-1} \\ -\mathbf{a} & 0 \end{pmatrix} \tag{4.9}$$

$$= Image \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \eta_\infty \tag{4.10}$$

-iii) It is sufficient to verify the commutativity of the diagram on the complement of $2U$

$$\begin{array}{ccc} \xi_\infty & \xrightarrow{a} & \eta_\infty \\ \downarrow & & \downarrow \\ \tilde{\xi}_\infty = \eta_\infty & \xrightarrow{Id} & \eta_\infty \end{array} \tag{4.11}$$

where the first vertical arrow is l_∞ and the second vertical arrow is the identity. In fact, this is true because

$$l_\infty\ p_1 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \tag{4.12}$$

Corollary 4.2. *-i) The operator $\boldsymbol{R} = \boldsymbol{P} - \boldsymbol{P}_2$ represents the quantization of the quasi-local residue bundle* $\mathbf{r}$.

-ii) Passing to tne Chern character, one has

$$Ch(\ \xi,\ \eta,\ a)\ =\ Ch(\ \tilde{\xi},\ \eta,\ \mathbf{I}_\eta\ )\ \in H^{ev}_{comp}(T^*(M)) \tag{4.13}$$

-iii) For any two linear or direct [19] , [20] connections $\nabla_{\tilde{\xi}}$, resp. ∇_η, on the bundle $\tilde{\xi}$, resp. on η, which coincide on the complement of $2U$, one has

$$Ch(\ \xi,\ \eta,\ \mathbf{a})\ =\ Ch(\ \tilde{\xi}, \nabla_{\tilde{\xi}}\ )\ -\ Ch(\ \eta, \nabla_\eta\ ) \tag{4.14}$$

-iv) $Ch(\ \xi,\ \eta,\ \mathbf{a})$ depends only on the the bundle homomorphism $\mathbf{r}$, which has support on $2U$.

The bundle $\mathbf{r}$ prepares the triple $(\ \xi,\ \eta,\ a)$ for computing its Chern character via Chern-Weil theory.

By stabilization of the symbol $\sigma\ =\ (\ \xi,\ \eta,\ a)$ we may assume that the bundles F and η are trivial. Then

$$Ch(\ \sigma\ ) = Ch(\ \tilde{\xi}) - Rank(\xi). \tag{4.15}$$

If $\mathbf{r}(x,y)$ is a direct connection [19] , [20] on the bundle $\tilde{\xi}$, we may assume that it is flat on the complement of $2U$. Using the results of [19], [20], [11], we obtain the

Theorem 4.3. *The components of the Chern character of σ, seen as a periodic cyclic homology classes, are*

$$Ch_q(\ \sigma\ ) = (-1)^{dim M}\frac{(2\pi i)^q(2q)!}{q!}.\ Tr\ [\mathbf{r}(x_0,x_1)\mathbf{r}(x_1,x_2)...\mathbf{r}(x_{k-1},x_q)\mathbf{r}(x_q,x_0)], \tag{4.16}$$

for q even number.

5 Review of Hochschild and Cyclic Homology.

In this section we review results due to Connes [2], [3] and Connes-Moscovici [4].

5.1 Hochschild and Cyclic Homology.

Let A be an associative algebra with unit over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Suppose A is a left and right module over L, which is a unitary ring over $\mathbb{K}$.

Define for any $k = 0, 1, 2, ...$

$$C_{L,k}(A) := \otimes_L^{k+1} A. \tag{5.1}$$

We have assumed here that the tensor product $\otimes_L^{k+1}$ is circular, i.e., for any $f_0, f_1, ..., f_q \in A$ and $f \in L$, one has

$$f_0 \otimes_L f_1 \otimes_L, ..., \otimes_L f_k.f \otimes_L = f.f_0 \otimes_L f_1 \otimes_L, ..., \otimes_L f_k \otimes_L, \tag{5.2}$$

see [12], Sect. 1.2.11. In particular,

$$C_{L,0}(A) := \otimes_L^1 A = \frac{A}{[A,L]}. \tag{5.3}$$

For negative integers k one defines $C_{L,k}(A) := 0$.

The bar operator $b'_k : C_{L,k}(A) \longrightarrow C_{L,k-1}(A)$ is defined by

$$b'_k(f_0 \otimes_L f_1 \otimes_L, ..., \otimes_L f_k \otimes_L) :=$$

$$:= \sum_{r=0}^{r=k-1} (-1)^r f_0 \otimes_L f_1 \otimes_L, ..., \otimes_L (f_r . f_{r+1}) \otimes_L, ..., \otimes_L f_k \otimes_L . \qquad (5.4)$$

In particular, $b'_0 = 0$.

The Hochschild operator $b_k : C_{L,k}(A) \longrightarrow C_{L,k-1}(A)$ is defined by

$$b_k(f_0 \otimes_L f_1 \otimes_L, ..., \otimes_L f_k \otimes_L) :=$$

$$:= b'_k(f_0 \otimes_L f_1 \otimes_L, ..., \otimes_L f_k \otimes_L) + (-1)^k (f_k . f_0) \otimes_L f_1 \otimes_L, ..., \otimes_L f_{k-1} \otimes_L). \quad (5.5)$$

Both operators b' and b satisfy $(b')^2 = b^2 = 0$. The corresponding complex $\{C_{L,*}(A), b\}$ is called Hochschild complex of the algebra A over the ground ring L, see [12], 1.2.11. The homology of this complex is called *Hochschild homology of the algebra A over the ground ring L* and is denoted by $HH_{L,*}(A)$.

The *cyclic permutation* $T : C_{L,k}(A) \longrightarrow C_{L,k}(A)$ is defined by

$$T(f_0 \otimes_L f_1 \otimes_L, ..., \otimes_L f_k \otimes_L) :=$$

$$:= (-1)^k f_1 \otimes_L f_2 \otimes_L, ..., \otimes_L f_k \otimes_L f_0 \otimes_L . \qquad (5.6)$$

An element of $f \in C_{L,k}(A)$ is called *cyclic* if $T(f) = f$.

If the algebra A has unit, the corresponding bar complex is acyclic. However, the cyclic elements form the *cyclic sub-complex* of the bar complex

$$C^{\lambda}_{L,k}(A) := \{f | f \in C^{\lambda}_{L,k}(A), \ \ T(f) = f\} \qquad (5.7)$$

and its homology is interesting. Its homology is called *cyclic homology* of the algebra A over the ring L, and it is denoted $H^{\lambda}_{L,k}(A)$.

If the ring L coincides with the ground field $\mathbb{K}$, then L is omitted from the notation and the corresponding homology is called *cyclic homology* of A.

Theorem 5.1. *(Connes' I.S.B. Long Exact Sequence, see [2] Part II, Sect. 4, p. 119. , [12] Sect. 2.1.4. and in co-homological context [3] Pg. 205).*

There exist functorial homomorphisms I, B and S such that the following long sequence

$$... \xrightarrow{B} HH_{L,k}(A) \xrightarrow{I} H^{\lambda}_{L,k}(A) \xrightarrow{S} HH^{\lambda}_{L,k-2}(A) \xrightarrow{B} HH_{L,k-1}(A) \xrightarrow{I} .... \qquad (5.8)$$

is exact.

Recall $M_n(A)$ denotes the algebra of $n \times n$ matrices with entries in A.

Theorem 5.2. *(Morita invariance Theorem) (see e.g. [12] Sect. 1.2.5-7, [7] Sect. 2.8)*

Replacing the algebra A with its matrix algebra $M_n(A)$ does not change its Hochshild and cyclic homology.

Definition 5.3. *A unital L-algebra is called* separable *over $\mathbb{K}$ if the multiplication mapping $\mu : L \otimes_{\mathbb{K}} L^{op} \longrightarrow L$ has a L-bimodule splitting, see [12] Sect. 1.2.12.*

Lemma 5.4. *Let $e \in A$ be an idempotent ($e^2 = e$) and $\Lambda := \mathbb{K} + \mathbb{K}e$.*

Then Λ is separable over $\mathbb{K}$.

Proof. Indeed, the splitting $s : \Lambda \longrightarrow \Lambda \otimes_K \Lambda^{op}$ is defined on the generators 1 and e by

$$\begin{aligned} s(\ 1\ ) &= e \otimes_K e + (1-e) \otimes_K (1-e) \\ s(\ e\ ) &= e \otimes_K e. \end{aligned} \tag{5.9}$$

□

Theorem 5.5. *(see [12] Theorem 1.2.12-13.)*

Let Λ be a separable algebra over K and $\mathcal{U}$ be a unital Λ-algebra.

Then there is a canonical isomorphism

$$HH_k(\ \mathcal{U}\ ) \ \cong\ HH_{\Lambda,k}(\ \mathcal{U}\ ). \tag{5.10}$$

If, in addition, Λ is a subalgebra of $\mathcal{U}$, then the canonical epimorphism

$$\phi_\Lambda : C_k(\ \mathcal{U}\ ) \longrightarrow C_{\Lambda,k}(\ \mathcal{U}\ ) \tag{5.11}$$

induces isomorphisms in homology.

Corollary 5.6. *The canonical epimorphism*

$$\phi : C_k(\ \mathcal{U}\ ) \longrightarrow C_{\Lambda,k}(\ \mathcal{U}\ ) \tag{5.12}$$

induces isomorphisms

$$\phi_{k*} : HH_k(\ \mathcal{U}\ ) \ \cong\ HH_{\Lambda,k}(\ \mathcal{U}\ ) \tag{5.13}$$

$$\phi^\lambda_{k*} : H^\lambda_{\Lambda,k}(\ \mathcal{U}\ ) \ \cong\ H^\lambda_{\Lambda,k}(\ \mathcal{U}\ ). \tag{5.14}$$

If the algebra A is a Fréchét algebra, then the algebraic tensor products are usually replaced with *projective* tensor products, see Connes [2], Part II, Sect. 6. The homologies of the projective tensor product completions are called, respectively, *continuous* Hochschild and cyclic homologies; in these cases, the adjective *continuous* is tacitly understood.

The following result constitutes the basic link between the Hochschild and cyclic homology, on a one side, and the classical differential forms $\Omega^*(M)$ and the *de* Rham (Alexander-Spanier) cohomology, on the other side.

Theorem 5.7. *Connes [2], [3], Teleman [18] For the Frèchèt algebra $C^\infty((M))$ on any paracompact smooth manifold M*

$$-i) \qquad HH_k C^\infty((M)) \;=\; \Omega^k(M) \tag{5.15}$$

$$-ii) \quad H_k^\lambda(C^\infty(M)) = \frac{\Omega^k(M)}{d\Omega^k(M)} \oplus H_{dR}^{k-2}(M)\oplus, ..., \oplus H_{dR}^{\epsilon}(M), \;\; \epsilon = 0,\ 1 \tag{5.16}$$

where ϵ has the parity of k.

Part -i) of this theorem is due to Connes [2] Part II, Lemma 45, for M compact, in the context of Hochschild *co*-homology. For paracompact manifolds, the result is due to Teleman [18], in the context of Hochschild homology.

Part -ii) of the theorem is a formal consequence of -i) combined with Connes' $(I.S.B)$-exact sequence, Theorem 5.1.

5.2 Chern Character of Idempotents.

In this section we consider the Hochschild and cyclic homology of the arbitrary associative algebra with unit A (the ring $L = \mathbb{K}$), see the previous section.

Proposition 5.8. *Suppose $p \in M_n(A)$ is an idempotent, $p^2 = p$. Then for any even number q, the chain*

$$\Psi_q(p) := \frac{(2\pi i)^q q!}{(q/2)!} p \otimes_C p \otimes_C, ..., \otimes_C p \quad (q+1 \;\; factors) \tag{5.17}$$

is a cyclic cycle with cyclic homology class

$$[\Psi_q(p)] \in H_q^\lambda(M_n(A)) \equiv H_q^\lambda(A). \tag{5.18}$$

Definition 5.9. *The system of cyclic homology classes*

$$Ch(p) := \{[\Psi_q(p)]\}_{q=ev} \tag{5.19}$$

is called the Chern character of the idempotent p.

5.3 Pairing of Cyclic Homology of Algebras of Operators with Alexander-Spanier Co-homology

Let Op denote some algebra of bounded operators in the Hilbert space H of L_2 sections in a vector bundle ξ over the smooth compact manifold M. In our applications, Op will be one Schatten ideal of operators on H.

Definition 5.10. *Let*

$$\phi_k = \sum A_0 \otimes_{\mathbb{C}} A_1 \otimes_{\mathbb{C}}, ..., \otimes_{\mathbb{C}} A_k \in C_k^{\lambda}(Op) \tag{5.20}$$

and

$$\eta^k = \sum f_0 \otimes_{\mathbb{C}} f_1 \otimes_{\mathbb{C}}, ..., \otimes_{\mathbb{C}} f_k, \quad f_i \in C^{\infty}(M), \tag{5.21}$$

be convergent series in the projective tensor product spaces. One assumes also that in each monomial at least one of the factors A_i is trace class.

The pairing $\phi \cap \eta$, called cap product *is defined on chains by the convergent double series*

$$\phi_k \ \cap \ \eta^k := \sum \sum Tr(A_0 f_0 A_1 f_1, ..., A_k f_k) \in \mathbb{C}, \tag{5.22}$$

where Tr denotes operator trace.

In this definition both the cyclic homology class and the Alexander-Spanier co-homology class should have the same degree. For this purpose, we introduce

$$H_{ev}^{\lambda}(Op) \ := \sum_{q=even} H_q^{\lambda}(Op) \tag{5.23}$$

and

$$H_{AS}^{ev}(M) \ = \sum_{q=even} H_{AS}^{q}(M) \tag{5.24}$$

Definition 5.11.

$$H_{ev}^{\lambda}(Op) \lozenge H_{AS}^{ev}(M) \ = \ \{ \sum \phi_q \otimes \eta^q \mid \{\phi_q \otimes \eta^q\}_{q=0,2,4,...} \in H_{ev}^{\lambda}(Op) \otimes H_{AS}^{ev}(M)\} \tag{5.25}$$

It is important to notice that the outcome of the $\lozenge$ product consists of systems of tensor products of homology by co-homology classes rather than scalars. In addition, one has the relation

$$Tr(\phi \lozenge \eta) \ = \ \phi \ \cap \ \eta.$$

Lemma 5.12. *(Connes-Moscovici [4], Lemma 2.1 (ii))*
Under the hypotheses of the above lemma, one has

$$b'(\phi_{k+1}) \ \cap \ \eta^k := \phi_{k+1} \ \cap \ (\delta \eta^k), \tag{5.26}$$

where η is anti-symmetric Alexander-Spanier k-cochain and δ denotes the Alexander-Spanier co-boundary.

Corollary 5.13. *The pairing* $\cap$ *passes to homology*

$$\cap : H_k^{\lambda}(Op) \otimes_{\mathbb{C}} H^k_{AS}(M) \longrightarrow \mathbb{C} \tag{5.27}$$

Corollary 5.14. *The cap product* $\cap$ *being linear in each factor, if one fixes one of the factors and make variable the other, one obtains an element of its dual. In particular, if* ϕ_k *is a cyclic* co-cycle*, then the correspondence*

$$\eta^k \to \phi_k \ \cap \ \eta^k \tag{5.28}$$

induces a mapping

$$H^k_{AS}(M) \to \mathbb{C} \tag{5.29}$$

and therefore it defines an Alexander-Spanier homology *class on* M*.*

6 ***Local*** Cyclic Homology and ***Local*** K-Theory of Schatten Ideals.

Remark 6.1. It is clear that none of the Schatten ideals $\mathcal{L}^p$ of compact operators in the Hilbert space of L_2 sections in the fibre bundle ξ over the manifold M carries any kind of information about the space M because all such Schatten ideals are *independent* of the space M. To restore information about the base space M, it would be necessary to take into account the module structure of these spaces over some algebra of functions over M. A partial solution to this problem is to consider the distributional kernel of such operators and keep track of their *supports*, as the Alexander-Spanier co-homology does.

The leading idea of this paper is to introduce, in analogy with the Alexander-Spanier co-homology, the *local* K-theory ($K^{i,loc}(A)$), *local* Hochschild homology and *local* cyclic homology (HH^{loc}_*, $H^{\lambda,loc}_*(A)$), etc. We believe that these new structures fit more naturally (than the classical counter-parts) in many interesting problems which involve Alexander-Spanier homology and co-homology, including the local index theorem.

The basic ideas of this program were announced at the "Trieste - 2007 Workshop on Noncommutative Geometry", October 2007.

We stress that our *local* cyclic homology definitely differs from the local cyclic homology introduced by Puschnigg [15]. Here, the word *local* refers to the supports of the operators involved.

The first algebras of interest onto which we intend to apply our *local* structures are the Schatten ideals $\mathcal{L}^p$, $1 \leq p \leq \infty$. It is well known and it is not surprising that the K-theory and cyclic homology of Banach algebras and, in particular, of these algebras are trivial or do not describe well the topology of the spaces onto which they are defined, see [3] , [4] Sect. 4, pg.371, [7] Corollary 3.6.

Our arguments are independent of the notion of *entire* cyclic co-homology or *asymptotic* co-homology, due to Connes (see [3] for more information) and Connes-Moscovici [4].

To proceed, for any element A belonging to the Schatten ideal $\mathcal{L}^p$ of operators on the Hilbert space of L_2 sections in a complex vector bundle over the compact smooth manifold M we consider its support $Supp(A) \subset M \times M$ given by the support of its distributional kernel.

The support filtration of $Supp(A) \subset M \times M$ gives a filtration of the elements of the Schatten ideal.

The support-filtration on each of the factors of the $\gamma = \sum A_0 \otimes_{\mathbb{C}} A_1 \otimes_{\mathbb{C}}, ..., \otimes_{\mathbb{C}} A_k \in C_k^{\lambda}(\mathcal{L}^p)$ gives a filtration in $C_k^{\lambda}(\mathcal{L}^p)$ in the same manner in which Alexander-Spanier co-homology is defined. To be more specific, we will say that γ has support in $U \subset M \times M$ iff $Supp(A_i) \subset U$ for any i.

By definition, $C_k^{\lambda,U}(\mathcal{L}^p)$ consists of those chains γ whose supports lay in U.

By analogy with the definition of the Alexander-Spanier co-homology,

$$H_*^{\lambda,loc}(\mathcal{L}^p) \tag{6.1}$$

is the homology of the sub-complex $C_*^{\lambda,U}(\mathcal{L}^p)$ consisting of those chains γ whose supports are contained in the subset $U \subset M \times M$, with U sufficiently small.

The K-theory groups $K^{loc,i}(\mathcal{L}^p)$ are defined analogously.

These groups will be called *local* cyclic homology, resp. *local* K-theory.

An extension of these constructions to more general Banach algebras will be discussed elsewhere.

We will show in Section 8, Proposition 8.2 and Propsition 8.3, that the *local* cyclic homology and the real *local* K-theory of the algebra of smoothing operators is at least as big as the de Rham cohomology and ordinary real K-theory, respectively.

7 Applications of the *Local* Cyclic Homology

In this section we show that the Connes-Moscovici local index Theorem 3.1 may be reformulated by using our notions of *local* structures introduced in Section 6.

In Section 3 we summarised the Connes-Moscovici construction of the index class. It is obtained as the result of the composition of two constructions

$$K^1(C(S^*M)) \xrightarrow{\mathbf{R}} K^0(\mathcal{K}_M) \xrightarrow{\tau} H_{ev}^{AS}(M), \tag{7.1}$$

where $K^0(\mathcal{K}_M)$ consists of differences of stably homotopy classes of smooth idempotents with *arbitrary* supports.

In [4], Pg. 352 it is clearly stated that the connecting homomorphism ∂_1 takes values in $K^0(\mathcal{K}_M) \simeq \mathbb{Z}$ and that the information carried by it is solely the index of the operator. However, [4] states also that by pairing the residue operator $\mathbf{R}$ with the Alexander-Spanier cohomology (which is *local*) one recovers the whole co-homological information carried by the symbol, as stated by Theorem 3.1 of Section 3.

Let S denote the unitarized algebra of smoothing operators on the compact smooth manifold M

$$S := \mathbb{C}.1 + \mathcal{L}^{\infty}(M). \tag{7.2}$$

All results of this section will remain valid if we replace smoothing operators by trace class operators.

The leading ideas of the paper are: -1) to consider $\mathbf{R}$ not only as element of $K^0(S)$, but rather as an element of $K^{0,loc}(S)$, -2) show that R has a genuine Chern character belonging to the *local* cyclic homology $H_{\Lambda}^{\lambda,loc,ev}(M_2(S))$ over the separable ring $\Lambda := \mathbb{C}+\mathbb{C}e$, where $e = \mathbf{P}_2$ is the trivial idempotent involved in the construction of the residue operator R, see formula (20) and Lemma 5.4, Section 5.1.

7.1 *Local* Chern Character of the Residue Operator R

Let the ring L from Section 5.1 be given by

$$\Lambda = \mathbb{C} + \mathbb{C}e \tag{7.3}$$

We assume that $\mathcal{U}$ is quasi-stable under products and stable under products by elements of Λ; (here, by quasi-stable we intend a property analogous to that refering to products of compact operators). This is the space of operators with *small* supports. We apply next the considerations made in Section 5 to produce *local* cyclic homology $H_{\Lambda,*}^{\lambda,loc}(S)$; here, $S) := \mathcal{S} + \mathbb{C}.1$.

Remark 7.1. Let $\mathbf{R} = \mathbf{P} - \mathbf{e}$, where $\mathbf{P}$ and $\mathbf{e}$ are idempotents. Then $\mathbf{R}$ satisfies the identity

$$\mathbf{R}^2 = \mathbf{R} - (\mathbf{eR} + \mathbf{Re}). \tag{7.4}$$

Notice that if R were an idempotent, the term $\mathbf{eR} + \mathbf{Re}$ would be absent.

Theorem 7.2. *Let $\mathbf{P}, \mathbf{e}$ be idempotents in $\mathcal{U}$ and $\mathbf{R} = \mathbf{P} - \mathbf{e}$.*

Then, for any even number q

$$\tau_q(\mathbf{R}) := R \otimes_{\Lambda} \mathbf{R} \otimes_{\Lambda} ..... \otimes_{\Lambda} \mathbf{R} \otimes_{\Lambda} \in C_q^{\Lambda}(\mathcal{U}) \tag{7.5}$$

is a local cyclic cycle of the algebra S over the ring Λ.

Proof. It is clear that $\tau_q(\mathbf{R})$ is cyclic.

To simplify the notation, limited to this proof, we write $\otimes = \otimes_\Lambda$ and we omit the last tensor product. We have

$$b'(\mathbf{R} \otimes \mathbf{R} \otimes ..... \otimes \mathbf{R}) = \sum_{r=0}^{r=q-1} (-1)^r \mathbf{R} \otimes \mathbf{R} \otimes ... \otimes \mathbf{R}^2 \otimes ... \otimes \mathbf{R} \tag{7.6}$$

$$= \sum_{r=0}^{r=q-1} (-1)^r \mathbf{R} \otimes ... \otimes \mathbf{R} \otimes [\mathbf{R} - (\mathbf{eR} + \mathbf{Re})] \otimes \mathbf{R} \otimes ... \otimes \mathbf{R} \tag{7.7}$$

$$= \sum_{r=0}^{r=q-1} (-1)^r \mathbf{R} \otimes ... \otimes \mathbf{R} \otimes \mathbf{R} \otimes \mathbf{R} \otimes ... \otimes \mathbf{R} \tag{7.8}$$

$$- \sum_{r=0}^{r=q-1} (-1)^r \mathbf{R} \otimes ... \otimes \mathbf{R} \otimes (\mathbf{eR} + \mathbf{Re}) \otimes \mathbf{R} \otimes ... \otimes \mathbf{R} \tag{7.9}$$

$$= - \sum_{r=0}^{r=q-1} (-1)^r \mathbf{R} \otimes ... \otimes \mathbf{R} \otimes (\mathbf{eR} + \mathbf{Re}) \otimes \mathbf{R} \otimes ... \otimes \mathbf{R} \tag{7.10}$$

$$= -\{ \sum_{r=0}^{r=q-1} (-1)^r \mathbf{R} \otimes ... \otimes \mathbf{R} \otimes \mathbf{eR} \otimes \mathbf{R} \otimes ... \otimes \mathbf{R} \tag{7.11}$$

$$+ \quad \sum_{r=0}^{r=q-1} (-1)^r \mathbf{R} \otimes ... \otimes \mathbf{R} \otimes \mathbf{R} \otimes \mathbf{eR} \otimes ... \otimes \mathbf{R}\} \tag{7.12}$$

$$= -\{ \, \mathbf{eR} \otimes ... \otimes \mathbf{R} \otimes \mathbf{R} \otimes \mathbf{R} \otimes ... \otimes \mathbf{R} \tag{7.13}$$

$$+(-1)^{q-1} \mathbf{R} \otimes ... \otimes \mathbf{R} \otimes \mathbf{R} \otimes \mathbf{R} \otimes ... \otimes \mathbf{Re} \, \} = 0, \tag{7.14}$$

which completes the proof. □

Definition 7.3. *The Chern character of the residue operator* $\mathbf{R}$ *is the system of* local *cyclic homology classes of the algebra* S *over the ring* Λ

$$Ch(\mathbf{R}) := \{[\, Ch_q(\mathbf{R}) \,]\}_{q=0,1,...} \in H^{loc}_{\Lambda,ev} \tag{7.15}$$

where

$$Ch_q(\mathbf{R}) := \frac{(2\pi i)^q (2q)!}{q!} \otimes_\Lambda^{2q+1} \mathbf{R} \tag{7.16}$$

7.2 Pairing of $H^{\lambda,loc}_{\Lambda,*}$ with Alexander-Spanier Co-homology.

Given that the multiplication of operators on M by smooth functions on M does not increase distributional supports of operators and that smooth functions on M commute with Λ, the Connes-Moscovici pairing Lemma 5.12, Section 5.3 passes to the the factor spaces $C^{\lambda,loc}_{\Lambda,*}$. The same formula (63) gives an induced pairing

$$\cap_{\Lambda,loc} : C^{\lambda,loc}_{\Lambda,k} \otimes_{\mathbb{C}} \Omega^k_{AS}(M) \longrightarrow \mathbb{C}, \tag{7.17}$$

where $\Omega^k_{AS}(M)$ denotes Alexander-Spanier k-cochains on M. It is understood that in order for the cup product to be defined it is required that both factors have the same degree.

$Ch((R))$ being already a *local* cyclic cycle of the algebra S over the ring Λ, we get the following interpretation of the formulas (24), (25)

$$Ind(\mathbf{a} \otimes_{\mathbb{C}} \phi) = \frac{q!}{(2\pi i)^q (2q)!} Ch_q(\mathbf{R}) \ \cap_{\Lambda,loc} \ \phi. \tag{7.18}$$

Here, ϕ is any Alexander-Spanier co-homology class of degree $2q$ represented by an anti-symmetric function defined on M^{2q+1} with small support about the diagonal.

In the context of *local* cyclic homology, the formula (64) and Definition 5.11 of Section 5.3 become

$$H^{\lambda,loc}_{ev}(Op) \ := \sum_{q=even} H^{\lambda,loc}_q(Op) \tag{7.19}$$

and

Definition 7.4.

$$H^{\lambda,loc}_{ev}(Op) \lozenge H^{ev}_{AS}(M) \ =$$
$$= \{ \ \phi_q \ \otimes \ \eta^q \mid \{\phi_q \ \cap \ \eta^q\}_{q=0,2,4,\ldots} \in H^{\lambda,loc}_{ev}(Op) \otimes H^{ev}_{AS}(M)\} \tag{7.20}$$

8 Connes-Moscovici Local Index Theorem.

The Connes-Moscovici Local Index Theorem 3.1 (Connes-Moscovici [4], Theorem 3.9) becomes

Theorem 8.1. *(Connes-Moscovici* Local *Index Theorem)*

Let $\mathbf{A}$ *be an elliptic pseudo-differential operator on the smooth, compact manifold* M *of even dimension and let* $[\phi] \in H^{2q}_{comp}(M)$. *Let* $\mathbf{a}$ *be its symbol and let* $\mathbf{R_a}$ *be its corresponding local residue smoothing operator. Then*

$$Ch(\mathbf{R_a}) \ \cap \ [\phi] \ = \ (-1)^{dimM} < \ Ch \, \sigma(\mathbf{A}) \tau(M)[\phi], \ [T^*M] \ > \tag{8.1}$$

where $\tau(M) = Todd(TM) \otimes C$ *and* $H^*(T^*M)$ *is seen as a module over* $H^*_{comp}(M)$.

We will show that this reformulation of the Connes-Moscovici local index theorem, combined with Poincaré duality, imply that $H^{\lambda,loc}_{\Lambda,*}(S)$ contains $K^0_{comp}((T^*M)) \otimes \mathbb{R}$.

To show this, we introduce two homomorphisms

$$\alpha : K^1(C^0(S(T^*M)) \longrightarrow H^{\lambda,loc}_{\Lambda,ev}(S)$$
$$\alpha(\mathbf{a}) := Ch(\mathbf{R_a}) \tag{8.2}$$

$$\beta : H^{\lambda,loc}_{\Lambda,ev}(S) \longrightarrow H^{AS}_{ev}(M)$$
$$\beta(\phi) \in Hom_{\mathbb{C}}(H^{ev}_{AS}(M), \mathbb{R}) = H^{AS}_{ev}(M)$$
$$(\beta(\phi))(\eta) := \phi \cap \eta \in \mathbb{C}. \tag{8.3}$$

Proposition 8.2. *Let* M *be a smooth, compact manifold* M *of even dimension.*

Then the composition

$$K^1(C^0(S(T^*M)) \otimes \mathbb{C} \xrightarrow{\alpha} H^{\lambda,loc}_{\Lambda,ev}(S) \xrightarrow{\beta} H^{AS}_{ev}(M, \mathbb{C}) \tag{8.4}$$

is an isomorphism.

Proof. We have

$$[(\beta \circ \alpha)(\mathbf{a}))](\phi) = [\ \beta\ (Ch(\mathbf{R}(\mathbf{a}))\ ](\phi) =$$

(Theorem 7.2)

$$= \ (-1)^{dimM} < \ Ch\, \sigma(\mathbf{A})\tau(M)[\phi],\ [T^*M]\ > \tag{8.5}$$

Suppose $\mathbf{a}$ is in the kernel of $\beta \circ \alpha$. This means that the RHS of (96) is zero for any even degree Alexander-Spanier cohomology class ϕ. Poicaré duality implies that the co-homology class $Ch\ \sigma(\mathbf{A})\tau(M)$ is zero. Given that the Todd class $\tau(M)$ is invertible, it follows that $Ch\, \sigma(\mathbf{A})$ is zero. As the Chern character is an isomorhism (over reals) from equivalence classes of virtual bundles to $H^{AS,ev}_{comp}(T^*M)$, it follows that $\mathbf{a} \in K^0(T^*M)$ is the zero element. Therefore, $\beta \circ \alpha$ is a monomorphism.

Given that $K^0(T^*M) \otimes \mathbb{R}$ and $H^{AS}_{ev}(M)$ have the same dimension, it follows that $\beta \circ \alpha$ is an isomorphism. □

The same result extends to the Schatten ideal $\mathcal{L}^1$.

Proposition 8.3. *The* local *cyclic homology, resp. the* local *real K-theory, of the algebra of smoothing operators is at least as big as the de Rham cohomology, resp. ordinary real K-theory.*

The same result extends to the Schatten ideal $\mathcal{L}^1$.

Remark 8.4. The condition $dimM = even$ in Theorem 8.1 may be dropped. In fact, our result follows from the Connes-Moscovici local index theorem, which is valid for manifolds of any dimension (see [4] Pg. 368).

Remark 8.5. Replacing the cyclic homology of the algebra of smoothing operators (which is trivial) by its *local* cyclic homology, one recovers the possibility to control at least that homological information which makes the index formula interesting.

Given that the local cyclic homology $H^{\lambda,loc}_{\Lambda,*}(S)$ is interesting, we are entitled to introduce a new pairing $\sqcap$, called *square cap* product.

Definition 8.6. *The* square *cup product*

$$\sqcap : H^{\lambda,loc}_{\Lambda,k}(S) \otimes_{\mathbb{C}} H^k_{AS}(M) \longrightarrow H^{\lambda,loc}_{\Lambda,k}(S) \tag{8.6}$$

is defined on the elements

$$\phi_k = \sum A_0 \otimes_{\mathbb{C}} A_1 \otimes_{\mathbb{C}}, ..., \otimes_{\mathbb{C}} A_k \in C^{\lambda,loc}_k(S) \tag{8.7}$$

$$\eta^k = \sum f_0 \otimes_{\mathbb{C}} f_1 \otimes_{\mathbb{C}}, ..., \otimes_{\mathbb{C}} f_k, \quad f_i \in C^\infty(M), \tag{8.8}$$

by the convergent double series

$$\phi_k \ \sqcap \ \eta^k := \sum \sum A_0 f_0 A_1 f_1, ..., A_k f_k \in C^{\lambda,loc}_k(S). \tag{8.9}$$

It is important to notice that square cap product $\sqcap$ does not require to manufacture trace class operators. The outcome of the square cup product is not a system of scalars, but rather a system of tensor products of homology and co-homoogy classes. The definition makes sense for S replaced by more general operator algebras.

9 Further Extensions and Problems

The study of the connections between our procedure presented above and the previously known techniques and results, as well as a more detailed study of the arguments presented in Sections 9.1-9.3 will be discussed elsewhere.

9.1 *Local* Chern Character on Banach Algebras Extensions

In analogy with the short exact sequence (19) of operator algebras considered in Section 2, we might want to consider the more general situation consisting of an exact sequence of Banach algebras

$$0 \longrightarrow \mathcal{S} \xrightarrow{\iota} \mathcal{B} \xrightarrow{\sigma} \mathcal{C} \longrightarrow 0; \tag{9.1}$$

where $\mathcal{B}$ and $\mathcal{C}$ are unitary.

Let $[\mathbf{a}] \in K^1(\mathcal{C})$ be the K-theory class of $\mathbf{a} \in GL_n(\mathcal{C})$. Let $A, B \in \mathcal{M}_n(\mathcal{B})$ such that $\sigma(A) = \mathbf{a}$ and $\sigma(B) = \mathbf{a}^{-1}$. In analogy with the construction described in Section 2, we define $S_0 = 1 - BA$ and $S_1 = 1 - AB$. Clearly, $S_0, S_1 \in \mathcal{M}_n(\mathcal{S})$. Next, with these elements one manufacture the elements $\mathbf{L}, \mathbf{P}, \mathbf{R}$ and $\mathbf{e}$.

Then, $\mathbf{R} \in \mathcal{M}_{2n}(\mathcal{S})$, $\quad \mathbf{R} = \mathbf{P} - \mathbf{e}$ where $\mathbf{P}$ and $\mathbf{e}$ are idempotents.

We assume also that for the elements of the algebras $\mathcal{S}, \mathcal{B}, \mathcal{C}$ there is some sort of notion of *locality*, as for pseudo-differential operators, and hence that one may define *local* cyclic homology of the algebra S. The notion of *locality* extends to matrices with entries in these algebras. Therefore, we may define the *local* Chern character $Ch(\mathbf{R}) \in H^{\lambda,loc}_{\Lambda,ev}$ of the class $\mathbf{a}$ as explained in Section 7.1.

9.2 *Local* Cyclic Homology of the Algebras of Schatten Operators

Proposition 8.2, Section 8 shows that the *local* cyclic homology of the ideal of smoothing operators or of trace class operators is interesting. It is important to compute this *local* cyclic homology.

9.3 Homological Local Index Theorem

It is clear that our construction of the Chern character discussed in Section 7.1 applies also in the case of K-homology.

As our Chern character $Ch(\mathbf{R})$ belongs to $H^{\lambda,loc}_{even}$, and given that for its calculation it is not necessary to work with *trace class* operators, we expect our considerations to apply in more general situations.

Baum and Douglas [1] define the Chern character in *K-homology*. It is important to study the connection between their Chern character and the Chern character we may produce as explained above.

It is an interesting problem to express $Ch(\mathbf{R})$ in local data, where $\mathbf{R}$ is the residue operator defined in Section 3, i.e to prove a dual version of the Connes-Moscovici local index theorem (Theorem 3.1).

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