An. Şt. Univ. Ovidius Constanţa Vol. 20(2), 2012, 71–78

First and second cohomology group of a bundle

Adelina Manea

Abstract

Let (E, π, M) be a vector bundle. We define two cohomology groups associated to π using the first and second order jet manifolds of this bundle. We prove that one of them is isomorphic with a Čech cohomology group of the base space. The particular case of trivial bundle is studied.

1 Preliminaries

In this paper we define two cohomology groups associated to a vector bundle, using the first and the second derivative of its global sections. These groups generalize the Mastrogiacomo cohomology introduced in [5]. In the last years there are concerns about the cohomology related to jets of manifolds [1]. We prove that the first cohomology group is isomorphic with a certain Čech cohomology group of the base manifold (Theorem 2.1) and then we prove that the first and the second cohomology groups of π are isomorphic (Theorem 3.1). In the case of a trivial bundle these groups are isomorphic with the first de Rham cohomology group of the base space (Theorem 4.1).

For the begining, we introduce some notions about first and second order jets of a bundle, using [2], [6].

Let M be a m-dimensional paracompact manifold and $\left\{(U, u = (x^i)_{i=\overline{1,m}})\right\}$ a differentiable atlas on it. Let (E, π, M) be a vector bundle, with the fiber dimension equal to n. For a local chart $\left(U, \left(x^i\right)\right)$ in M, the adapted coordinate system in $\pi^{-1}(U) \subset E$ is $\left(x^i, y^\alpha\right)$, where $i = \overline{1,m}$, $\alpha = \overline{1,n}$. We shall use

Key Words: cohomology, jet manifolds.
2010 Mathematics Subject Classification: 55R10, 58A20.
Received: August, 2011.
Accepted: February, 2012.

the same notation x^i for the coordinate functions x^i from M and $x^i \circ \pi$ from the manifold E. For two local charts $(\pi^{-1}(U), (x^i, y^\alpha))$, $(\pi^{-1}(\tilde{U}), (\tilde{x}^{i_1}, \tilde{y}^{\alpha_1}))$ which domains overlap, in $U \cap \tilde{U}$ we have

$$\tilde{x}^{i_1} = \tilde{x}^{i_1}(x^i), \quad \tilde{y}^{\alpha_1} = M_\alpha^{\alpha_1}(x) y^\alpha, \quad M_\alpha^{\alpha_1}(x) \in GL(n, \mathbf{R}). \tag{1}$$

A local section of the bundle π in $x \in M$ is a map $\Phi : V \to E$, $x \in V \subset M$ such that $\pi \circ \Phi = 1_V$. The set of all local sections of π in x is denoted by $\Gamma_x(\pi)$. The local representation of a section is (x^i, Φ^α), where $\Phi^\alpha = y^\alpha \circ \Phi$. In every local chart on E there are the local sections $e_\alpha : \pi^{-1}(U) \to M$ given by $y^\alpha \circ e_\beta = \delta_\beta^\alpha$, hence a local section Φ could be written as a sum

$$\Phi = \Phi^\alpha e_\alpha,$$

where δ_β^α is the Kronecker symbol.

We say that two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are *1-equivalent* at x if $\Phi(x) = \Psi(x)$ and if in some adapted coordinate system $\left(x^i, y^\alpha\right)$ around $\Phi(x)$, $y^\alpha(\Phi(x)) = 0$

$$\frac{\partial \Phi^\alpha}{\partial x^i}(x) = \frac{\partial \Psi^\alpha}{\partial x^i}(x), \tag{2}$$

for $i = \overline{1, m}$ and $\alpha = \overline{1, n}$. The equivalence class containing Φ is called the *1-jet* of the section Φ at x and is denoted $j_x^1 \Phi$. Of course the above conditions have geometrical meaning.

Remark 1.1. *The usual definition of the jet of a section of a bundle is slightly modificated in this paper, where we asked that $j_x^1\Phi$ to contain those sections whose coordinate representation vanishes at x. This condition has geometrical meaning for vector bundles.*

We say that two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are *2-equivalent* at x if they are *1-equivalent* at x and if in some adapted coordinate system $\left(x^i, y^\alpha\right)$ around $\Phi(x)$,

$$\frac{\partial^2 \Phi^\alpha}{\partial x^i \partial x^j}(x) = \frac{\partial^2 \Psi^\alpha}{\partial x^i \partial x^j}(x), \tag{3}$$

for every $1 \leq i \leq j \leq m$ and $\alpha = \overline{1, n}$. The equivalence class containing Φ is called the *2-jet* of the section Φ at x and it is denoted $j_x^2 \Phi$.

The *first jet manifold* of π is the set

$$J^1\pi = \left\{ j_x^1 \Phi \mid x \in M, \Phi \in \Gamma_x(\pi) \right\}.$$

It is a $(m + mn)$-dimensional manifold with local charts $\left(U^1, u^1\right)$, where $U^1 = \left\{ j_x^1 \Phi \in J^1\pi \mid \Phi(x) \in U \right\}$, $(U, (x^i, y^\alpha))$ a local chart on E, and the functions $u^1 = \left(x^i, y_i^\alpha\right)$, are defined by

$$x^i\left(j_x^1 \Phi\right) = x^i(x), \quad y_i^\alpha\left(j_x^1 \Phi\right) = \frac{\partial \Phi^\alpha}{\partial x^i}(x). \tag{4}$$

By a straightforward calculation, taking into account relation (1), we obtain the following changing rules for local coordinates in $J^1\pi$

$$\tilde{y}_{i_1}^{\alpha_1} = \frac{\partial x^i}{\partial \tilde{x}^{i_1}} M_\alpha^{\alpha_1} y_i^\alpha, \tag{5}$$

Moreover, $J^1\pi$ has a structure of a fiber bundle over M, with the surjection submersion

$$\pi_1 : J^1\pi \to M, \quad \pi_1 \left(j_x^1 \Phi\right) = x, \tag{6}$$

The *second order jet manifold* of π is the set

$$J^2\pi = \left\{ j_x^k \Phi \mid x \in M, \Phi \in \Gamma_x (\pi) \right\}.$$

It is a manifold with local charts $\left(U^2, u^2\right)$, where $U^2 = \left\{ j_x^2 \Phi \in J^2\pi \mid \Phi (x) \in U \right\}$, and the functions $u^2 = \left(x^i, y_i^\alpha, y_{ij}^\alpha\right)_{1 \le i \le j \le m}$, are defined by

$$x^i \left(j_x^k \Phi\right) = x^i (x), \quad y_i^\alpha \left(j_x^2 \Phi\right) = \frac{\partial \Phi^\alpha}{\partial x^i} (x), \quad y_{ij}^\alpha \left(j_x^2 \Phi\right) = \frac{\partial^2 \Phi^\alpha}{\partial x^i \partial x^j} (x), \tag{7}$$

for $1 \le i \le j \le m$ and $\alpha = \overline{1, n}$. We have that $\left(J^2\pi, \pi_2, M\right)$ and $\left(J^2\pi, \pi_{2,1}, J^1\pi\right)$ are bundles, with the surjection submersions

$$\pi_2 : J^2\pi \to M, , \quad , \pi_2 \left(j_x^2 \Phi\right) = x, \tag{8}$$

$$\pi_{2,1} : J^2\pi \to J^1\pi, \quad \pi_{2,1}(j_x^2 \Phi) = j_x^1 \Phi$$

which satisfy

$$\pi_1 \circ \pi_{2,1} = \pi_2. \tag{9}$$

2 The first cohomology group of a vector bundle

The first prolongation of a section $\Phi \in \Gamma(\pi)$ is a section denoted by $j^1\Phi \in \Gamma(\pi_1)$ where $j^1\Phi$ is locally given by

$$(x^i, \frac{\partial \Phi^\alpha}{\partial x^i}).$$

The map

$$j^1 : \Gamma(\pi) \to \Gamma(\pi_1), \quad j^1(\Phi) = j^1\Phi, \quad (\forall)\Phi \in \Gamma(\pi),$$

is called the first derivation of sections of π.

Let U be an open domain in M. A section $\psi \in \Gamma_U(\pi_1)$ has the coordinate representation (x^i, ψ_i^α). If it is a prolongation of a local section Φ of π, then in U we have

$$\psi_i^\alpha = \frac{\partial \Phi^\alpha}{\partial x^i}.$$

Definition 2.1. *A global section ψ of π_1 is called j^1-**exact** if it is a prolongation of a global section Φ in π.*

*A global section ψ of π_1 is called j^1-**closed** if for every open domain $U \in M$ there is a local section Φ_U of π such that $\psi|_U = j^1\Phi_U$.*

We denote by $C^1(\pi)$ the set of j^1-*closed* sections of π_1. Obviously, there is the following inclusion of subgroup $Imj^1 \subseteq C^1(\pi)$. We call *the first cohomology group* of the bundle π the factor group

$$H^1(\pi) = \frac{C^1(\pi)}{Imj^1}. \tag{10}$$

We consider now the presheaf S of germs of local sections of π whose first jet vanishes, and the 1-dimensional Čech cohomology group of M with coefficients in S, $\check{H}^1(M, S)$.

For a local finite open covering of M, $\mathcal{U} = \{U_k\}_{k\in I}$, where U_k are domains of local charts on M with local adapted coordinates (x_k^i, y_k^α), in E, we have

$$S(U_k) = \{\Phi : U_k \to E| \quad j^1\Phi = 0, \}.$$

Remark 2.1. *Condition $j^1\Phi = 0$ for $\Phi \in S(U_k)$ means $\frac{\partial \Phi^\alpha}{\partial x_k^i} = 0, (\forall) i = \overline{1,m}, \alpha = \overline{1,n}$.*

Let be $\psi \in C^1(\pi)$. For a local finite open covering of M, $\mathcal{U} = \{U_k\}_{k\in I}$, where U_k are domains of local charts, there are some local sections $\{\Phi_k \in \Gamma_{U_k}(\pi)\}_{k\in I}$ such that in U_k, $\psi|_{U_k} = j^1\Phi_k$. Hence we obtain a familly $\{\Phi_k\}_{k\in I}$ of local sections corresponding to ψ. In $U_k \cap U_l$ we have $j^1\Phi_k = j^1\Phi_l$. The sections $\Phi_{kl} = \Phi_k - \Phi_l$, defined in $U_k \cap U_l$ are satisfying

$$j^1(\Phi_k - \Phi_l) = 0,$$

so they belong to $S(U_k \cap U_l)$. Moreover, the family $\{\Phi_{kl}\}$ is a 1-cocycle and its cohomology class belongs to $\check{H}^1(\mathcal{U}, S)$. We denote by ψ^1 the inductive limit of the cohomology class of $\{\Phi_{kl}\}$ over the set of open coverings of M, filtred to right.

We obtain a function

$$\varphi : C^1(\pi) \to \check{H}^1(M, S), \quad \varphi(\psi) = \psi^1.$$

Proposition 2.1.

$$Ker\varphi = Imj^1.$$

Proof: Let $\psi \in Ker\varphi$, hence it is a j^1-closed section of π_1 such that $\psi^1 = 0$. It follows that the family of sections $\{\Phi_{kl}\}$ corresponding to ψ in a covering $\mathcal{U} = \{U_k\}_{k\in I}$ is a coborder of a 0-cochain $\{p_k\}_{k\in I}$, $p_k \in S(U_k)$, so

$$\Phi_{kl} = p_l - p_k.$$

But $\Phi_{kl} = \Phi_k - \Phi_l$, where $\psi|_{U_k} = j^1\Phi_k$. We obtain a family of sections $\{\Phi_k + p_k\}_{k\in I}$ such that in $U_k \cap U_l$ we have

$$\Phi_k + p_k = \Phi_l + p_l. \tag{11}$$

We define a section $\Phi : M \to E$ putting $\Phi|_{U_k} = \Phi_k + p_k$. This is a global section and, taking into account that $j^1 p_k = 0$, it results that $j^1\Phi|_{U_k} = \psi|_{U_k}$ for every $k \in I$. Hence $\psi \in Imj^1$.

Conversely, for every $\psi = j^1\Phi$, the family of sections Φ_{kl} is a family of nulls sections, so the cohomology class vanishes, q.e.d.$\square$

Using the fundamental theorem of isomorphism for groups, a consequance of Proposition 2.1 is:

Proposition 2.2. *The map φ induces an injective morphism $\varphi^* : H^1(\pi) \to \check{H}^1(M, S)$ given by $\varphi^*([\psi]) = \psi^1$.*

Theorem 2.1. *The morphism φ^* is an isomorphism between the first cohomology group of the bundle π and the 1-dimensional Čech cohomology group $\check{H}^1(M, S)$.*

Proof: Let ψ^1 be a cohomology class from $\check{H}^1(M, S)$. For a local finite open covering of M, $\mathcal{U} = \{U_k\}_{k\in I}$, there is a familly of sections $\{\Phi_{kl} \in S(U_k \cap U_l)\}_{k,l\in I}$ which is a 1-cocycle and the inductive limite of its cohomology class is ψ^1. We have in an intersection of domains which overlap $U_k \cap U_l \cap U_t$ that

$$\Phi_{kl} = \Phi_{kt} - \Phi_{lt}. \tag{12}$$

For a partition of unity subordonated to $\mathcal{U}$, we can define the sections $s_k = \sum_{t\in I} \Phi_{kt} a_t$. We obtain a familly of sections $\{s_k\}_{k\in I}$ and, taking into account relation (12) in $U_k \cap U_l \cap U_t$ we obtain:

$$s_k - s_l = \sum_{t\in I} \Phi_{kl} a_t = \Phi_{kl}.$$

We define a section ψ in π_1 putting $\psi|_{U_k} = j^1 s_k$. Since $\Phi_{kl} \in S(U_k \cap U_l)$, we obtain ψ is a global section. Moreover, it is j^1-closed and $\varphi(\psi) = \psi^1$. $\square$

3 The second cohomology group of a vector bundle

The second prolongation of a section $\Phi \in \Gamma(\pi)$ is a section denoted by $j^2\Phi \in \Gamma(\pi_1)$, where $j^2\Phi(x)$ is locally given by

$$(x^i(x), \frac{\partial \Phi^\alpha}{\partial x^i}(x), \frac{\partial^2 \Phi^\alpha}{\partial x^i \partial x^j}(x)).$$

The map

$$j^2 : \Gamma(\pi) \to \Gamma(\pi_2), \quad j^2(\Phi) = j^2\Phi, \quad (\forall)\Phi \in \Gamma(\pi),$$

is called the second derivation of sections of π.

Let U be an open domain in M. A section $\Psi \in \Gamma_U(\pi_2)$ has the coordinate representation $(x^i, \Psi_i^\alpha, \Psi_{ij}^\alpha)$. If it is a prolongation of a local section Φ of π, then in U we have

$$\Psi_i^\alpha = \frac{\partial \Phi^\alpha}{\partial x^i}, \quad \Psi_{ij}^\alpha = \frac{\partial^2 \Phi^\alpha}{\partial x^i \partial x^j},$$

for every $1 \le i \le j \le m$.

Definition 3.1. *A global section Ψ of π_2 is called j^2-**exact** if it is a prolongation of a global section Φ of π.*

*A global section Ψ of π_2 is called j^2-**closed** if for every open domain $U \in M$ there is a local section Φ_U of π such that $\Psi|_U = j^2\Phi_U$.*

We denote by $C^2(\pi)$ the set of j^2-*closed* sections of π_2. Obviously, there is the following inclusion of subgroup $Imj^2 \subseteq C^2(\pi)$. We call *the second cohomology group* of the bundle π the factor group

$$H^2(\pi) = \frac{C^2(\pi)}{Imj^2}. \tag{13}$$

We define the injective map $D : C^1(\pi) \to C^2(\pi)$, by $D(\psi)$ is locally given by

$$D(\psi_i^\alpha) = (\psi_i^\alpha, \frac{1}{2}(\frac{\partial \psi_i^\alpha}{\partial x^j} + \frac{\partial \psi_j^\alpha}{\partial x^i})).$$

This map is well-defined, as it follows: for an arbitrary $\psi \in C^1(\pi)$, in every domain U from M there is a local section Φ_U such that $\psi|_U = j^1\Phi_U$. Local representation of $D(\psi)$ is $(\frac{\partial \Phi_U^\alpha}{\partial x^i}, \frac{\partial^2 \Phi_U^\alpha}{\partial x^j \partial x^i})$. We obtain that in U we have $D(\psi)|_U = j^2\Phi_U$. Hence $D(\psi) \in C^2(\pi)$.

Proposition 3.1. *The following equalities hold:*

$$D(C(\pi_1)) = C(\pi_2), \quad D(Imj^1(\pi)) = Imj^2(\pi).$$

Proof: The first equality says that D is surjective. Let γ be a j^2-closed section of π_2. In every open domain $U \subset M$ there is a section Φ_U of π such that $\gamma|_U = j^2\Phi_U$. Taking into account relation (9), $\psi = \pi_{2,1} \circ \gamma$ is a global section of π_1 which in every open domain $U \subset M$ verifies $\psi|_U = j^1\Phi_U$, so $\psi \in C^1(\pi)$. Moreover, $D(\psi) = \gamma$.

The second equality folllows by similar arguments.$\square$

As a consequence of the above Proposition we obtain that the map D induces an isomorphism between the cohomology groups $H^1(\pi)$ and $H^2(\pi)$. So, we proved that:

Theorem 3.1. *The first and the second cohomology groups of a bundle are isomorphic.*

4 Case of trivial bundle

When π is the trivial bundle $(M \times \mathbf{R}, \pi, M)$, we can find the Mastrogiacomo's results. The set of global sections of trivial bundle is the space of $\mathbf{C}^\infty$-differentiable functions on M and we denote it by $\Omega^0(M)$. The first jet of a function f at a point x depends only by the first derivatives at that point, and not by the value $f(x)$. The first jet manifold of the trivial bundle, called the first jet manifold of M and denoted J^1M, is:

$$J^1M = \left\{ j^1_x f \mid x \in M, f \in \Omega^0_x(M) \right\},$$

The second jet manifold of M is:

$$J^2M = \left\{ j^2_x f \mid x \in M, f \in \Omega^0_x(M) \right\}.$$

where $\Omega^0_x(M)$ is the set of local sections defined on a domain containing $x \in M$.

The module of sections of J^1M is isomorphic to space of 1-forms $\Omega^1(M)$ of the manifold M. We consider the exterior derivative $d : \Omega^0(M) \to \Omega^1(M)$.

Local coordinates in J^1M are (x^i, ω_i) with $\omega_i(j^1_x f) = \frac{\partial f}{\partial x^i}$. The j^1-closed jets correspond to the closed 1-forms by Poincare Lemma for d, and the cohomology group $H^1(\pi)$ is isomorphic with the first de Rham cohomology group of M. The sheaf S contains functions local constant, so it is the constant sheaf $\mathbf{R}$.

In this case, Theorem 2.1. is exactly the de Rham theorem for the 1-dimensional de Rham cohomology group of M.

The injective map by section 3 is in this case

$$D : \Omega^1(M) \to \Gamma(\pi_2^M), \tag{14}$$

locally given by

$$D(\omega_i) = (\omega_i, \frac{1}{2}(\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i})).$$

We remark that the map D is well-defined since for $(U, (x^i))$, $(\widetilde{U}, (\widetilde{x}^{i_1}))$ two local charts which domains overlap, we have

$$\widetilde{\omega}_{i_1} = \frac{\partial x^i}{\partial \widetilde{x}^{i_1}} \omega_i,$$

$$\frac{1}{2}(\frac{\partial \widetilde{\omega}_{i_1}}{\partial \widetilde{x}^{j_1}} + \frac{\partial \widetilde{\omega}_{j_1}}{\partial \widetilde{x}^{i_1}}) = \frac{\partial^2 x^i}{\partial \widetilde{x}^{i_1} \partial \widetilde{x}^{j_1}} \omega_i + \frac{\partial x^i}{\partial \widetilde{x}^{i_1}} \frac{\partial x^j}{\partial \widetilde{x}^{j_1}} \frac{1}{2}(\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i}),$$

which are exactly the local coordinates changing rules in J^2M. Hence Theorems 2.1 and 3.1 say that:

Theorem 4.1. *The second cohomology group of the trivial bundle* $(M \times \boldsymbol{R}, \pi, M)$, $C^2(M)/Imj^2_M$, *is isomorphic with the first de Rham cohomology group of manifold* M.

References

[1] G. Giachetta, L. Mangiarotti, G. Sardanashvily, *Cohomology of the infinite-order jet space and the inverse problem*, Journal of Mathematical Physics, 42(2001), no. 9, p. 4272-4282.

[2] I. Kolar, P.W. Michor and J. Slovak, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin Heidelberg, 1993.

[3] A. Manea, *Some remarks about first order jets of bundles over foliated manifold,* BSG Proceedings 18, 2011.

[4] R. Miron, I.Pop, *Topologie algebrica. Omologie, omotopie, spatii de acoperire*, Ed. Academiei R.S.R., Bucuresti 1976.

[5] P. Mastrogiacomo, *Coomologia dei campi di getti e coomologia dei campi di tensori covarianti su una varieta a connessione lineare*, Le Matematiche, 31(1976), p. 349-370.

[6] D.J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc. Lecture Note Series, 142, Cambridge U.P., 1989.

[7] D.J. Saunders, *Horizontal forms on jet bundles*, Balkan Journal of Geometry and Its Applications, 15(2010), pp. 149-154.

Adelina MANEA,
Department of Mathematics,
University of Brasov, Romania.
Email: amanea28@yahoo.com