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On the non-generic Tzitzeica-Johnson's Configuration

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Abstract

Working in the context of Hilbert's axiomatic system, we prove that the Tzitzeica-Johnson configuration has sense in neutral geometry. The elements of a Tzitzeica-Johnson configuration allow us to analyse the Euclidean or the non-Euclidean character of the metric of the plane.

1 Introduction

The statement of the analogue of Tzitzeica-Johnson theorem in the neutral geometry is described by the following theorem as it appears in [2].

Theorem 1. *Let H, O_1, O_2, O_3 four points in the neutral plane such that $|O_1H| =$
$= |O_2H| = |O_3H|$ and $H \in Int(\Delta O_1O_2O_3)$. Consider*

$$A = S_{O_1O_2}H, \quad B = S_{O_3O_1}H, \quad C = S_{O_2O_3}H.$$

Then there exists a point O in the neutral plane such that $|OA| = |OB| = |OC|$.

Therefore in the case when $H \in Int(\Delta O_1O_2O_3)$, called the *generic case*, it exists a Tzitzeica-Johnson configuration in the neutral plane and it was denoted by $[H, O_1O_2O_3 \mid O, ABC]$ (see [2]).

In the Euclidean plane, the generic Tzitzeica-Johnson's configuration has the property that $[OA], [OB], [OC], [HO_1], [HO_2], [HO_3]$ all are congruent segments.

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In the neutral plane we have that $[OA], [OB], [OC]$ are congruent and separately that $[HO_1], [HO_2], [HO_3]$ are congruent.

It is natural to compare $[OA]$ and $[HO_1]$ in order to establish certain properties of the neutral plane itself.

To formulate the next result we use the comparison of the angles and of the segments as introduced in [10]. By the notation R we mean the right angle in the plane. By the sign $\Re$ we mean any of the relations $>, =, <$.

The result which allows to highlight the importance of a generic Tzitzeica-Johnson's configuration in the neutral plane in establishing the type of the metric is the following (see [2]).

Theorem 2. *Let $[H, O_1O_2O_3 \mid O, ABC]$ be a generic Tzitzeica-Johnson's configuration in the neutral plane. Let Σ be the sum of the angles of $\triangle O_1O_2O_3$. If $[AO] \ \Re \ [HO_1]$ then $\Sigma \ \Re \ 2R$.*

At this point we introduce a new definition. We call *non-generic Tzitzeica-Johnson configuration*, the geometric figure in the neutral plane such that the point H doesn't lie in the interior of $\Delta O_1O_2O_3$. Our goal becomes to obtain an equivalent theorem in the case of non-generic Tzitzeica-Johnson's configuration. We have to analyze two possibilities.

2 Undecidability in the case $H \in \partial(\Delta O_1O_2O_3)$

Suppose now that H lies on the boundary of $\Delta O_1O_2O_3$. Denote this case by $[H, \partial\,(O_1O_2O_3)\,|O, ABC]$
Without loss of generality (see Figure 1) we can assume that H lies on the segment $[O_1O_3]$, such that $|HO_1| = |HO_2| = |HO_3|$. Then

$$S_{O_1O_3}H = H = B, \quad S_{O_1O_2}H = A \quad S_{O_2O_3}H = C.$$

Denote $\{T\} = AH \cap O_1O_2$ and $\{K\} = CH \cap O_2O_3$. Remark that $\Delta ATO_2 \equiv \Delta HTO_2$, which yields $|O_2A| = |O_2H| (= |O_2B|)$. Similarly from $\Delta CKO_2 \equiv \Delta HKO_2$ we get$|CO_2| = |O_2H| (= |O_2B|)$. Therefore O_2 is equidistant from $A, B = H$, and C.

This shows that when H lies on the boundary, the Tzitzeica-Johnson configuration has the same structure in both Euclidean and non-Euclidean cases. In fact, in this case, one can not decide if the geometry is Euclidean or hyperbolic.

3 The case when H lies in the exterior of $\Delta O_1O_2O_3$: Analysis

Now we analyze the Tzitzeica-Johnson configuration when H lies in the exterior of $\Delta O_1O_2O_3$. For this discussion we refer to Figure 2.

Suppose that $[HO_1$ is between $[HO_2$ and $[HO_3$ as in Figure 4. Let $A = S_{O_1O_2}H$, $B = S_{O_1O_3}H$, and $C = S_{O_2O_3}H$. The rays $[HA, [HB$ and $[HC$ bisect $\angle O_2HO_1, \angle O_1HO_3$, and $\angle O_2HO_3$, respectively. Consider now the perpendicular bisectors of $[AB]$ and $[AC]$. One of them is perpendicular on $[AB]$, the other one is slant. In the Hilbert axiomatic sistem corresponding to Euclidean geometry is easy to see that these perpendicular bisectors meet. On the other hand, in the axiomatic framework corresponding to neutral geometry there is no criterion to decide if the two lines intersect.

So, in the case of a non-generic Tzitzeica-Johnson configuration if there is not O such that $|OA| = |OB| = |OC|$ it results that our geometry is hyperbolic. Let us denote such configuration by $[H, ext\,(O_1O_2O_3)\,|\nexists O, ABC]$.

Let us analyze the Tzitzeica-Johnson configuration in the case when the perpendicular bisectors of $[AB]$ and $[AC]$ meet in a point O in neutral geometry. Let us denote this case by $[H, ext\,(O_1O_2O_3)\,|O, ABC]$. First, let us describe the construction of these perpendicular bisectors. Let L be the midpoint of $[AB]$. Then O_1L is perpendicular on AB since ΔO_1LB and ΔO_1LA are congruent (case SSS). Similarly, denote by D the midpoint of $[AC]$. In the same way, $O_2D \perp AC$ by the same congruence case in $\Delta O_2DA \equiv \Delta O_2DC$. Thus, the perpendicular bisector of $[AB]$ and $[AC]$ are O_1L and O_2D, respectively. By our assumption, let O be their point of intersection. Then the perpendicular bisector of $[BC]$ passes also through O. The construction of the perpendicular bisector of $[BC]$ is similar: the midpoint Y of $[BC]$ determines the line O_3Y which is the perpendicular bisector of $[BC]$. Furthermore, $O \in [O_3Y$. At this point, let us focus on ΔO_2O_3O. We denote by x the equal angles

$$\angle HO_1O_2 = \angle HO_2O_1 = \angle AO_1O_2 = \angle AO_2O_1.$$

Then remark that $\angle OO_2O_3$ has also measure x, since $[O_2O_3$ bisects $\angle HO_2C$ and $[O_2O$ bisects $\angle CO_2A$. By analogy, denoting by y the measure of equal angles $\angle HO_1O_3 = \angle HO_3O_1 = \angle BO_1O_3 = \angle BO_3O_1$, we obtain that $\angle O_2O_3O$ has measure y. For $\angle O_2OO_3$, the reasoning is different. Denote by z the measure of $\angle AO_1O = \angle BO_1O$, and by u the measure of $\angle AOO_1 = \angle BOO_1$. The reasoning made to measure $\angle O_2OO_3$ still holds true due to the fact that $[OO_2$ bisects $\angle COA$ and $[OO_3$ bisects COB. Thus $\angle O_2OO_3$ has measure u. The sum of the angles around O_1 yields $2x+2y+2z = 4R$ (we use the standard notation with R for the measure of a right angle), thus $x + y + z = 2R$. The sum of angles in ΔO_2OO_3 is $x + y + u$.

Denote by $\Re$ one of the symbols $>, =, <$. Denote by Σ the sum of the angles of the triangle O_2OO_3.

Theorem 3. *In the case of the non-generic Tzitzeica-Johnson configuration* $[H, ext\,(O_1O_2O_3)\,|O, ABC]$ *we have* $|O_1H|\Re|OA|$ *if and only if* $\Sigma\Re 2R$.

Proof: In order to complete the proof let us observe that $u\Re z$ if and only if $|O_1H|\Re|OA|$, that is if and only if $\Sigma\Re 2R$.$\square$

One more comment can be made. Since the sum of the angles of a triangle in Hilbert's neutral plane is less than or equal $2R$ (Legendre's theorem) we obtain both in the case of a generic configuration and in the case of a non-generic configuration that the metric can be Euclidean of Hyperbolic only. Except the case when $H \in \partial(\Delta O_1O_2O_3)$, one can decide the type of the metric knowing the type of the configuration and looking at the comparison between the segments $|O_1H|$ and $|OA|$. All these are syntetised in the following.

Theorem 4. *(Main Theorem) Consider a Tzitzeica-Johnson configuration.*

a) $[H, O_1O_2O_3|O, ABC]$*:* $|AO|\Re|HO_1| \iff \Sigma\Re 2R$*, where* Σ *is the sum of angles of the triangle* $O_1O_2O_3$*; the metric can be Euclidean or hyperbolic.*

b) $[H, \partial\,(O_1O_2O_3)\,|O, ABC]$*: the metric can not be decided between Euclidean and hyperbolic.*

c) $[H, ext\,(O_1O_2O_3)\,|\nexists O, ABC]$*: the metric is hyperbolic.*

d) $[H, ext\,(O_1O_2O_3)\,|O, ABC]$*:* $|O_1H|\Re|OA| \iff \Sigma\Re 2R$*, where* Σ *is the sum of angles of the triangle* O_2OO_3*; the metric can be Euclidean or hyperbolic.*

This complets the study of the Tzitzeica-Johnson configuration according to Barbilian's point of view.

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