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On infinitesimal conformal transformations with respect to the Cheeger-Gromoll metric

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Abstract

The present paper deals with the classification of infinitesimal fibre-preserving conformal transformations on the tangent bundle, equipped with the Cheeger-Gromoll metric.

1 Introduction

Let M be an $n-$dimensional manifold and TM its tangent bundle. We denote by $\Im_s^r(M)$ the set of all tensor fields of type (r,s) on M. Similarly, we denote by $\Im_s^r(TM)$ the corresponding set on TM. We also note that in the present paper everything will be always discussed in the $C^\infty-$category, and manifolds will be assumed to be connected and of dimension $n>1$.

Let M be a Riemannian manifold with a Riemannian metric g and X be a vector field on M. Let us consider the local one-parameter group $\{\phi_t\}$ of local transformations of M generated by X. The vector field X is called an *infinitesimal conformal transformation* if each ϕ_t is a local conformal transformation of M. As is well known, the vector field X is an *infinitesimal conformal transformation* or *conformal vector field* on M if and only if there exist a scalar function ρ on M satisfying $L_X g = 2\rho g$, where L_X denotes the Lie derivation with respect to X. Especially, the vector field X is called an *infinitesimal homothetic one* when ρ is constant.

Let TM be the tangent bundle over M and Φ be a transformation of TM. If the transformation Φ preserves the fibres, it is called a *fibre-preserving*

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transformation. Consider a vector field $\tilde{X}$ on TM and the local one-parameter group $\{\Phi_t\}$ of local transformations of TM generated by $\tilde{X}$. The vector field $\tilde{X}$ is called an *infinitesimal fibre-preserving transformation* if each Φ_t is a local fibre-preserving transformation of TM. An infinitesimal fibre-preserving transformation $\tilde{X}$ on TM is called an *infinitesimal fibre-preserving conformal transformation* if each Φ_t is a local fibre-preserving conformal transformation of TM. Let $\tilde{g}$ be a Riemannian or pseudo-Riemannian metric on TM. $\tilde{X}$ is an infinitesimal conformal transformation of TM if and only if there exist a scalar function Ω on TM such that $L_{\tilde{X}}\tilde{g} = 2\Omega\tilde{g}$, where $L_{\tilde{X}}$ denotes the Lie derivation with respect to $\tilde{X}$. An infinitesimal conformal transformation $\tilde{X}$ is called *essential* if Ω depends only on (y^i) with respect to the induced coordinates (x^i, y^i) on TM, and is called *inessential* if Ω depends only (x^i), that is, Ω is a constant on each fibre of TM. In this case, Ω induces a function on M.

The geometry of tangent bundles goes back to the fundamental paper [27] of Sasaki published in 1958. He uses a given Riemannian metric g on a differentiable manifold M to construct a metric $\tilde{g}$ on the tangent bundle TM of M. Today this metric is a standard notion in the differential geometry called the *Sasaki metric (*or *the metric I+III).* For a given Riemannian metric g on a differentiable manifold M, there are well known Riemannian or pseudo-Riemannian metrics on TM, constructed from the metric g, as follows:

1. The complete lift metric or the metric II
2. The metric $I + II$
3. The Sasaki metric or the metric $I + III$
4. The metric $II + III$

where $I = g_{ij}dx^i dx^j$, $II = 2g_{ij}dx^i\delta y^j$, $III = g_{ij}\delta y^i \delta y^j$ are all quadratic differential forms defined globally on the tangent bundle TM over M(for details, see [[33], p.137-177]). Yamauchi [30] proved that every infinitesimal fibre-preserving conformal transformation on TM with the metric $I + III$ is homothetic and it induces an infinitesimal homothetic transformation on M. Also, in the case when M is a complete, simply connected Riemannian manifold with a Riemannian metric, Hasegawa and Yamauchi [11] showed that the Riemannian manifold M is isometric to the standard sphere when the tangent bundle TM equipped with the metric $I + II$ admits an essential infinitesimal conformal transformation. In [9], the first author has studied the similar problem in [30, 31] with respect to the synectic lift metric on the tangent bundle.

All the preceding metrics belong to the wide class of the so-called $g-$*natural metrics* on the tangent bundle, initially classified by Kowalski and Sekizawa

[13] and fully characterized by Abbassi and Sarih [1, 2, 3, 4] (see also [12, 6] for other presentation of the basic result from [13] and for more details about the concept of naturality). Another well-known $g-$natural Riemannian metric g_{CG} had been defined, some years before, by Muso and Tricerri [15] who, inspired by the paper [7] of Cheeger and Gromoll, called it the *Cheeger-Gromoll metric*. The metric was defined by Cheeger and Gromoll; yet, there were Musso and Tricerri who wrote down its expression, constructed it in a more "comprehensible" way, and gave it the name. The Levi-Civita connection of g_{CG} and its Riemannian curvature tensor are calculated by Sekizawa in [28] (for more details see [10]). In [4], Abbassi and Sarih classified Killing vector fields on $(TM,\ g_{CG})$; that is, they found general forms of all Killing vector fields on $(TM,\ g_{CG})$. Also, they showed that if $(TM,\ g_{CG})$ is the tangent bundle with the Cheeger-Gromoll metric g_{CG} of a Riemannian, compact and orientable manifold $(M,\ g)$ with vanishing first and second Betti numbers, then the Lie algebras of Killing vector fields on $(M,\ g)$ and on $(TM,\ g_{CG})$ are isomorphic. Finally, they showed that the sectional curvature of the tangent bundle $(TM,\ g_{CG})$ with the Cheeger-Gromoll metric g_{CG} of a Riemannian manifold $(M,\ g)$ is never constant. In [26], Salimov and Kazimova investigated geodesics on the tangent bundle with respect to the Cheeger-Gromoll metric g_{CG}. Different types of metrics on the tangent bundle of a Riemannian manifold were also studied in [5, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]

The purpose of the present paper is to characterize infinitesimal fibre-preserving conformal transformations with respect to the Cheeger-Gromoll metric g_{CG} on the tangent bundle TM of a Riemannian manifold M. In Theorem 3.2, we give a necessary and sufficient condition for the vector field $\tilde{X}$ on the tangent bundle with the Cheeger-Gromoll metric g_{CG} to be an infinitesimal fibre-preserving conformal transformation. This condition is represented by a set of relations involving certain tensor fields on M of type $(1,0)$ and $(1,1)$. We obtain these relations by giving the formula $L_{\tilde{X}} g_{CG} = 2\Omega\ g_{CG}$ in an adapted frame. The paper ends two Corollaries which follow immediately from Theorem 3.2 and its Proof.

2 Preliminaries

2.1 Cheeger-Gromoll metric on the tangent bundle

Let TM be the tangent bundle over an n-dimensional manifold M, and π the natural projection $\pi : TM \to M$. Let the manifold M be covered by a system of coordinate neighborhoods (U, x^i), where (x^i), $i = 1, ..., n$ is a local coordinate system defined in the neighborhood U. Let (y^i) be the Cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to the natural base $\left\{\frac{\partial}{\partial x^i}|_P\right\}$, P being an arbitrary point in U whose coordinates are (x^i). Then we can introduce local coordinates (x^i, y^i) on open set $\pi^{-1}(U) \subset TM$. We call them *induced coordinates* on $\pi^{-1}(U)$ from (U, x^i). The projection π is represented by $(x^i, y^i) \to (x^i)$. We use the notions $x^I = (x^i, x^{\bar{i}})$ and $x^{\bar{i}} = y^i$. The indices $i, j, ...$ run from 1 to n, the indices $\bar{i}, \bar{j}, ...$ run from $n+1$ to $2n$. Summation over repeated indices is always implied.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the vertical lift ${}^V X$, the horizontal lift ${}^H X$ and the complete lift ${}^C X$ of X are given, with respect to the induced coordinates, by

$$ {}^V X = X^i \partial_{\bar{i}}, \tag{2.1} $$

$$ {}^H X = X^i \partial_i - y^s \Gamma^i_{sk} X^k \partial_{\bar{i}}, \tag{2.2} $$

and

$$ {}^C X = X^i \partial_i + y^s \partial_s X^i \partial_{\bar{i}}, \tag{2.3} $$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ^i_{sk} are the coefficients of the Levi-Civita connection ∇ of g.

Suppose that we are given a tensor field $S \in \Im^p_q(M), q > 1$, on M. We define a tensor field $\gamma S \in \Im^p_{q-1}(TM)$ on $\pi^{-1}(U)$ by

$$ \gamma S = (y^e S^{j_1 \ldots j_p}_{e i_2 \ldots i_q}) \partial_{\bar{j}_1} \otimes ... \otimes \partial_{\bar{j}_p} \otimes dx^{i_2} \otimes ... \otimes dx^{i_q} $$

with respect to the induced coordinates (x^i, y^i)([33],p.12). The tensor field γS defined on each $\pi^{-1}(U)$ determines a global tensor field on TM. We easily see that γA has components, with respect to the induced coordinates (x^i, y^i),

$$ (\gamma A) = \begin{pmatrix} 0 \\ y^i A^j_i \end{pmatrix} $$

for any $A \in \Im^1_1(M)$and $(\gamma A)({}^V f) = 0$, $f \in \Im^0_0(M)$, i.e. γA is a vertical vector field on TM.

Explicit expression for the Lie bracket $[,]$ of the tangent bundle TM is given by Dombrowski [8]. The bracket products of vertical and horizontal vector fields are given by the formulas:

$$ \begin{aligned} \left[{}^H X, {}^H Y\right] &= {}^H [X, Y] - \gamma(R(X, Y)) \\ \left[{}^H X, {}^V Y\right] &= {}^V (\nabla_X Y) \\ \left[{}^V X, {}^V Y\right] &= 0 \end{aligned} $$

for all vector fields X and Y on M, where R is the Riemannian curvature of g defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and $\gamma(R(X, Y))$ is a tensor field of type (1,0) on TM, which is locally expressed as $\gamma(R(X,Y)) = y^s R_{jks}{}^i X^j Y^k \partial_{\bar{i}}$ with respect to the induced coordinates.

Let us consider a vector field $X = X^i\partial_i$ and the corresponding covector field $g_X = g_{ij}X^i dx^j$ on U. Then $\gamma g_X \in \Im^0_0(M)$ is a function on $\pi^{-1}(U)$ defined by $\gamma g_X = y^i g_{ij} X^j$ with respect to the induced coordinates (x^i, y^i). Now, denote by r the norm a vector $y = (y^i)$, i.e. $r^2 = g_{ji}y^j y^i$. The Cheeger-Gromoll metric g_{CG} on the tangent bundle TM is given by

$$
\begin{aligned}
&g_{CG}({}^H X, {}^H Y) = {}^V(g(X,Y)),\\
&g_{CG}({}^H X, {}^V Y) = 0,\\
&g_{CG}({}^V X, {}^V Y) = \frac{1}{1+r^2}\left[{}^V(g(X,Y)) + (\gamma g_X)(\gamma g_Y)\right],
\end{aligned}
$$

for all $X,\ Y \in \Im^1_0(M)$, where ${}^V(g(X,Y))$=$(g(X,Y)) \circ \pi$.

2.2 Basic formulas in adapted frames

With a torsion-free affine connection ∇ given on M, we can introduce on each induced coordinate neighborhood $\pi^{-1}(U)$ of TM a frame field which is very useful in our computation. In each local chart $U \subset M$, we put $X_{(j)} = \dfrac{\partial}{\partial x^j}, j = 1, ..., n$. Then from (2.1) and (2.2), we see that these vector fields have, respectively, local expressions

$$
\begin{aligned}
{}^H X_{(j)} &= \delta^h_j \partial_h + (-y^s \Gamma^h_{sj})\partial_{\bar{h}}\\
{}^V X_{(j)} &= \delta^h_j \partial_{\bar{h}}
\end{aligned}
$$

with respect to the natural frame $\{\partial_h, \partial_{\bar{h}}\}$, where δ^h_j-Kronecker delta. These $2n$ vector fields are linear independent and generate, respectively, the horizontal distribution of ∇ and the vertical distribution of TM. We have call the set $\left\{{}^H X_{(j)}, {}^V X_{(j)}\right\}$ the frame adapted to the affine connection ∇ in $\pi^{-1}(U) \subset TM$. On putting

$$
\begin{aligned}
E_j &= {}^H X_{(j)},\\
E_{\bar{j}} &= {}^V X_{(j)},
\end{aligned}
$$

we write the adapted frame as $\{E_\lambda\} = \left\{E_j, E_{\bar{j}}\right\}$. $\left\{dx^h, \delta y^h\right\}$ is the dual frame of $\{E_i, E_{\bar{i}}\}$, where $\delta y^h = dy^h + y^b \Gamma^h_{ba} dx^a$. By the straightforward calculation, we have the following:

2.3 Lemma. *The Lie brackets of the adapted frame of TM satisfy the following identities:*

$$
\begin{cases}
[E_j, E_i] = y^b R_{ijb}{}^a E_{\bar{a}}\\
[E_j, E_{\bar{i}}] = \Gamma^a_{ji} E_{\bar{a}}\\
\left[E_{\bar{j}}, E_{\bar{i}}\right] = 0
\end{cases}
$$

where $R_{ijb}^{\ \ a}$ *denote the components of the curvature tensor of* M *[31].*

Using (2.1), (2.2) and (2.3), we have

$$\begin{aligned} {}^{H}X &= \begin{pmatrix} X^j\delta_j^h \\ -X^j\Gamma_{sj}^h y^s \end{pmatrix} = X^j \begin{pmatrix} \delta_j^h \\ -\Gamma_{sj}^h y^s \end{pmatrix} = X^j E_j \\ {}^{V}X &= \begin{pmatrix} 0 \\ X^h \end{pmatrix} = \begin{pmatrix} 0 \\ X^j\delta_j^h \end{pmatrix} = X^j \begin{pmatrix} 0 \\ \delta_j^h \end{pmatrix} = X^j E_{\bar{j}}, \end{aligned}$$

and

$$\begin{aligned} {}^{C}X &= \begin{pmatrix} X^j\delta_j^h \\ y^s\partial_s X^j \end{pmatrix} \\ &= X^j \begin{pmatrix} \delta_j^h \\ -\Gamma_{sj}^h y^s \end{pmatrix} + y^m\nabla_m X^j \begin{pmatrix} 0 \\ \delta_j^h \end{pmatrix} = X^j E_j + y^m \nabla_m X^j E_{\bar{j}} \end{aligned}$$

with respect to the adapted frame $\{E_\lambda\}$.

We shall need a new lift of vector fields on M. For any vector field $Y \in \Im_0^1(M)$ with the components (Y^h), ${}^{V'}Y$ is a vector field on TM defined by

$${}^{V'}Y = \{(1-r^2)Y^a + g_{kr}Y^k y^r y^a\}E_{\bar{a}},$$

with respect to the adapted frame $\{E_\lambda\}$. Clearly the lift ${}^{V'}Y$ is a smooth vector field on TM. Remark that ${}^{V'}Y$ is a vertical vector field on TM. In fact, $f \in \Im_0^0(M)$; ${}^{V'}Y\ {}^{V}(f) = 0$.

Let $\tilde{X}$ be a vector field on TM with components $(v^h, v^{\bar{h}})$ with respect to the adapted frame $\{E_h, E_{\bar{h}}\}$. Then $\tilde{X}$ is a fibre-preserving vector field on TM if and only if v^h depend only on the variables (x^h). Therefore, every fibre-preserving vector field $\tilde{X}$ on TM induces a vector field $X = v^h \frac{\partial}{\partial x^h}$ on M. Also, it is well-known that ${}^{C}X$, ${}^{V}X$, ${}^{V'}X$ and ${}^{H}X$ are fibre-preserving vector fields on TM.

Let $L_{\tilde{X}}$ be the Lie derivation with respect to the fibre-preserving vector field $\tilde{X}$, then we have the following Lemma:

2.4 Lemma. *(see [30, 31]) The Lie derivations of the adapted frame and its dual basis with respect to* $\tilde{X} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ *are given as follows:*

$$\begin{aligned} &(1)\quad L_{\tilde{X}}E_h = -\partial_h v^a E_a + \left\{ y^b v^c R_{hcb}^{\ \ \ a} - v^{\bar{b}}\Gamma_{b\,h}^{a} - E_h(v^{\bar{a}}) \right\} E_{\bar{a}} \\ &(2)\quad L_{\tilde{X}}E_{\bar{h}} = \left\{ v^b \Gamma_{b\,h}^{a} - E_{\bar{h}}(v^{\bar{a}}) \right\} E_{\bar{a}} \\ &(3)\quad L_{\tilde{X}}dx^h = \partial_m v^h dx^m \\ &(4)\quad L_{\tilde{X}}\delta y^h = -\left\{ y^b v^c R_{mcb}^{\ \ \ \ h} - v^{\bar{b}}\Gamma_{b\,m}^{h} - E_m(v^{\bar{h}}) \right\} dx^m \\ &\qquad\qquad - \left\{ v^b \Gamma_{b\,m}^{h} - E_{\bar{m}}(v^{\bar{h}}) \right\} \delta y^m. \end{aligned}$$

3 Results

If $g = g_{ij}dx^i dx^j$ is the expression of the Riemannian metric g, the Cheeger-Gromoll metric g_{CG} is expressed in the adapted local frame by

$$g_{CG} = g_{ij}dx^i dx^j + h_{ij}\delta y^i \delta y^j$$

where h_{ij} is the function on $\pi^{-1}(U)$ defined by $h_{ij} = \frac{1}{1+r^2}(g_{ij} + y^s y^t g_{is} g_{tj})$. For shortness we set $G_1 = g_{ij}dx^i dx^j$ and $G_2 = h_{ij}\delta y^i \delta y^j$. Therefore the Cheeger-Gromoll metric g_{CG} can be expressed as follows:

$$g_{CG} = G_1 + G_2.$$

We shall first state the following Lemma which is needed later on.

3.1 Lemma. *The Lie derivatives $L_{\tilde{X}}G_1$ and $L_{\tilde{X}}G_2$ with respect to the fibre-preserving vector field $\tilde{X}$ are given as follows:*

$$\begin{aligned}
&(1) \quad L_{\tilde{X}}G_1 = (L_X g_{ij})dx^i dx^j \\
&(2) \quad L_{\tilde{X}}G_2 = -2h_{mj}\left\{ y^b v^c R_{icb}^{\;m} - v^{\bar{b}}\Gamma_{b\,i}^{m} - E_i(v^{\bar{m}}) \right\} dx^i \delta y^j \\
&\quad + \Big\{ L_X h_{ij} - 2h_{mj}\nabla_i v^m + 2h_{mj}E_{\bar{i}}(v^{\bar{m}}) \\
&\quad + \frac{1}{1+r^2} v^{\bar{m}} y^s(-2g_{ms}h_{ij} + g_{mj}g_{is} + g_{sj}g_{im}) \Big\} \delta y^i \delta y^j
\end{aligned}$$

where $L_X g_{ij}$ denote the components of the Lie derivative $L_X g$, and also $\nabla_i v^m$ denote the components of the covariant derivative of X.

Proof. Proof of this Lemma is similar to proof of the Proposition 2.3 of Yamauchi [31]. □

3.2 Theorem. *Let $(TM,\ g_{CG})$ be the tangent bundle with the Cheeger-Gromoll metric of a Riemannian manifold $(M,\ g)$. Let*

(i) X be an infinitesimal homothetic transformation on (M, g), with $L_X g = \Omega g$, for some constant Ω;

(ii) Y be a parallel vector field on (M, g);

(iii) A be a $(1,1)$-tensor field on M which satisfies the followings

$$\begin{aligned}
&(A_1) \quad g_{ik}A_j^k + g_{kj}A_i^k = 2\Omega g_{ij}, \\
&(A_2) \quad \nabla_i A_j^k + X^l R_{lij}^k = 0.
\end{aligned}$$

Then the vector field $\tilde{X}$ on TM defined by

$$(\sharp) \quad \tilde{X} = {}^C X + \gamma A + {}^{V'}Y$$

is an infinitesimal fibre-preserving conformal transformation on (TM, g_{CG}).

Conversely, every infinitesimal fibre-preserving conformal transformation on (TM, g_{CG}) *is of the form* ($\sharp$).

Let TM be the tangent bundle over M with the Cheeger-Gromoll metric g_{CG}, and let $\tilde{X}$ be an infinitesimal fibre-preserving conformal transformation on (TM, g_{CG}) such that

$$L_{\tilde{X}} g_{CG} = 2\Omega\, g_{CG}. \tag{3.1}$$

By means of Lemma 3.1, we have

$$\begin{aligned} & (L_X g_{ij}) dx^i dx^j - 2h_{mj} \left\{ y^b v^c R_{icb}^{\ \ \ m} - v^{\bar{b}} \Gamma_{b\,i}^{m} - E_i(v^{\bar{m}}) \right\} dx^i \delta y^j \\ + & \ [L_X h_{ij} - 2h_{mj} \nabla_i v^m + 2h_{mj} E_{\bar{i}}(v^{\bar{m}}) \\ + & \ \frac{1}{1+r^2} v^{\bar{m}} y^s (-2g_{ms} h_{ij} + g_{mj} g_{is} + g_{sj} g_{im})] \delta y^i \delta y^j \\ = & \ 2\Omega g_{ij} dx^i dx^j + 2\Omega h_{ij} \delta y^i \delta y^j. \end{aligned}$$

Comparing both sides of the above equation, we obtain the following three relations:

$$L_X g_{ij} = 2\Omega g_{ij} \tag{3.2}$$

$$y^b v^c R_{icb}^{\ \ \ m} - v^{\bar{b}} \Gamma_{b\,i}^{m} - E_i(v^{\bar{m}}) = 0 \tag{3.3}$$

$$\begin{aligned} & L_X h_{ij} - 2h_{mj} \nabla_i v^m + 2h_{mj} E_{\bar{i}}(v^{\bar{m}}) \\ & + \frac{1}{1+r^2} v^{\bar{m}} y^s (-2g_{ms} h_{ij} + g_{mj} g_{is} + g_{sj} g_{im}) = 2\Omega h_{ij}. \end{aligned} \tag{3.4}$$

First all, we shall study the particular cases ${}^C X$, γA, ${}^{V'} Y$. Using (3.2)-(3.4) and the local expressions of ${}^C X$, γA, ${}^{V'} Y$ with respect to the adapted frame, one easily proves, by direct computation, the following Lemmas.

3.3 Lemma. *In order that a complete lift* ${}^C X$ *to* TM *of a vector field* X *on* M *be an infinitesimal fibre-preserving conformal transformation of* (TM, g_{CG}), *it is necessary and sufficient that* X *is an infinitesimal homothetic transformation of* (M, g).

3.4 Lemma. *Let* A *be a* $(1,1)$*-tensor field on* (M, g) *satisfying the conditions* (A_1) *and* (A_2) *in Theorem 3.2. Then* γA *is an infinitesimal fibre-preserving conformal transformation on* (TM, g_{CG}).

3.5 Lemma. *Let* Y *be a vector field on* (M, g) *which is parallel with respect to the Levi-Civita connection of* g. *Then* ${}^{V'} Y$ *is an infinitesimal fibre-preserving conformal transformation on* (TM, g_{CG}).

Proof. Since sufficiency is shown by Lemma 3.3, Lemma 3.4 and Lemma 3.5, we now show necessity. We consider the 0-section ($y^i = 0$) in the coordinate neighborhood $\pi^{-1}(U)$ in TM and its neighborhood W. For a vector field $\tilde{X} = v^i E_i + v^{\bar{i}} E_{\bar{i}}$ on TM, and $(x, y) = (x^i, y^i)$ in W, we can write, by Taylor's theorem,

$$v^i(x,y) = v^i(x,0) + (\partial_{\bar{r}} v^i)(x,0) y^r + \frac{1}{2}(\partial_{\bar{r}}\partial_{\bar{s}} v^i)(x,0) y^r y^s + ... + [*]^i_\lambda, \quad (3.5)$$

$$v^{\bar{i}}(x,y) = v^{\bar{i}}(x,0) + (\partial_{\bar{r}} v^{\bar{i}})(x,0) y^r + \frac{1}{2}(\partial_{\bar{r}}\partial_{\bar{s}} v^{\bar{i}})(x,0) y^r y^s + ... + [*]^{\bar{i}}_\lambda, \quad (3.6)$$

where $[*]^I_\lambda$ $(I = 1, 2, ..., 2n)$ is of the form:

$$[*]^I_\lambda = \frac{1}{\lambda!}(\partial^\lambda v^I / \partial y^{i_1} \partial y^{i_2} ... \partial y^{i_\lambda})(x^a, \theta(x,y) y^b) y^{i_1} y^{i_2} ... y^{i_\lambda};\ 1 \leq i_1, ..., i_\lambda \leq n.$$

The following lemma is valid.

3.6 Lemma. *In the above situation, the following*

$$\begin{aligned} &X = (X^i(x)) = (v^i(x,0)), \\ &Y = (Y^i(x)) = (v^{\bar{i}}(x,0)), \\ &K = (K^i_r(x)) = ((\partial_{\bar{r}} v^i)(x,0)), \\ &E = (E^i_{rs}(x)) = ((\partial_{\bar{r}}\partial_{\bar{s}} v^i)(x,0)), \\ &P = (P^i_r(x)) = ((\partial_{\bar{r}} v^{\bar{i}})(x,0) - (\partial_r v^i)(x,0)) \end{aligned}$$

are tensor fields on M [29].

For a fibre-preserving vector field $\tilde{X} = v^i E_i + v^{\bar{i}} E_{\bar{i}}$ on TM, with the notations of Lemma 3.6, we can write:

$$v^i(x,y) = X^i \quad (3.7)$$

$$v^{\bar{i}}(x,y) = Y^i + \tilde{P}^i_r y^r + \frac{1}{2} Q^i_{rs} y^r y^s + ... + [*]^{\bar{i}}_\lambda, \quad (3.8)$$

where $\tilde{P}^i_r$ and Q^i_{rs} are given by $\tilde{P}^i_r = (\partial_{\bar{r}} v^{\bar{i}})(x,0)$ and $Q^i_{rs} = (\partial_{\bar{r}}\partial_{\bar{s}} v^{\bar{i}})(x,0)$.

Substituting (3.7) into (3.2), we have

$$X^m \partial_m g_{ij} + (\partial_i X^m) g_{mj} + (\partial_j X^m) g_{im} = 2\Omega g_{ij}. \quad (3.9)$$

The equation (3.9) reduces to

$$\nabla_i X_j + \nabla_j X_i = 2\Omega g_{ij}. \quad (3.10)$$

Raising j and contracting with i in (3.10), it is easily seen that

$$\Omega = \frac{1}{n}(\nabla_i X^i),$$

i.e. the scalar function Ω on TM depends only on the variables (x^i) with respect to the induced coordinates (x^i, y^i). Further, the vector field X with the components (X^i) is an infinitesimal conformal transformation on M. Since, by Lemma 3.3, ${}^C X = X^a E_a + (y^m \nabla_m X^a) E_{\bar{a}}$ is an infinitesimal fibre-preserving conformal transformation on (TM, g_{CG}), $\tilde{X} - {}^C X$ is also an infinitesimal fibre-preserving conformal transformation. Therefore, in the following, denoting $\tilde{X} - {}^C X$ by the same letter $\tilde{X}$, one may assume that $X^i = 0$ in (3.7). Then $(\tilde{P}_i^r) = (P_i^r)$ is a tensor field on M by lemma 3.6.

Putting (3.7) and (3.8) into (3.3) [from now on, we omit this statement] and taking the part which does not contain y^r, we get

$$\nabla_i Y^m = 0. \tag{3.11}$$

Taking the part which does not contain y^r in (3.4), we get

$$g_{mj} \tilde{P}_i^m + g_{im} \tilde{P}_j^m = 2\Omega g_{ij}. \tag{3.12}$$

On differentiating $\partial_{\bar{k}}$ to the both sides of the equation (3.12), we obtain

$$g_{im} \partial_{\bar{k}} \tilde{P}_j^m + g_{mj} \partial_{\bar{k}} \tilde{P}_i^m = 0. \tag{3.13}$$

Using (3.13) and the last equation in Lemma 2.3, we have

$$\begin{aligned} g_{im} \partial_{\bar{k}} \partial_{\bar{j}} (v^{\bar{m}}) &= -g_{mj} \partial_{\bar{k}} \partial_{\bar{i}} (v^{\bar{m}}) g_{mj} = -g_{mj} \partial_{\bar{i}} \partial_{\bar{k}} (v^{\bar{m}}) \\ &= g_{mk} \partial_{\bar{i}} \partial_{\bar{j}} (v^{\bar{m}}) = g_{mk} \partial_{\bar{j}} \partial_{\bar{i}} (v^{\bar{m}}) \\ &= -g_{mi} \partial_{\bar{j}} \partial_{\bar{k}} (v^{\bar{m}}) = -g_{mi} \partial_{\bar{k}} \partial_{\bar{j}} (v^{\bar{m}}), \end{aligned}$$

which gives

$$\partial_{\bar{k}} \tilde{P}_j^m = 0.$$

This shows that $\tilde{P}_j^m$ depends only on the variables (x^h). Hence $\tilde{P}_j^m$ can be written as

$$\tilde{P}_j^m = A_j^m, \tag{3.14}$$

where A_j^m is a certain function which depends only on the variables (x^h).

The coefficient of y^r in (3.3), by (3.14), gives

$$X^c R_{icr}^m - \nabla_i A_r^m = 0$$

or equivalently

$$X^c R_{icrl} - \nabla_i A_{rl} = 0. \tag{3.15}$$

By (3.14), (3.12) is written as

$$g_{mj} A_i^m + g_{im} A_j^m = 2\Omega g_{ij} \tag{3.16}$$

Applying the covariant derivative ∇_k to the both sides of the last equation, we obtain

$$\nabla_k(A_{ij}) + \nabla_k(A_{ji}) = 2(\nabla_k \Omega) g_{ij}.$$

Combining the last identity with (3.15), we get $\nabla_k \Omega = \partial_k \Omega = 0$. This together with connectedness of M shows that the scalar function Ω is constant.

Taking the coefficient of y^r in (3.4), we get

$$Y^m(-2g_{mr}g_{ij} + g_{mj}g_{ir} + g_{rj}g_{im}) + g_{mj}Q_{ri}^m + g_{mi}Q_{rj}^m = 0. \tag{3.17}$$

We put $Q_{rs}^i = -2Y^i g_{rs} + (Y^k g_{kr}\delta_s^i + Y^k g_{ks}\delta_r^i) + T_{rs}^i$. By a simple calculation, using (3.17), we can verify that $g_{mj}T_{ir}^m + g_{mi}T_{jr}^m = 0$. If we put $T_{irj} = g_{mj}T_{ir}^m$, then T_{irj} is symmetric in i and r, and skew-symmetric i and j. Hence $T_{irj} = 0$. That is

$$Q_{rs}^i = -2Y^i g_{rs} + (Y^k g_{kr}\delta_s^i + Y^k g_{ks}\delta_r^i). \tag{3.18}$$

Finally, we consider the coefficient of $y^r y^s$ in (3.3), we get by virtue of (3.18)

$$-2(\nabla_i Y^m) g_{rs} + (\nabla_i Y^k) g_{kr}\delta_s^m + (\nabla_i Y^k) g_{ks}\delta_r^m = 0.$$

In view of (3.11), the last equation holds.

Now, by (3.15) and (3.16), we see that γA is an infinitesimal fibre-preserving conformal transformation on (TM, g_{CG}) by Lemma 3.4. By (3.11) and Lemma 3.5, ${}^{V'}Y$ is an infinitesimal fibre-preserving conformal transformation on (TM, g_{CG}).

Summing up we find that $\tilde{X} \in \Im_0^1(TM)$ is an infinitesimal fibre-preserving conformal transformation with respect to the Chegeer-Gromoll metric iff

$$\tilde{X} = X^i E_i + (Y^i + \tilde{P}_s^i y^s + \frac{1}{2} Q_{sr}^i y^s y^r) E_{\bar{i}}$$

$$= X^i E_i + (Y^i + y^s(\nabla_s X^i + A_s^i) + (1 - r^2) Y^i + g_{ks} Y^k y^s y^i) E_{\bar{i}}$$

$$= {}^C X + \gamma A + {}^{V'}Y$$

for each local coordinate systems (x^i), $i = 1, ..., n$ on M. This proves the assertion and the conditions (i), (ii) and (iii) are direct consequences of (3.9), (3.11), (3.15), (3.16). □

The result follows immediately from Theorem 3.2 and from its Proof.

3.7 Corollary. *Every infinitesimal fibre-preserving conformal transformation on* (TM, g_{CG}) *is homothetic and it induces an infinitesimal homothetic transformation. Consequently, it is of the form* ($\sharp$).

It is known that an infinitesimal homothetic transformation in a compact Riemannian manifold is a Killing vector field [32]. Theorem 3.2 and Corollary 3.7 deliver a simple and surprising result on compact manifolds:

3.8 Corollary. *Let* (M, g) *be a compact Riemannian manifold and* TM *be the tangent bundle of* M. $\tilde{X} \in \Im_0^1(TM)$ *is an infinitesimal fibre-preserving conformal transformation with respect to the Chegeer-Gromoll metric on* TM *iff* $\tilde{X}$ *is a Killing vector field with respect to the the Chegeer-Gromoll metric on* TM.

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