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Topological properties of the attractors of iterated function systems

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Abstract

In this paper we investigate necessary conditions for an attractor of an iterated function system to have a finite number of connected components. Then we prove that each connected component of an attractor of an iterated function system which has a finite number of connected components is also arcwise connected. We also emphasize by a counterexample, that the result does not hold when the attractor has an infinite number of connected components.

1 Introduction

Iterated function systems were conceived in the present form by John Hutchinson in ([4]), popularized by Michael Barnsley in ([1]) and are one of the most common and general ways to generate fractals. Many of the important examples of functions and sets with special and unusual properties turn out to be fractal sets or functions whose graphs are fractal sets and a great part of them are attractors of iterated function systems. There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems or, more generally, to multifunction systems and to study them ([2], [6], [7], [8], [9]). A recent such example can be found in ([6]), where the Lipscomb's space, which was an important example in dimension theory, can be obtain as an attractor of an infinite iterated function system. The topological properties of fractal sets have a great importance in analysis on fractals as we can see in ([5], [6]). In this article we study attractors of iterated function systems which have a finite number of connected components

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and we prove that each of these components is also arcwise connected. The result does not hold when the attractor has an infinite number of connected components.

For a metric space (X, d) we denote by $\mathcal{K}(X)$ the set of nonempty compact subsets of X and by $\mathcal{P}(X)$ the set of nonempty subsets of X.

Definition 1.1. Let (X, d) be a metric space and $\mathcal{K}(X)$ the set of all nonempty compact subsets of X. The application $h : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow [0, +\infty)$ defined by $h(A, B) = \max(d(A, B), d(B, A))$, where $d(A, B) = \sup\limits_{x \in A} d(x, B) = \sup\limits_{x \in A}(\inf\limits_{y \in B} d(x, y))$ is called the *Hausdorff-Pompeiu metric*. When $h : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow [0, +\infty]$, then it is called the *Hausdorff-Pompeiu semimetric.*

Definition 1.2. Let (X, d) be a metric space. For a function $f : X \longrightarrow X$, the constant $Lip(f) = \sup\limits_{x,y \in X \,;\, x \neq y} \dfrac{d(f(x), f(y))}{d(x, y)} \in [0, +\infty]$ is called the *Lipschitz constant* associated to f. We also say that f is a *Lipschitz function* if $Lip(f) < +\infty$ and a *contraction* if $Lip(f) < 1$.

Proposition 1.1. *([1]) The Hausdorff-Pompeiu semimetric, h, has the following properties:*

1). If H and K are two nonempty subsets of X then $h(H, K) = h(\overline{H}, \overline{K})$;

2). If $(H_i)_{i \in I}$ are $(K_i)_{i \in I}$ are two famillies of nonempty subsets of X then:

$$h(\bigcup_{i \in I} H_i, \bigcup_{i \in I} K_i) = h(\overline{\bigcup_{i \in I} H_i}, \overline{\bigcup_{i \in I} K_i}) \leq \sup_{i \in I} h(H_i, K_i).$$

3). If H and K are two nonempty subsets of X and $f : X \longrightarrow Y$ is a function then

$$h_X(f(K), f(H)) \leq Lip(f) \cdot h_Y(K, H).$$

Remark 1.1. ([1], [3]) If (X, d) is a complete metric space then $(\mathcal{K}(X), h)$ is a complete metric space and if (X, d) is a compact metric space then $(\mathcal{K}(X), h)$ is a compact metric space.

Definition 1.3. An *iterated function system* on a metric space (X, d) consists of a finite family of contractions $(f_k)_{k=\overline{1,n}}$ on X and it is denoted by $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$.

Definition 1.4. For an iterated function system $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, the function $F_{\mathcal{S}} : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$ defined by $F_{\mathcal{S}}(B) = \bigcup\limits_{k=1}^{n} f_k(B)$ is called the *fractal operator* associated with the iterated function system $\mathcal{S}$.

Proposition 1.2. *([1], [3]) Let $S = (X, (f_k)_{k=\overline{1,n}})$ be an iterated function system. Then the function $F_{\mathcal{S}}$ is a contraction satisfying $Lip(F_{\mathcal{S}}) \leq \max_{k=\overline{1,n}} Lip(f_k) < 1$.*

Using Banach's contraction theorem there exists, for an iterated function system $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, a unique set A such that $F_{\mathcal{S}}(A) = A$. More precisely we have the following well-known result:

Theorem 1.1. *([1], [5]) Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an iterated function system with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$. Then there exists a unique set $A \in \mathcal{K}(X)$ such that $F_{\mathcal{S}}(A) = A$. Moreover, for any $H_0 \in \mathcal{K}(X)$ the sequence $(H_n)_{n\geq 1}$ defined by $H_{n+1} = F_{\mathcal{S}}(H_n)$ is convergent to A. For the speed of the convergence we have the following estimation: $h(H_n, A) \leq \dfrac{c^n}{1-c} h(H_0, H_1)$.*

Definition 1.5. The unique set $A \in \mathcal{K}(X)$ from theorem 1.1. is called the *attractor* of iterated function system $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$.

Definition 1.6. A metric space (X, d) is *arcwise connected* if for every $x, y \in X$ there exists a continuous function $\varphi : [0, 1] \longrightarrow X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

Definition 1.7. Let X be a nonempty set and $(A_i)_{i\in I}$ a family of nonempty subsets of X. Then the family $(A_i)_{i\in I}$ is said to be *connected* if for every $i, j \in I$ there exists $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_n = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1, 2, ..., n-1\}$. If a family $(A_i)_{i\in I}$ is not connected we say that it is *disconnected*.

Definition 1.8. Let X be a nonempty set. On X we consider the following relation: xRy if and only daca there exists a connected set $B \subset X$ such that $x, y \in B$. The relation R is a equivalence relation. The equivalence classes are called the *connected components* of X.

Concerning the connectedness of the attractor of an iterated function system we have the following result:

Theorem 1.2. *([5], Theorem 1.6.2., page 33.) Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an iterated function system with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$ and A the attractor of $\mathcal{S}$. Then the following are equivalent:*

1) *The family $(A_i)_{i=\overline{1,n}}$ is connected, where $A_i = f_i(A)$ for every $k \in \{1, ..., n\}$.*

2) *A is arcwise connected.*

3) *A is connected.*

Next we briefly present the shift space of an iterated function system. For more details one can see ([5]). We start with some set notations: $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}^* = \mathbb{N} - \{0\}$, $\mathbb{N}_n^* = \{1, 2, ..., n\}$. For two nonempty sets A and B, B^A denotes the set of functions from A to B. By $\Lambda = \Lambda(B)$ we will understand the set $B^{\mathbb{N}^*}$ and by $\Lambda_n = \Lambda_n(B)$ we will understand the set $B^{\mathbb{N}_n^*}$. The elements of $\Lambda = \Lambda(B) = B^{\mathbb{N}^*}$ will be written as infinite words $\omega = \omega_1\omega_2...\omega_m\omega_{m+1}...$, where $\omega_m \in B$ and the elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbb{N}_n^*}$ will be written as finite words $\omega = \omega_1\omega_2...\omega_n$. By λ we will understand the empty word. Let us remark that $\Lambda_0(B) = \{\lambda\}$. By $\Lambda^* = \Lambda^*(B)$ we will understand the set of all finite words $\Lambda^* = \Lambda^*(B) = \bigcup_{n\geq 0} \Lambda_n(B)$. For $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$ by $\alpha\beta$ we will understand the joining of the words α and β namely $\alpha\beta = \alpha_1\alpha_2...\alpha_n\beta_1\beta_2...\beta_m$ and respectively $\alpha\beta = \alpha_1\alpha_2...\alpha_n\beta_1\beta_2...\beta_m\beta_{m+1}...$.

On $\Lambda = \Lambda(\mathbb{N}_n^*) = (\mathbb{N}_n^*)^{\mathbb{N}^*}$ we can consider the metric $d_s(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1-\delta_{\alpha_k}^{\beta_k}}{3^k}$, where $\delta_x^y = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}$ and $\alpha = \alpha_1\alpha_2...$ and $\beta = \beta_1\beta_2...$.

Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an iterated function system on X and A the attractor of $\mathcal{S}$. For an element $\omega = \omega_1\omega_2...\omega_m \in \Lambda_m(\mathbb{N}_n^*)$, f_ω denotes $f_{\omega_1} \circ f_{\omega_2} \circ ... \circ f_{\omega_m}$ and H_ω denotes $f_\omega(H)$ for a any subset $H \subset X$. By H_λ we will understand the set H. In particular $A_\omega = f_\omega(A)$.

2 Main results

We prove some neccesary conditions for an attractor of an iterated function systems to have a finite number of connected components.

Theorem 2.1. *Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an iterated function system with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$ and A the attractor of $\mathcal{S}$. We denote by $A_i = f_i(A)$ for every $i \in \{1, ..., n\}$. Then the following are equivalent:*

1). The set A_i is arcwise connected for every $i \in \{1, ..., n\}$.

2). The set A_i is connected for every $i \in \{1, ..., n\}$.

3). The set A_ω is arcwise connected for every $\omega \in \Lambda_m$ and $m \in \mathbb{N}^*$.

4). The set A_ω is connected for every $\omega \in \Lambda_m$ and $m \in \mathbb{N}^*$.

Proof: Obviously 3).$\Longrightarrow$1).$\Longrightarrow$2). and 3).$\Longrightarrow$4).$\Longrightarrow$2).

1).$\Longrightarrow$3). If $m = 1$ the statement is true. Suppose now that $m \geq 2$. Let $\omega \in \Lambda_m$, $\omega = i_1 i_2 ... i_m$ where $i_j \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. Then $A_\omega = f_\omega(A) = f_{i_1 i_2 ... i_{m-1}}(f_{i_m}(A)) = f_{i_1 i_2 ... i_{m-1}}(A_{i_m})$ and hence A_ω is arcwise connected, because A_{i_m} is arcwise connected and $f_{i_1 i_2 ... i_{m-1}}$ is continuous.

2).$\Longrightarrow$4). If $m = 1$ the statement is true. Suppose now that $m \geq 2$. Let $\omega \in \Lambda_m$, $\omega = i_1 i_2 ... i_m$ where $i_j \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. Then $A_\omega = f_\omega(A) = f_{i_1 i_2 ... i_{m-1}}(f_{i_m}(A)) = f_{i_1 i_2 ... i_{m-1}}(A_{i_m})$ and hence A_ω is connected, because A_{i_m} is connected and $f_{i_1 i_2 ... i_{m-1}}$ is continuous.

2).$\Longrightarrow$1). We have that $A = \bigcup_{j=1}^{n} A_j$ which implies $A_i = f_i(A) = f_i\left(\bigcup_{j=1}^{n} A_j\right) = \bigcup_{j=1}^{n} f_i(A_j) = \bigcup_{j=1}^{n} A_{ij}$. We also have that A_i is a connected set for every $i \in \{1, ..., m\}$. That means that the family of sets $(A_{ij})_{j=\overline{1,n}}$ is connected for every $i \in \{1, ..., n\}$.

We define now the following set: $P = \{f : A \times A \times [0,1] \longrightarrow A$ such that $f(A_i \times A_i \times [0,1]) \subset A_i$, $f(p,q,0) = p$ and $f(p,q,1) = q$ for every $(p,q) \in A_i \times A_i$ and $i \in \{1, ..., n\}\}$. On P we define the following metric:

$$d_P(f,g) = \sup_{(x,t,y) \in A \times A \times [0,1]} d(f(x,y,t), g(x,y,t)) \text{ for every } f, g \in P.$$

In that way (P, d_P) becomes a complete metric space. Let $i \in \{1, ..., n\}$ be fixed. For every $(p,q) \in A_i \times A_i$ there exist $n^{p,q} \in \{1, ..., n\}$, $\{i_k^{p,q}\}_{k=\overline{0,n^{p,q}-1}} \subset \{1, ..., n\}$ and $\{x_k^{p,q}\}_{k=\overline{0,n^{p,q}}} \subset A_i$ such that $p = x_0^{p,q}$, $q = x_{n^{p,q}}^{p,q}$ and $x_k^{p,q}, x_{k+1}^{p,q} \in A_{i i_k^{p,q}}$ for every $k \in \{0, ..., n^{p,q}-1\}$. Now for every $f \in P$ we define the function $Tf \in P$ in the following way: $Tf(p,q,t) = f_i(f(y_k^{p,q}, z_k^{p,q}, n^{p,q}t - k))$ for every $t \in [\frac{k}{n^{p,q}}, \frac{k+1}{n^{p,q}}]$, where $y_k^{p,q} \in {f_i}^{-1}(x_k^{p,q}) \in A_{i_k^{p,q}}$, $z_k^{p,q} \in {f_i}^{-1}(x_{k+1}^{p,q}) \in A_{i_k^{p,q}}$ for every $k \in \{0, ..., n^{p,q} - 1\}$. We will make some notations: $T^0 f = f$ and $T^m f = \underbrace{Tf \circ Tf \circ ... \circ Tf}_{m \text{ times}}$, where $m \in \mathbb{N}^*$. Then $d_P(Tf, Tg) \leq c \cdot d_P(f,g)$ and inductively it results that $d_P(T^m f, T^m g) \leq \ c^m \cdot d_P(f,g) \overset{m\to\infty}{\longrightarrow} 0$ for every $f, g \in P$. Hence there exists $f^* \in P$ such that $T^m f \overset{m\to\infty}{\longrightarrow} f^*$ in P. We denote now by $\omega_t(f) = \lim_{\varepsilon \to 0} \sup_{x,y \in (t-\varepsilon, t+\varepsilon)} d(f(x), f(y))$ the oscilation of a function $f \in P$ in the point $t \in [0,1]$, by $f_{p,q}(t) = f(p,q,t)$, where $(p,q,t) \in A_i \times A_i \times [0,1]$ and by $\Omega(Tf) = \sup_{(p,q,t) \in A_i \times A_i \times [0,1]} \omega_t(Tf_{p,q})$. Hence we obtain that $\Omega(Tf) = \sup_{(p,q,t) \in A_i \times A_i \times [0,1]} \omega_t(Tf_{p,q}) \leq Lip(f_i) \cdot \sup_{(p,q,t) \in A_i \times A_i \times [0,1]} \omega_t(f_{p,q}) \leq c \cdot \Omega(f)$ and inductively $\Omega(T^m f) \leq c^m \cdot \Omega(f) \overset{m\to\infty}{\longrightarrow} 0$. Thus $\Omega(f^*) = 0$ and so f^* is continuous with respect to t. Hence as f^* is a continuous function between p and q, A_i is arcwise connected.

Remark 2.1. Let (X,d) be a complete metric space, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an iterated function system with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$ and A the attractor of S. We denote by $A_i = f_i(A)$ for every $i \in \{1, ..., n\}$. If A_i is connected for every $i \in \{1, ..., n\}$, then the attractor A has a finite number of connected components. Indeed, because $A = \bigcup_{i=1}^{n} A_i$ and A_i is connected for every $i \in \{1, ..., n\}$, it follows that A has a finite number of connected components.

We prove now that each connected component of an attractor with a finite number of connected components is also arcwise connected. The result does not remain true when the attractor has an infinite number of components.

Theorem 2.2. *Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_i)_{i=\overline{1,n}})$ an iterated function system with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$ and A the attractor of $\mathcal{S}$. If A has a finite number of connected components then each connected component is arcwise connected.*

Proof: Let $K_1, K_2, ..., K_m$ be the connected components of A. Since $K_i \subset A$ for every $i \in \{1, ..., m\}$ is a closed set included in a compact one, we obtain that each connected component is compact. We consider now the functions $F_i^j : K_j \longrightarrow X$, $F_i^j(x) = f_i(x)$ for every $x \in K_j$, $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. Because the functions F_i^j are continuous and K_j is connected for every $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$, we have that $F_i^j(K_j)$ is a connected set in X and hence has to be included in one connected component.

We consider the product space $K = \prod_{i=1}^{m} K_i$ and for every $(x_1, ..., x_m), (y_1, ..., y_m) \in K$ we define the metric:

$$d_{\max}((x_1, ..., x_m), (y_1, ..., y_m)) = \max\{d(x_1, y_1), ..., d(x_m, y_m)\}.$$

Thus $(K, d_{\max})$ becomes a metric space. Because $K_1, K_2, ..., K_m$ are compact sets, it follows that $(K, d_{\max})$ is a compact metric space, hence it is complete.

We introduce now some notations:

a). $p_1 : \{1, ..., n\} \times \{1, ..., m\} \longrightarrow \{1, ..., n\}$, $p_1(x, y) = x$ is the projection on the first component and $p_2 : \{1, ..., n\} \times \{1, ..., m\} \longrightarrow \{1, ..., m\}$, $p_2(x, y) = y$ is the projection on the second component.

b). $J = \{\varphi : \{1, ..., m\} \longrightarrow \{1, ..., n\} \times \{1, ..., m\} \mid \theta(\varphi(j)) = j$ for every $j \in \{1, ..., m\}$, where $\theta : \{1, ..., n\} \times \{1, ..., m\} \longrightarrow \{1, ..., m\}$ is the unique indice such that $F_i^j(K_j) = f_i(K_j) \subset K_{\theta(i,j)}\}$.

c). $M_r = \{(i, j) \in \{1, ..., n\} \times \{1, ..., m\} | \ r = \theta(i, j)\}$ for every $r \in \{1, ..., m\}$.

d). For $\varphi \in J$ we define the functions $F_\varphi : K \longrightarrow K$ for every $(x_1, ..., x_m) \in K$ by:

$$F_\varphi(x_1,...,x_m) = (F^{p_2(\varphi(1))}_{p_1(\varphi(1))}(x_{p_2(\varphi(1))}),...,F^{p_2(\varphi(m))}_{p_1(\varphi(m))}(x_{p_2(\varphi(m))})) = (f_{p_1(\varphi(1))}(x_{p_2(\varphi(1))}),...,f_{p_1(\varphi(m))}(x_{p_2(\varphi(m))}))$$

With these notations we have that:

$$Lip(F_\varphi) = \sup_{\max\{d(x_1,y_1),...,d(x_m,y_m)\}>0} \frac{d(F_\varphi(x_1,x_2,...,x_m),F_\varphi(y_1,y_2,...,y_m))}{\max\{d(x_1,y_1),...,d(x_m,y_m)\}} \leq \max_{i=1}^{n} Lip(f_i) = c < 1.$$

Thus F_φ is a contraction for every $\varphi \in J$ and we can consider now the iterated function system $\mathcal{S}_1 = (K,(F_\varphi)_{\varphi\in J})$. We will prove that K is the attractor of $\mathcal{S}_1$. It is obvious that $F_\varphi(K) \subset K$ for every $\varphi \in J$. Hence $\underset{\varphi\in J}{\cup} F_\varphi(K) \subset K$.

On the other hand, let $x = (x_j)_{j=\overline{1,m}} \in K$. We remark that $F_\varphi(K) = \prod_{j=1}^{m} f_{p_1(\varphi(j))}(K_{p_2(\varphi(j))})$. Then $\bigcup_{j=1}^{m} K_j = A = \bigcup_{i=1}^{n} f_i(A) = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m} f_i(K_j)$, which implies that $K_r = \bigcup_{(i,j)\in M_r} f_i(K_j)$ for every $r \in \{1,...,m\}$. Hence there exist $i_r, j_r \in \{1,..,n\}$ such that $x_r \in f_{i_r}(K_{j_r})$ for all $r \in \{1,...,m\}$. Let $\varphi_x : \{1,...,m\} \longrightarrow \{1,...,n\}\times\{1,...,m\}$ be a function defined by $\varphi_x(r) = (i_r, j_r)$. It results that $x \in F_{\varphi_x}(K)$ and hence $K \subset \bigcup_{\varphi\in J} F_\varphi(K)$.

In this way we proved that K is the attractor of $\mathcal{S}_1$. Since K is connected, it follows from theorem 1.3. that K is arcwise connected. Thus K_i is arcwise connected for every $i \in \{1,...,m\}$.

3 Examples

We give now some examples of attractors with a finite number of connected components.

Example 3.1. Let $n \in \mathbb{N}$, $n \geq 3$. We consider the function $f : [0,1] \longrightarrow [0,1]$ defined by

$$f(x) = \begin{cases} \frac{x}{2}, \text{ if } x \in [0,\frac{1}{n}] \\ \frac{1}{2n}, \text{ if } x \in [\frac{1}{n},\frac{n-1}{n}] \\ \frac{nx-n+2}{2n}, \text{ if } x \in [\frac{n-1}{n},1] \end{cases}.$$

Then $Lip(f) = \frac{1}{2}$. Let $g : [0,1] \longrightarrow [0,1]$ be a function defined by $g(x) = 1 - f(1-x)$. Then $Lip(g) = \frac{1}{2}$. We consider now the following iterated function system $\mathcal{S} = ([0,1],\{f,g\})$. Then $A = [0,\frac{1}{n}] \cup [\frac{n-1}{n},1]$ is the attractor of $\mathcal{S}$. Indeed:

$$f(A) = f([0, \tfrac{1}{n}] \cup [\tfrac{n-1}{n}, 1]) = f([0, \tfrac{1}{n}]) \cup f([\tfrac{n-1}{n}, 1]) =$$
$$[0, \tfrac{1}{2n}] \cup [\tfrac{1}{2n}, \tfrac{1}{n}] = [0, \tfrac{1}{n}]$$

and

$$g(A) = g([0, \tfrac{1}{n}] \cup [\tfrac{n-1}{n}, 1]) = g([0, \tfrac{1}{n}]) \cup g([\tfrac{n-1}{n}, 1]) =$$
$$(1 - f([\tfrac{n-1}{n}, 1])) \cup (1 - f([0, \tfrac{1}{n}])) =$$
$$(1 - [\tfrac{1}{2n}, \tfrac{1}{n}]) \cup (1 - [0, \tfrac{1}{2n}] =$$
$$[\tfrac{n-1}{n}, \tfrac{2n-1}{2n}] \cup [\tfrac{2n-1}{2n}, 1] = [\tfrac{n-1}{n}, 1].$$

So $F_{\mathcal{S}}(A) = A$. Thus the attractor of $\mathcal{S}$ is the set $A = [0, \frac{1}{n}] \cup [\frac{n-1}{n}, 1]$. We have that $A_1 = f(A) = [0, \frac{1}{n}]$ and $A_2 = g(A) = [\frac{n-1}{n}, 1]$. Since $n \geq 3$, it follows that A it is not a connected set but it has two connected components.

Example 3.2. We consider the space $\mathbb{R}^2$ endowed with the euclidean metric and the iterated function system $\mathcal{S} = (\mathbb{R}^2, (f_k)_{k=\overline{0,n-1}})$, where $f_k(x, y) = (\frac{x+k}{n}, k)$ for every $k \in \{0, ..., n-1\}$. Then the attractor of $\mathcal{S}$ is the set $A = \bigcup_{k=0}^{n-1} [\frac{k}{n}, \frac{k+1}{n}] \times \{k\}$. Indeed, for every $k \in \{0, ..., n-1\}$, we have that:

$$f_k(A) = f_k\left(\bigcup_{i=0}^{n-1} [\tfrac{i}{n}, \tfrac{i+1}{n}] \times \{i\}\right) = \bigcup_{i=0}^{n-1} f_k([\tfrac{i}{n}, \tfrac{i+1}{n}], i) =$$
$$\bigcup_{i=0}^{n-1} \left([\tfrac{i+nk}{n^2}, \tfrac{i+1+nk}{n^2}] \times \{k\}\right) = [\tfrac{k}{n}, \tfrac{k+1}{n}] \times \{k\}.$$

Thus $A = \bigcup_{k=0}^{n-1} \left([\frac{k}{n}, \frac{k+1}{n}] \times \{k\}\right) = \bigcup_{k=0}^{n-1} f_k(A)$. We remark that A has n connected components which are arcwise connected.

Remark 3.1. Theorem 2.2. does not remain true when the attractor has a infinite number of connected components. The following example is an attractor of an iterated function system with a countable number of connected components such that one of the connected components is not arcwise connected.

Example 3.3. We consider the following set of the plane $\mathbb{R}^2$ endowed with the euclidian metric $A = (\{0\} \times [0, 1]) \cup \bigcup_{n \geq 0} \left(\{\frac{1}{2^n}\} \times [0, 1]\right) = A^* \cup \left(\bigcup_{n \geq 0} A_n\right)$, where $A^* = \{0\} \times [0, 1]$ and $A_n = \{\frac{1}{2^n}\} \times [0, 1]$ for every $n \in \mathbb{N}$. Then A is the attractor of the iterated function system $\mathcal{S}_1 = (A, \{f_0, f_1, f_2, f_3\})$, where $f_i : A \longrightarrow \mathbb{R}^2$ for every $i \in \{0, 1, 2, 3\}$ are defined by $f_0(x, y) = (\frac{x}{2}, \frac{y}{2})$, $f_1(x, y) = (\frac{x}{2}, \frac{y+1}{2})$, $f_2(x, y) = (1, \frac{y}{2})$, $f_3(x, y) = (1, \frac{y+1}{2})$. We remark that A has an infinite number of connected components. We consider now the function $f_4 : A \longrightarrow \mathbb{R}^2$ defined by:

$$f_4(x,y)=\begin{cases}(\frac{x}{2}-\frac{y}{2^{2n+2}},\frac{y}{2}+3), \text{ if } (x,y)\in A_{2n} \text{ for every } n\in\mathbb{N},\\(\frac{x}{4}+\frac{y}{2^{2n+3}},\frac{y}{2}+3), \text{ if } (x,y)\in A_{2n+1} \text{ for every } n\in\mathbb{N},\\(0,\frac{y}{2}+3)\,, \text{ if } (x,y)\in A^*.\end{cases}$$

Then f_4 is a contraction and $B=f_4(A)$ is a connected set which is not arcwise connected. In fact f_4 transforms any vertical line of A into an oblique line of B such that the segment determined by $P(1,0)$ and $Q(1,1)$ goes into the segment determined by $P^{'}(\frac{1}{2},3)$ and $Q^{'}(\frac{1}{4},\frac{7}{2})$, the segment determined by $P_1(\frac{1}{2},0)$ and $Q_1(\frac{1}{2},1)$ goes into the segment determined by $P_1^{'}(\frac{1}{8},3)$ and $Q_1^{'}(\frac{1}{4},\frac{7}{2})=Q^{'}(\frac{1}{4},\frac{7}{2})$, and so on. In the end the segment determined by $O(0,0)$ and $R(0,1)$ goes into the segment determined by $O^{'}(0,3)$ and $R^{'}(0,\frac{7}{2})$.

Let $g_0,g_1,g_2,g_3,g_4 : A\cup B\longrightarrow A\cup B$ be the functions defined by $g_i|_A=f_i$ for $i\in\{0,1,2,3,4\}$, $g_0(B)=\{(0,0)\}$, $g_1(B)=\{(0,1)\}$, $g_2(B)=\{(1,0)\}$, $g_2(B)=\{(1,1)\}$ and $g_4(B)=\{(0,3)\}$. Then the set $A\cup B$ is the attractor of the iterated function system $\mathcal{S}_2=(A\cup B,\{g_0,g_1,g_2,g_3,g_4\})$ and we remark that $A\cup B$ has an infinite number of connected components, but not all the components are arcwise connected.

References

[1] M.F. Barnsley, *Fractals everywhere,* Academic Press Professional, Boston, 1993.

[2] D. Dumitru, A. Mihail, *The shift space of an iterated function system containing Meir-Keeler functions*, An. Stiint. Univ. Bucuresti, Matematica, LVII, **1**(2008), 75-89.

[3] K.J. Falconer, *Fractal Geometry-Foundations and Applications,* John Wiley, 1990.

[4] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. J. Math., **30**(1981), 713-747.

[5] J. Kigami, *Analysis on fractals,* Cambridge univ. Press., 2001.

[6] R. Miculescu, A. Mihail, *Lipscomb's space ϖ^A is the attractor of an IFS containing affine transformation of* $l^2(A)$, Proceedings of A.M.S., **136**(2008), 587-592.

[7] A. Mihail, *On the connectivity of attractors of iterated multifunction systems,* Real Analysis Exchange, **34**(2008-2009), 195-206.

[8] N.A. Secelean, *Countable iterated function systems,* Far East J. Dyn. Syst., **3**(2001), 149-167.

[9] M. Yamaguti, M. Hata, J. Kigami, *Mathematics of fractals*, American Mathematical Society, Translations of Mathematical Monographs, Vol 167, 1997.

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