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Warped product pseudo-slant submanifolds of nearly Kaehler manifolds

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Abstract

In this paper, we study warped product pseudo-slant submanifolds of nearly Kaehler manifolds. We prove the non-existence results on warped product submanifolds of a nearly Kaehler manifold.

1 Introduction

Slant submanifolds of an almost Hermitian manifold were defined by B.Y. Chen [3] as a natural generalization of both holomorphic and totally real submanifolds. Since then many researchers have studied these submanifolds in complex as well as contact setting [2, 8]. The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by N. Papaghiuc [9], and is in fact a generalization of CR-submanifolds. Pseudo-slant submanifolds were introduced by A. Carriazo [2] as a special case of bi-slant submanifolds.

Recently, B. Sahin [10] introduced the notion of warped product hemi-slant (pseudo-slant) submanifolds of Kaehler manifolds. He showed that there does not exist any warped product hemi-slant submanifolds in the form $M_\perp \times {}_f M_\theta$. He considered warped product hemi-slant submanifolds in the form $M_\theta \times {}_f M_\perp$ where $M_\perp$ is a totally real submanifold and M_θ is a proper slant submanifold of a Kaehler manifold, and gave some examples for their existence. In this paper we prove that there do not exist warped product submanifolds of the types $N_\perp \times {}_f N_\theta$ and $N_\theta \times {}_f N_\perp$ in a nearly Kaehler manifold $\bar{M}$, where $N_\perp$ is a totally real submanifold and N_θ is a proper slant submanifold of $\bar{M}$.

Key Words: Warped product, slant submanifold, pseudo-slant submanifold, nearly Kaehler manifold
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2 Preliminaries

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure J and a Riemannian metric g such that

$$(a)\quad J^2 = -I, \qquad\qquad (b)\quad g(JX, JY) = g(X, Y) \tag{2.1}$$

for all vector fields X, Y on $\bar{M}$.

Further let $T\bar{M}$ denote the tangent bundle of $\bar{M}$ and $\bar{\nabla}$, the covariant differential operator on $\bar{M}$ with respect to g. If the almost complex structure J satisfies

$$(\bar{\nabla}_X J)X = 0 \tag{2.2}$$

for any $X \in T\bar{M}$, then the manifold $\bar{M}$ is called a *nearly Kaehler manifold*. Equation (2.2) is equivalent to $(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0$. Obviously, every Kaehler manifold is nearly Kaehler manifold.

For a submanifold M of a Riemannian manifold $\bar{M}$, the Gauss and Weingarten formulae are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.3}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.4}$$

for all $X, Y \in TM$, where ∇ is the induced Riemannian connection on M, N is a vector field normal to $\bar{M}$, h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator of the second fundamental form. They are related as in [11] by

$$g(A_N X, Y) = g(h(X, Y), N) \tag{2.5}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the metric induced on M. The mean curvature vector H of M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) \tag{2.6}$$

where n is the dimension of M and $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of vector fields on M.

A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a *totally umbilical submanifold* if the second fundamental form satisfies

$$h(X, Y) = g(X, Y)H \tag{2.7}$$

for all $X, Y \in TM$. The submanifold M is *totally geodesic* if $h(X,Y)=0$, for all $X, Y \in TM$ and minimal if $H=0$.

For any $X \in TM$ and $N \in T^{\perp}M$, the transformations JX and JN are decomposed into tangential and normal parts respectively as

$$JX = TX + FX \tag{2.8}$$

$$JN = BN + CN. \tag{2.9}$$

Now, denote by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ the tangential and normal parts of $(\bar{\nabla}_X J)Y$, respectively. That is,

$$(\bar{\nabla}_X J)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y \tag{2.10}$$

for all $X, Y \in TM$. Making use of equations (2.8), (2.9) and the Gauss and Weingarten formulae, the following equations may be obtained easily.

$$\mathcal{P}_X Y = (\bar{\nabla}_X T)Y - A_{FY}X - Bh(X,Y) \tag{2.11}$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X F)Y + h(X,TY) - Ch(X,Y) \tag{2.12}$$

Similarly, for any $N \in T^{\perp}M$, denoting tangential and normal parts of $(\bar{\nabla}_X J)N$ by $\mathcal{P}_X N$ and $\mathcal{Q}_X N$ respectively, we obtain

$$\mathcal{P}_X N = (\bar{\nabla}_X B)N + TA_N X - A_{CN}X \tag{2.13}$$

$$\mathcal{Q}_X N = (\bar{\nabla}_X C)N + h(BN,X) + FA_N X \tag{2.14}$$

where the covariant derivatives of T, F, B and C are defined by

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \tag{2.15}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y \tag{2.16}$$

$$(\bar{\nabla}_X B)N = \nabla_X BN - B\nabla^{\perp}_X N \tag{2.17}$$

$$(\bar{\nabla}_X C)N = \nabla^{\perp}_X CN - C\nabla^{\perp}_X N \tag{2.18}$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$.

It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$, which we enlist here for later use:

(p_1) (i) $\mathcal{P}_{X+Y}W = \mathcal{P}_X W + \mathcal{P}_Y W$, (ii) $\mathcal{Q}_{X+Y}W = \mathcal{Q}_X W + \mathcal{Q}_Y W$,

(p_2) (i) $\mathcal{P}_X(Y+W) = \mathcal{P}_X Y + \mathcal{P}_X W$, (ii) $\mathcal{Q}_X(Y+W) = \mathcal{Q}_X Y + \mathcal{Q}_X W$,

(p_3) (i) $g(\mathcal{P}_X Y,\ W) = -g(Y, \mathcal{P}_X W)$, (ii) $g(\mathcal{Q}_X Y,\ N) = -g(Y, \mathcal{P}_X N)$,

$(p_4)\quad \mathcal{P}_X JY + \mathcal{Q}_X JY = -J(\mathcal{P}_X Y + \mathcal{Q}_X Y)$

for all $X, Y, W \in TM$ and $N \in T^\perp M$.

On a submanifold M of a nearly Kaehler manifold, by equations (2.2) and (2.10), we have

$$(a)\ \mathcal{P}_X Y + \mathcal{P}_Y X = 0, \quad (b)\ \mathcal{Q}_X Y + \mathcal{Q}_Y X = 0 \tag{2.19}$$

for any $X,\ Y \in TM$.

The submanifold M is said to be *holomorphic* if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand, M is said to be *totally real* if T is identically zero, that is $\phi X \in T^\perp M$, for any $X \in TM$.

A distribution D on a submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a *slant distribution* if for each $X \in D_x$, the angle θ between JX and D_x is constant i.e., independent of $x \in M$ and $X \in D_x$. In this case, a submanifold M of $\bar{M}$ is said to be a *slant submanifold* if the tangent bundle TM of M is slant.

Moreover, for a slant distribution D, we have

$$T^2 X = -\cos^2\theta X \tag{2.20}$$

for any $X \in D$. The following relations are straightforward consequences of equation (2.20):

$$g(TX, TY) = \cos^2\theta g(X, Y) \tag{2.21}$$

$$g(FX, FY) = \sin^2\theta g(X, Y) \tag{2.22}$$

for all $X, Y \in D$.

A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a *pseudo-slant submanifold* if there exist two orthogonal complementary distributions D_1 and D_2 satisfying:

(i) $TM = D_1 \oplus D_2$

(ii) D_1 is a slant distribution with slant angle $\theta \neq \pi/2$

(iii) D_2 is totally real i.e., $JD_2 \subseteq T^\perp M$.

A pseudo-slant submanifold M of an almost Hermitian manifold $\bar{M}$ is *mixed geodesic* if

$$h(X, Z) = 0 \tag{2.23}$$

for any $X \in D_1$ and $Z \in D_2$.

If μ is the invariant subspace of the normal bundle $T^\perp M$, then in the case of pseudo-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows:

$$T^\perp M = \mu \oplus FD_1 \oplus FD_2. \tag{2.24}$$

3 Warped product pseudo-slant submanifolds

In 1969 Bishop and O'Neill [1] introduced the notion of warped product manifolds. These manifolds are natural generalizations of Riemannian product manifolds. They defined these manifolds as: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times {}_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times {}_f N_2$ is said to be *trivial* if the warping function f is constant. We recall the following general formula on a warped product [1].

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z, \tag{3.2}$$

where X is tangent to N_1 and Z is tangent to N_2.

Let $M = N_1 \times {}_f N_2$ be a warped product manifold. This means that N_1 is totally geodesic and N_2 is a totally umbilical submanifold of M [1].

Throughout this section, we consider warped product pseudo-slant submanifolds which are either in the form $N_\perp \times {}_f N_\theta$ or $N_\theta \times {}_f N_\perp$ in a nearly Kaehler manifold $\bar{M}$, where N_θ and $N_\perp$ are proper slant and totally real submanifolds of $\bar{M}$, respectively. In the following theorem we consider the warped product pseudo-slant submanifolds in the form $M = N_\perp \times {}_f N_\theta$ of a nearly Kaehler manifold $\bar{M}$.

Theorem 3.1. *Let $\bar{M}$ be a nearly Kaehler manifold. Then the warped product submanifold $M = N_\perp \times {}_f N_\theta$ is a Riemannian product of $N_\perp$ and N_θ if and only if $\mathcal{P}_X TX$ lies in TN_θ, for any $X \in TN_\theta$, where $N_\perp$ and N_θ are totally real and proper slant submanifolds of $\bar{M}$, respectively.*

Proof. Let $M = N_\perp \times {}_f N_\theta$ be a warped product pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$. For any $X \in TN_\theta$ and $W \in TN_\perp$, we have

$$g(h(TX, W), FX) = g(\bar{\nabla}_W TX, FX) = -g(TX, \bar{\nabla}_W FX).$$

Using (2.8), we derive

$$g(h(TX, W), FX) = g(TX, \bar{\nabla}_W TX) - g(TX, \bar{\nabla}_W JX).$$

Then from (2.3) and the covariant derivative property of J, we obtain

$$g(h(TX, W), FX) = g(TX, \nabla_W TX) - g(TX, (\bar{\nabla}_W J)X) - g(TX, J\bar{\nabla}_W X).$$

Thus, using (2.1), (2.10) and (3.2) we get

$$g(h(TX,W),FX) = (W\ln f)g(TX,TX) - g(TX,\mathcal{P}_W X) + g(JTX,\bar{\nabla}_W X).$$

Using (2.3), (2.8), (2.19) (a) and (2.21), we obtain

$$g(h(TX,W),FX) = (W\ln f)\cos^2\theta\|X\|^2 + g(TX,\mathcal{P}_X W) + g(T^2X,\nabla_W X) + g(h(X,W),FTX).$$

Thus by property p_3 (i), (2.20) and (3.2), we derive

$$g(h(TX,W),FX) = (W\ln f)\cos^2\theta\|X\|^2 - g(\mathcal{P}_X TX,W) - (W\ln f)\cos^2\theta\|X\|^2 + g(h(X,W),FTX).$$

Hence the above equation takes the form

$$g(\mathcal{P}_X TX,W) = g(h(X,W),FTX) - g(h(TX,W),FX). \tag{3.3}$$

On the other hand for any $X \in TN_\theta$ and $W \in TN_\perp$, we have

$$g(h(X,TX),JW) = g(\bar{\nabla}_{TX}X,JW) = -g(J\bar{\nabla}_{TX}X,W).$$

Using the covariant differentiation formula of J, we get

$$g(h(X,TX),JW) = g((\bar{\nabla}_{TX}J)X,W) - g(\bar{\nabla}_{TX}JX,W).$$

Then by (2.10) and property of $\bar{\nabla}$, we derive

$$g(h(X,TX),JW) = g(\mathcal{P}_{TX}X,W) + g(JX,\bar{\nabla}_{TX}W).$$

Thus from (2.3), (2.8) and (2.19) (a), we obtain

$$g(h(X,TX),JW) = -g(\mathcal{P}_X TX,W) + g(TX,\nabla_{TX}W) + g(h(TX,W),FX).$$

By (3.2), the above equation reduces to

$$g(h(X,TX),JW) = -g(\mathcal{P}_X TX,W) + (W\ln f)g(TX,TX) + g(h(TX,W),FX).$$

Hence, using (2.21), we get

$$g(h(X,TX),JW) = -g(\mathcal{P}_X TX,W) + (W\ln f)\cos^2\theta\|X\|^2 + g(h(TX,W),FX). \tag{3.4}$$

By property (p_3) (i), the above equation reduces to

$$g(h(X,TX),JW) = g(TX, \mathcal{P}_X W) + (W \ln f)\cos^2\theta \|X\|^2$$
$$+ g(h(TX,W),FX).$$

Interchanging X with TX and then using (2.20) and (2.21), we obtain

$$-\cos^2\theta g(h(X,TX),JW) = -\cos^2\theta g(X, \mathcal{P}_{TX}W) + (W\ln f)\cos^4\theta g(X,X)$$
$$-\cos^2\theta g(h(X,W),FTX).$$

Again, by first using property (p_3) (i) followed by (2.19) (a) we arrive at

$$-g(h(X,TX),JW) = -g(\mathcal{P}_X TX, W) + (W\ln f)\cos^2\theta\|X\|^2$$
$$-g(h(X,W),FTX). \tag{3.5}$$

Then from (3.4) and (3.5), we obtain

$$2(W\ln f)\cos^2\theta\|X\|^2 = 2g(\mathcal{P}_X TX, W) + g(h(X,W),FTX)$$
$$-g(h(TX,W),FX). \tag{3.6}$$

Thus, by (3.3) and (3.6), we conclude that

$$(W\ln f)\cos^2\theta\|X\|^2 = \frac{3}{2}g(\mathcal{P}_X TX, W). \tag{3.7}$$

Since N_θ is proper slant, thus we get $(W\ln f) = 0$, if and only if $\mathcal{P}_X TX$ lies in TN_θ for all $X \in TN_\theta$ and $W \in TN_\perp$. This shows that f is constant on $N_\perp$. The proof is thus compelete. ☐

Theorem 3.2. *The warped product submanifold $M = N_\theta \times {}_f N_\perp$ of a nearly Kaehler manifold $\bar{M}$ is simply a Riemannian product of N_θ and $N_\perp$ if and only if*

$$g(h(X,Z),FZ) = g(h(Z,Z),FX), \tag{3.7}$$

for any $X \in TN_\theta$ and $Z \in TN_\perp$, where N_θ and $N_\perp$ are proper slant and totally real submanifolds of $\bar{M}$, respectively.

Proof. Let $M = N_\theta \times {}_f N_\perp$ be a warped product submanifold of a nearly Kaehler manifold $\bar{M}$. Then for any $X \in TN_\theta$ and $Z \in TN_\perp$, we have

$$g(h(TX,Z),FZ) = g(\bar{\nabla}_Z TX, JZ).$$

Using (2.1), we get

$$g(h(TX,Z),FZ) = -g(J\bar{\nabla}_Z TX, Z).$$

Thus, on using the covariant differentiation property of J, we obtain

$$g(h(TX,Z),FZ) = g((\bar{\nabla}_Z J)TX, Z) - g(\bar{\nabla}_Z JTX, Z).$$

Then from (2.8) and (2.10), we derive

$$g(h(TX,Z),FZ) = g(\mathcal{P}_Z TX, Z) - g(\bar{\nabla}_Z T^2 X, Z) - g(\bar{\nabla}_Z FTX, Z).$$

Now, using (2.4), (p_3) (i) and (2.20) we obtain that

$$g(h(TX,Z),FZ) = -g(\mathcal{P}_Z Z, TX) + \cos^2\theta g(\nabla_Z X, Z) + g(A_{FTX}Z, Z).$$

Since on using (2.2) and (2.10) we have $\mathcal{P}_Z Z = 0$, then from (2.5) and (3.2), we get

$$g(h(TX,Z),FZ) = (X\ln f)\cos^2\theta\|Z\|^2 + g(h(Z,Z),FTX). \tag{3.8}$$

Interchanging X with TX in (3.8), we obtain

$$\cos^2\theta g(h(X,Z),FZ) = -(TX\ln f)\cos^2\theta\|Z\|^2 + \cos^2\theta g(h(Z,Z),FX).$$

The above equation can be written as

$$(TX\ln f)\|Z\|^2 = g(h(Z,Z),FX) - g(h(X,Z),FZ). \tag{3.9}$$

Thus, $(TX\ln f) = 0$ if and only if $g(h(Z,Z),FX) = g(h(X,Z),FZ)$. This proves the theorem. □

The following corollaries are consequences of the above theorem.

Corollary 3.1. *There does not exist any warped product pseudo-slant submanifolds $M = N_\theta \times {}_f N_\perp$ of a nearly Kaehler manifold $\bar{M}$, if the condition*

$$h(TM, D^\perp) \in \mu,$$

holds, where μ is the invariant normal subbundle of TM and $D^\perp$ is a distribution corresponding to the submanifold $N_\perp$.

Proof. The proof follows from (3.9). □

Corollary 3.2. *There does not exist any mixed totally geodesic pseudo-slant warped product submanifold $M = N_\theta \times {}_f N_\perp$ of a nearly Kaehler manifold $\bar{M}$ such that $h(Z,Z) \in \mu$ for all $Z \in D^\perp$.*

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