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Almost Paracontact Structure on Finslerian Indicatrix

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Abstract

Recently Girtu by using the Sasaki metric, proved that the indicatrix bundle of a Finsler manifold carries an almost paracontact structure. As a generalization of this fact, we introduce a framed $f(3,-1)$-structure of corank 2 on the slit tangent bundle of a Finsler manifold. Then we prove that there exists an almost paracontact structure on the indicatrix bundle, when this structure is restricted to the indicatrix bundle of tangent bundle of Finsler manifold.

1 Introduction

A Riemannian metric g on a smooth manifold M gives rise to several Riemannian metrics on the tangent bundle TM. Maybe the best known example is the Sasaki metric g_S introduced in [13]. Although the Sasaki metric is naturally defined, it is very rigid; for example TM with the Sasaki metric is never locally symmetric unless the metric g on the base manifold is flat [10]. On the other hand, the Sasaki metric is not a good metric in the sense of [4] since its Ricci curvature is not constant, that is, the Sasaki metric is not Einstein [11].

The Sasaki-Matsumoto lift G_{SM} to the manifold $TM_0 := TM \setminus \{0\}$ of a Finsler metric tensor g is extremely important in the study of the geometry of a Finsler space $F^n = (M, F(x,y))$ [7]. This metric determines a Riemannian structure on TM_0, which depends only on the fundamental function F. Although the Sasaki-Matsumoto metric is naturally defined, but it is very rigid. For example, it is not difficult to see that G_{SM} does not have a Finslerian meaning. More precisely, the Sasaki-Matsumoto metric is not homogeneous

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with respect to the vertical variables y^i. Consequently, we cannot study global properties- as the Gauss-Bonnet theorem-for the Finsler space F^n by means of this lift [3][5]. Also, since the two terms of the metric G_{SM} do not have the same physical dimensions, it does not satisfy the principles of the Post-Newtonian calculus and so it is not convenient for a gauge theory. For this reasons, Miron define a new lift G_M to TM_0, which is 0-homogeneous on the fibers of the tangent bundle TM [9]. In continue, Anastasiei introduced lift metric G to TM_0 of a Finsler metric tensor g [1]. Then he showed that this metric is generalization of Sasaki-Matsumoto metric, Miron Metric, Cheeger-Gromoll metric [12] and Antonelli-Hrimiuc metrical structure [2]. We call this metric with g-natural metric.

In [6], Girtu by using the Sasaki metric showed that the indicatrix bundle of a Finsler manifold carries an almost paracontact structure. In this paper, we introduce a framed $f(3,-1)$-structure on the slit tangent bundle TM_0 of a Finsler space. Then by considering g-natural metric G, we prove that the framed $f(3,-1)$-structure on TM_0 induces on indicatrix bumdle IM an almost paracontact structure.

2 Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left[F^2(y+su+tv)\right]|_{s,t=0}, \quad u,v \in T_xM.$$

The homogeneity of F implies

$$F^2(x,y) = g_{ij}(x,y)y^iy^j = y^iy_i,$$

where $y_i = g_{ij}y^j$. The functions $N^i_j(x,y) = \frac{1}{2}\frac{\partial}{\partial y^j}(\gamma^i_{kr}(x,y)y^ky^r)$ and $\gamma^i_{kr}(x,y)$ the *generalized* Christoffel symbols, are the local coefficients of the nonlinear Cartan connection.

Let $(x,y) = (x^i,y^i)$ be the local coordinates on TM_0. It is well known that the tangent space to TM_0 at (x,y) splits into the direct sum of the vertical subspace $VTM_{0(x,y)} = span\{\partial_{\bar{i}}\}$ and the horizontal subspace $HTM_{0(x,y)} = span\{\delta_i\}$ as follows

$$T_{(x,y)}TM_0 = VTM_{0(x,y)} \oplus HTM_{0(x,y)},$$

where

$$\delta_i = \partial_i - N_i^k \partial_{\bar{k}}, \tag{1}$$

and $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$, $\delta_i = \frac{\delta}{\delta x^i}$, $\partial_i = \frac{\partial}{\partial x^i}$. Its dual basis is $(dx^i, \delta y^i)$, where

$$\delta y^i = dy^i + N_j^i(x,y)dx^j.$$

In [7], Matsumoto extended to Finsler spaces F^n the notion of Sasaki lift metric, considering the tensor field

$$G_{SM}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j \quad \forall (x,y) \in TM_0. \tag{2}$$

It easily follows that G_{SM} is a Riemannian metric globally defined on TM_0 and depending only on the fundamental function F of the Finsler space F^n. Also, We see that the Sasaki-Matsumoto lift G_{SM} is not homogeneous on the fibers of the tangent bundle TM.

The Miron metric is defined uniquely by the following relations

$$G_M(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + \frac{c^2}{F^2} g_{ij}(x,y)\delta y^i \otimes \delta y^j, \tag{3}$$

for each $(x,y) \in TM_0$, where c is a constant. It is obvious that G_M is 0-homogeneous on the fibers of TM and it depends only on the fundamental function of the Finsler space [14].

A general metric is in fact a family of Riemannian metrics (depending on two parameters) and we call it G [15]. The Sasaki-Matsumoto metric and the Miron metric are particular cases of this metric. It is defined by the following formulas

$$G(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + (a(F^2)g_{ij}(x,y) + b(F^2)y_i y_j)\delta y^i \otimes \delta y^j, \tag{4}$$

for all $(x,y) \in TM_0$, where $a, b : [0,\infty] \longrightarrow [0,\infty]$ and $a > 0$. The Sasaki-Matsumoto metric is obtained for $a = 1$ and $b = 0$, while the Miron metric for $a = \frac{c^2}{F^2}$ and $b = 0$.

An almost paracontact structure on a manifold N is a set (ϕ, ξ, η) where ϕ is a tensor field of type (1,1), ξ a vector field and η an 1-form such that

$$\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \phi^2 = I - \eta \otimes \xi \tag{5}$$

where I denotes the Kronecker tensor field. This structure generalizes as follows. One considers on a manifold N of dimension $(2n+s)$ a tensor field f of type (1,1). If there exists on N the vector fields (ξ_α) and the 1- forms η^α $(\alpha = 1, 2, \ldots, s)$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad f(\xi_\alpha) = 0, \quad \eta^\alpha \circ f = 0, \quad f^2 = I - \sum_\alpha \eta^\alpha \otimes \xi_\alpha, \tag{6}$$

then $(f, (\xi_\alpha), (\eta^\alpha))$ is called a framed $f(3, -1)$- structure. The term was suggested by the equation $f^3 - f = 0$. This is in some sense dual to the framed f-structure which generalizes the almost contact structure and which may be called a framed $f(3, 1)$- structure. For an account of such kind of structures we refer to the book [8].

3 A framed $f(3, -1)$- structure on TM_0

As is well known, there are two remarkable vector fields defined on TM_0. One is the vertical Liouville vector field $\mathcal{C} = y^i \partial_{\bar{i}}$, which is globally defined on TM_0. The other is the horizontal Liouville vector field $\mathcal{S} = y^i \delta_i$ (also called the geodesic spray field of F).

Let us put

$$\xi_1 := \alpha \mathcal{S} = \alpha y^i \delta_i \quad \text{and} \quad \xi_2 := \beta \mathcal{C} = \beta y^i \partial_{\bar{i}},$$

where α and β are functions on TM_0 to be determined. Also we define the linear operator P in the local basis by

$$P(\delta_i) = \delta_i, \quad P(\partial_{\bar{i}}) = -\partial_{\bar{i}}. \tag{7}$$

By a direct calculation, we get

$$P(\xi_1) = \xi_1, \quad P(\xi_2) = -\xi_2. \tag{8}$$

We consider the following 1-forms

$$\eta^1 = \gamma y_i dx^i, \quad \eta^2 = \lambda y_i \delta y^i, \tag{9}$$

then we have

Lemma 3.1. $\eta^1 \circ P = \eta^1, \quad \eta^2 \circ P = -\eta^2.$

Proof. It is sufficient to check these equalities on the adapted basis $(\delta_i, \partial_{\bar{i}})$. We have

$$(\eta^1 \circ P)(\delta_i) = \eta^1(P(\delta_i)) = \eta^1(\delta_i), \quad (\eta^1 \circ P)(\partial_{\bar{i}}) = -\eta^1(\partial_{\bar{i}}) = 0.$$

Similarly we get

$$(\eta^2 \circ P)(\delta_i) = \eta^2(\delta_i) = 0, \quad (\eta^2 \circ P)(\partial_{\bar{i}}) = -\eta^1(\partial_{\bar{i}}).$$

This completes the proof. □

Lemma 3.2. *For $X \in \chi(TM_0)$, we have $\eta^1(X) = G(X, \xi_1)$ and $\eta^2(X) = G(X, \xi_2)$ if and only if*

$$\alpha = \gamma, \quad \lambda = \beta(a + bF^2). \tag{10}$$

Proof. In the adapted basis $(\delta_i, \partial_{\bar{i}})$, we have

$$\eta^1(\partial_{\bar{i}}) = \gamma y_k dx^k(\partial_{\bar{i}}) = 0, \quad G(\partial_{\bar{i}}, \xi_1) = G(\partial_{\bar{i}}, \alpha y^k \delta_k) = 0.$$

Further

$$\eta^1(\delta_i) = \gamma y_k dx^k(\delta_i) = \gamma y_i, \quad G(\delta_i, \alpha y^k \delta_k) = \alpha y^k g_{ik} = \alpha y_i.$$

Then $\eta^1(X) = G(X, \xi_1)$ if and only if $\eta^1(\delta_i) = G(\delta_i, \xi_1)$ or $\alpha = \gamma$. Similarly, we have

$$\eta^2(\delta_i) = \lambda y_k \delta y^k(\delta_i) = 0, \quad G(\delta_i, \xi_2) = G(\partial_{\bar{i}}, \beta y^k \partial_{\bar{k}}) = 0,$$

and

$$\eta^2(\partial_{\bar{i}}) = \lambda y_k \delta y^k(\partial_{\bar{i}}) = \lambda y_i, \quad G(\partial_{\bar{i}}, \beta y^k \partial_{\bar{k}}) = \alpha y^k g_{ik} = \beta y^k(a g_{ik} + b y_i y_k) = \beta(a + bF^2) y_i.$$

Then $\eta^2(X) = G(X, \xi_2)$ if and only if $\eta^1(\partial_{\bar{i}}) = G(\partial_{\bar{i}}, \xi_2)$ or $\lambda = \beta(a + bF^2)$. □

Now we define a tensor field p of type $(1,1)$ on TM_0 by

$$p(X) = P(X) - \eta^1(X)\xi_1 + \eta^2(X)\xi_2, \quad X \in \chi(TM_0). \tag{11}$$

This can be written in a more compact form as $p = P - \eta^1 \otimes \xi_1 + \eta^2 \otimes \xi_2$.

Theorem 3.3. *For the triple $(p, (\xi_k), (\eta^k))$, k=1,2, we have*
(i) $\eta^k(\xi_l) = \delta^k_l, \quad p(\xi_k) = 0, \quad \eta^k \circ p = 0,$
(ii) $p^2 = I - \eta^1 \otimes \xi_1 - \eta^2 \otimes \xi_2,$
(iii) *p is of rank* $2n - 2$
if and only if

$$\alpha\gamma = \frac{1}{F^2}, \quad \beta\lambda = \frac{1}{F^2}. \tag{12}$$

Proof. By using (9) we have

$$\eta^1(\xi_1) = \gamma y_i dx^i(\alpha y^k \delta_k) = \alpha\gamma y_i y^i = \alpha\gamma F^2.$$

Further, since $dx^i(\partial_{\bar{k}}) = 0$, then we result that $\eta^1(\xi_2) = 0$. Therefore $\eta^1(\xi_l) = \delta_l^1$ if and only if $\eta^1(\xi_1) = 1$ or $\alpha\gamma = \frac{1}{F^2}$. By similar way we get $\eta^2(\xi_1) = 0$ and $\eta^2(\xi_2) = \beta\lambda F^2$. Therefore $\eta^2(\xi_l) = \delta_l^2$ if and only if $\beta\lambda = \frac{1}{F^2}$. From (8), (9) and (11), we obtain

$$p(\xi_1) = (1 - \alpha\gamma F^2)\xi_1, \quad p(\xi_2) = (\beta\lambda F^2 - 1)\xi_2. \tag{13}$$

Hence $p(\xi_k) = 0$ if and only if (12) is hold. By using (9), (11) and Lemma 3.1, we obtain

$$(\eta^1 \circ p)(\delta_i) = \eta^1[P(\delta_i) - \eta^1(\delta_i)\xi_1] = \eta^1(\delta_i) - \eta^1(\delta_i)\eta^1(\xi_1) = \gamma y_i(1 - \alpha\gamma F^2), \tag{14}$$

$$(\eta^1 \circ p)(\partial_{\bar{i}}) = \eta^1[P(\partial_{\bar{i}}) + \eta^2(\partial_{\bar{i}})\xi_2] = -\eta^1(\partial_{\bar{i}}) + \eta^1(\partial_{\bar{i}})\eta^1(\xi_2) = 0. \tag{15}$$

Similarly we get

$$(\eta^2 \circ p)(\delta_i) = 0, \quad (\eta^2 \circ p)(\partial_{\bar{i}}) = \lambda y_i(\beta\lambda F^2 - 1). \tag{16}$$

From (14), (15) and (16) we result that $\eta^1 \circ p = \eta^2 \circ p = 0$ if and only if (12) is hold. For (ii) we have

$$\begin{aligned} p^2(X) = p(p(X)) &= P[P(X) - \eta^1(X)\xi_1 + \eta^2(X)\xi_2] - \eta^1[P(X) - \eta^1(X)\xi_1 + \eta^2(X)\xi_2]\xi_1 \\ &+\eta^2[P(X) - \eta^1(X)\xi_1 + \eta^2(X)\xi_2]\xi_2 = X - \eta^1(X)\xi_1 - \eta^2(X)\xi_2 - \eta^1(P(X))\xi_1 \\ &+\eta^1(X)\eta^1(\xi_1)\xi_1 - \eta^2(X)\eta^1(\xi_2)\xi_1 + \eta^2(P(X))\xi_2 - \eta^1(X)\eta^2(\xi_1)\xi_2 + \eta^2(X)\eta^2(\xi_2)\xi_2. \end{aligned} \tag{17}$$

Since $\eta^1(\xi_2) = \eta^2(\xi_1) = 0$, $\eta^1(\xi_1) = \alpha\gamma F^2$ and $\eta^2(\xi_2) = \beta\lambda F^2$, then by using Lemma 3.1 we can rewrite (17) as follows

$$p^2(X) = X + (\alpha\gamma F^2 - 2)\eta^1(X)\xi_1 + (\beta\lambda F^2 - 2)\eta^2(X)\xi_2. \tag{18}$$

Therefore (ii) is hold if and only if $\alpha\gamma F^2 - 2 = -1$ (or $\alpha\gamma = \frac{1}{F^2}$) and $(\beta\lambda F^2 - 2) = -1$ (or $\beta\lambda = \frac{1}{F^2}$). For proof of (iii), it is sufficient to show that $\ker p = span(\xi_1, \xi_2)$ if and only if (12) is hold. Let $X = X^i\delta_i + X^{\bar{i}}\partial_{\bar{i}} \in \ker p$. By using (11), we have

$$p(X) = X^i\delta_i - X^{\bar{i}}\partial_{\bar{i}} - \gamma y_i X^i\xi_1 + \lambda y_i X^{\bar{i}}\xi_2 = (X^i - \alpha\gamma y_k X^k y^i)\delta_i - (X^{\bar{i}} - \beta\lambda y_k X^{\bar{k}} y^i)\partial_{\bar{i}} = 0,$$

which is equal to

$$X^i = \alpha\gamma y_k X^k y^i, \quad X^{\bar{i}} = \beta\lambda y_k X^{\bar{k}} y^i.$$

Hence

$$X = \alpha\gamma y_k X^k y^i \delta_i + \beta\lambda y_k X^{\bar{k}} y^i \partial_{\bar{i}} = \gamma y_k X^k \xi_1 + \lambda y_k X^{\bar{k}} \xi_2,$$

that is X belongs to $span(\xi_1, \xi_2)$. In other words, $\ker p \subseteq span(\xi_1, \xi_2)$. Let $X = c_1\xi_1 + c_2\xi_2 \in span(\xi_1, \xi_2)$. Then by using (13), we obtain

$$p(X) = c_1 p(\xi_1) + c_2 p(\xi_2) = c_1(1 - \alpha\gamma F^2)\xi_1 + c_2(\beta\lambda F^2 - 1)\xi_2.$$

Therefore $p(X) = 0$ if and only if (12) is hold. In other words, $span(\xi_1, \xi_2) = \ker p$ if and only if (12) is hold. □

Theorem 3.4. *$p^3 - p = 0$ if and only if $\alpha\gamma = \frac{k}{F^2}$ and $\beta\lambda = \frac{l}{F^2}$, where $k, l = 1, 2$.*

Proof. By using (13) and (18), we have

$$\begin{aligned} p^3(X) &= p(X) + (\alpha\gamma F^2 - 2)\eta^1(X)p(\xi_1) + (\beta\lambda F^2 - 2)\eta^2(X)p(\xi_2) \\ &= p(X) + (\alpha\gamma F^2 - 2)(1 - \alpha\gamma F^2)\eta^1(X)\xi_1 + (\beta\lambda F^2 - 2)(\beta\lambda F^2 - 1)\eta^2(X)\xi_2. \end{aligned} \tag{19}$$

By attention the above equation, it results that $p^3 = p$ if and only if $(\alpha\gamma F^2 - 2)(1 - \alpha\gamma F^2) = 0$ and $(\beta\lambda F^2 - 2)(\beta\lambda F^2 - 1) = 0$. □

Theorem 3.5. *If (10) and (12) is hold then the Riemannian metric G satisfies*

$$G(pX, pX) = G(X, Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y), \quad \forall X, Y \in \chi(TM_0). \tag{20}$$

Proof. Let (10) and (12) is hold, then we have

$$G(\xi_1, \xi_1) = G(\alpha y^i \delta_i, \alpha y^j \delta_j) = \alpha^2 y^i y^j g_{ij} = \alpha^2 F^2 = 1, \tag{21}$$

$$G(\xi_2, \xi_2) = G(\beta y^i \partial_{\bar{i}}, \beta y^j \partial_{\bar{j}}) = \beta^2 y^i y^j (a g_{ij} + b y_i y_j) = \beta^2 F^2 (a + bF^2) = 1, \tag{22}$$

and

$$G(\xi_1, \xi_2) = G(\alpha y^i \delta_i, \beta y^j \partial_{\bar{j}}) = 0. \tag{23}$$

From (21), (22), (23), Lemma 3.1 and Lemma 3.2 we get

$$\begin{aligned} G(pX, pY) &= G(PX, PY) - G(PX, \xi_1)\eta^1(Y) + G(P(X), \xi_2)\eta^2(Y) - G(\xi_1, PY)\eta^1(X) \\ &\quad + \eta^1(X)\eta^1(Y) + \eta^2(X)G(\xi_2, PY) + \eta^2(X)\eta^2(Y) \\ &= G(X, Y) - \eta^1(PX)\eta^1(Y) + \eta^2(PX)\eta^2(Y) - \eta^1(PY)\eta^1(X) \\ &\quad + \eta^1(X)\eta^1(Y) + \eta^2(X)\eta^2(PY) + \eta^2(X)\eta^2(Y) \\ &= G(X, Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y) \end{aligned} \tag{24}$$

This completes the proof. □

Remark 3.6. It is easy to check that conditions (10) and (12) are equivalent to

$$\alpha = \gamma = \pm\frac{1}{F}, \quad \beta = \pm\frac{1}{F\sqrt{a+bF^2}}, \quad \lambda = \pm\frac{\sqrt{a+bF^2}}{F}. \tag{25}$$

Therefore by attention to the Theorem 3.5, we can conclude that if (25) is hold then the metric G satisfies in (20).

From (11), the following local expression of p hold

$$p(\delta_i) = P(\delta_i) - \eta^1(\delta_i)\xi_1 = (\delta_i^k - \alpha\gamma y_i y^k)\delta_k \tag{26}$$
$$p(\partial_{\bar{i}}) = P(\partial_{\bar{i}}) + \eta^2(\partial_{\bar{i}})\xi_2 = (\beta\lambda y_i y^k - \delta_i^k)\partial_{\bar{k}} \tag{27}$$

Using (26) and (27), one can obtains

$$\begin{aligned} G(p(\partial_{\bar{i}}), p(\partial_{\bar{j}})) &= (\beta\lambda y_i y^k - \delta_i^k)(\beta\lambda y_j y^r - \delta_j^r)(a g_{kr} + b y_k y_r) \\ &= a g_{ij} + [\beta\lambda(a+bF^2)(\beta\lambda F^2 - 2) + b] y_i y_j, \qquad (28) \\ G(p(\delta_i), p(\delta_j)) &= (\delta_i^k - \alpha\gamma y_i y^k)(\delta_j^r - \alpha\gamma y_j y^r) g_{kr} \\ &= g_{ij} + \alpha\gamma(\alpha\gamma F^2 - 2) y_i y_j, \qquad (29) \\ G(p(\delta_i), p(\partial_{\bar{j}})) &= 0. \qquad (30) \end{aligned}$$

Theorem 3.7. *(G, p) is almost product structure if and only if $\alpha\gamma = \frac{2}{F^2}$ and $\beta\lambda = \frac{2}{F^2}$.*

Proof. If $\alpha\gamma = \frac{2}{F^2}$ and $\beta\lambda = \frac{2}{F^2}$, then by using (18) we have $p^2 = I$. In this case, also by using (28), (29) and (30) we obtain

$$\begin{aligned} G(p(\partial_{\bar{i}}), p(\partial_{\bar{j}})) &= a g_{ij} + b y_i y_j = G(\delta_i, \delta_j), \\ G(p(\delta_i), p(\delta_j)) &= g_{ij} = G(\delta_i, \delta_j), \\ G(p(\delta_i), p(\partial_{\bar{j}})) &= 0 = G(\delta_i, \partial_{\bar{j}}). \end{aligned}$$

In other word we have $G(p(X), p(Y)) = G(X, Y)$, for all $X \in TM_0$, i.e., (G, p) is almost product structure.

Conversely, if (G, p) is almost product structure, then from condition $p^2 = I$ and (18) we result that $\alpha\gamma = \frac{2}{F^2}$ and $\beta\lambda = \frac{2}{F^2}$. □

Let us put

$$h(X, Y) = G(pX, Y), \quad X, Y \in \chi(TM_0). \tag{31}$$

Then we get the following.

Theorem 3.8. *The map h is a symmetric bilinear form on TM_0. Further, h is of $rank 2n-2$ with the null space $span(\xi_1, \xi_2)$ if and only if (12) is hold.*

Proof. By using (26), (27) and (31) we obtain

$$h(\partial_{\bar{i}}, \partial_{\bar{j}}) = (\beta\lambda y_i y^k - \delta_i^k)(a g_{kj} + b y_k y_j) = -a g_{ij} + [\beta\lambda(a + bF^2) - b] y_i y_j, \quad (32)$$
$$h(\delta_i, \delta_j) = (\delta_i^k - \alpha\gamma y_i y^k) g_{kj} = g_{ij} - \alpha\gamma y_i y_j, \quad (33)$$
$$h(\delta_i, \partial_{\bar{j}}) = 0. \quad (34)$$

Since G is bilinear, then from the above equations we conclude that h is symmetric bilinear form on TM_0. For proof the second part of theorem, first we let (12) is hold. Then by using (i) of the Theorem 3.3, we have $h(\xi_1, \xi_1) = h(\xi_2, \xi_2) = 0$. Thus $span(\xi_1, \xi_2)$ is contained in the null space of h. Now, if $X = X^i \delta_i$ is such that $h(X, X) = 0$ it results $X = \frac{y_k X^k \xi_1}{F^2}$ and similarly, if $X = X^{\bar{i}} \partial_{\bar{i}}$ is such that $h(X, X) = 0$, it results $X = \frac{y_k X^{\bar{k}} \xi_2}{F^2}$. Thus the null space of h is just $span(\xi_1, \xi_2)$. Conversely, we let the null space of h is $span(\xi_1, \xi_2)$. Then we have $h(\xi_1, \xi_1) = h(\xi_2, \xi_2) = 0$. By using (13), we obtain

$$h(\xi_1, \xi_1) = G(p(\xi_1), \xi_1) = (1 - \alpha\gamma F^2) G(\xi_1, \xi_1) = \alpha^2 F^2 (1 - \alpha\gamma F^2)$$
$$h(\xi_2, \xi_2) = G(p(\xi_2), \xi_2) = (\beta\lambda F^2 - 1) G(\xi_2, \xi_2) = (\beta\lambda F^2 - 1)[\beta^2 F^2 (a + bF^2)]$$

From the above equations and condition $h(\xi_1, \xi_1) = h(\xi_2, \xi_2) = 0$, we result $\alpha\gamma F^2 = 1$ and $\beta\lambda F^2 = 1$. □

If $\alpha\gamma = \beta\lambda = \frac{1}{F^2}$, then from (32), (33) and (34) the bilinear form h as follows

$$h = (g_{ij} - \frac{2}{F^2}) dx^i \otimes dx^j - (a g_{ij} - (\frac{2a + bF^2}{F^2}) y_i y_j) \delta y^i \otimes \delta y^j. \quad (35)$$

If $\alpha\gamma = \beta\lambda = \frac{1}{F^2}$, then by (35) and Theorem 3.7 we have the following.

Theorem 3.9. *If $\alpha\gamma = \beta\lambda = \frac{1}{F^2}$, then the map h is a singular pseudo-Riemannian metric on TM_0 and it is the twin tensor of almost product metric G.*

4 Almost Paracontact Structure on Indicatrix Bundle

The set $IM = \{(x, y) \in TM_0 | F(x, y) = 1\}$ is called the indicatrix bundle of F^n. This set is a submanifold of dimension $2n - 1$ of TM_0. We show that the framed $f(3, -1)$- structure on TM_0, given by Theorem 3.3, induces an almost paracontact structure on TM_0.

It is easy to show that $\xi_2 = \beta y^i \partial_{\bar{i}}$ is the unit normal vector field with respect to the metric G. Indeed, if the local equations of IM in TM_0 are

$$x^i = x^i(u^\alpha), \quad y^i = y^i(u^\alpha), \quad \alpha \in \{1, ..., 2m - 1\}, \quad (36)$$

then, we have

$$\frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha}+\frac{\partial F}{\partial y^i}\frac{\partial y^i}{\partial u^\alpha}=0. \tag{37}$$

As the h-covariant derivative of F vanishes, by using (1), we obtain

$$(N_i^k\frac{\partial x^i}{\partial u^\alpha}+\frac{\partial y^k}{\partial u^\alpha})\ell_k=0, \tag{38}$$

where $\ell_k=\frac{y_k}{F}$. The natural frame field on IM is represented by

$$\frac{\partial}{\partial u^\alpha}=\frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i}+\frac{\partial y^i}{\partial u^\alpha}\frac{\partial}{\partial y^i}=\frac{\partial x^i}{\partial u^\alpha}\frac{\delta}{\delta x^i}+(N_i^k\frac{\partial x^i}{\partial u^\alpha}+\frac{\partial y^k}{\partial u^\alpha})\frac{\partial}{\partial y^k}. \tag{39}$$

Then by (38), we deduce that

$$G(\frac{\partial}{\partial u^\alpha},\xi_2)=\beta(a+bF^2)(N_i^k\frac{\partial x^i}{\partial u^\alpha}+\frac{\partial y^k}{\partial u^\alpha})y_k=0. \tag{40}$$

Thus ξ_2 is orthogonal to any vector tangent to IM. The vector field $\xi_1=\alpha y^i\delta_i$ is tangent to IM since $G(\xi_1,\xi_2)=0$.

We restrict to IM all the objects introduced above and indicate this fact by putting a bar over the letters denoting those objects. We have

Lemma 4.1. *On indicatrix bundle IM, the following hold*

$$\bar{\xi}_1=\xi_1,\quad \bar{\eta}^2=0,\quad \bar{p}(X)=P(X)-\bar{\eta}^1(X)\xi_1,\quad \forall X\in\chi(IM).$$

Proof. Since ξ_1 is tangent to IM, then we result $\bar{\xi}_1=\xi_1$. From $\bar{\eta}^2(X)=G(X,\xi_2)=0$, the other relation of lemma will conclude. □

Since $G(\bar{p}(X),\xi_2)=0$, then we have the following.

Lemma 4.2. *The map $\bar{p}$ is an endomorphism of the tangent bundle to IM.*

We put $\bar{\xi}_1=\bar{\xi}$, $\bar{\eta}^1=\bar{\eta}$. Then Theorem 3.3, Theorem 3.4 and Lemmas 4.1 and 4.2, imply the following.

Theorem 4.3. *If (12) is hold, then triple $(\bar{p},\bar{\xi},\bar{\eta})$ defines an almost paracontact structure on IM, that is,*

(i) $\bar{\eta}(\bar{\xi})=1,\quad \bar{p}(\bar{\xi})=0,\quad \bar{\eta}\circ\bar{p}=0,$

(ii) $\bar{p}^2(X)=X-\bar{\eta}(X)\bar{\xi},\quad X\in\chi(IM),$

(iii) $\bar{p}^3-\bar{p}=0,\quad rank\bar{p}=2n-2=(2n-1)-1.$

Let $\bar{G} = G|_{IM}$. Using the restriction to IM and Theorem 3.5, one infers that if (10) and (12) are hold then the Riemannian metric $\bar{G}$ satisfies

$$\bar{G}(\bar{p}X, \bar{p}Y) = \bar{G}(X, Y) - \bar{\eta}(X)\bar{\eta}(Y), \quad X, Y \in \chi(IM). \tag{41}$$

By the equation (41) and Theorem 4.3, we conclude the following.

Theorem 4.4. *If (10) and (12) are hold then the ensemble $(\bar{p}, \bar{\xi}, \bar{\eta}, \bar{G})$ defines an almost metrical paracontact structure on IM.*

References

[1] M. Anastasiei, *Locally conformal Kaehler structures on tangent manifold of a space form*, Libertas Math. **19**(1999), 71-76.

[2] P. L. Antonelli and D. Hrimiuc, *A new class of Spray-generating Lagrangians. In the volume Lagrange and Finsler geometry. Application to Physics and Biology*, Eds. P.L. Antonelli and R. Miron, Kluwer Academic Publishers, TFTPH 76, 1996, 81-92.

[3] D. Bao, S. S. Chern and Z. Shen, *Contemporary Mathematics*, vol. 196, A.M.S., 1996.

[4] A. L. Besse, Einstein Manifolds, Springer-Verlag, Berlin, New York, 1987.

[5] M. P. Do Carmo, *Riemannian Geometry*, Birkhäuser, Boston, 1992.

[6] M. Girtu, *An Almost Paracontact Structure on the Indicatrix Bundle of a Finsler Space*, Balkan Journal of Geometry and Its Applications. **7**(2002), 43-48.

[7] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha, Japan, 1986.

[8] I. Mihai, R. Rosca and L. Verstraelen, *Some aspects of the differential geometry of vector fields*, PADGE, Katholieke Universitieit Leuven, Vol. 2, 1996.

[9] R. Miron, *The homogeneous lift to the tangent bundle of a Finsler metric*, Publ. Math. Debrecen. **57**(2000), 445-453.

[10] E. Peyghan and A. Tayebi, *Finslerian Complex and Kählerian Structures*, Journal of Non-Linear Analysis: Real World Applications, **11**(2010), 3021-3030.

[11] E. Peyghan and A. Tayebi, *A Kähler structures on Finsler spaces with non-zero constant flag curvature*, Journal of Mathematical Physics, **51**, 022904 (2010).

[12] M. Sekizawa, *Curvatures of tangent bundle with a Cheeger-Gromoll metric*, Tokyo J. Math. **14**(1991), 407-417

[13] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tohoku Math. J. **10**(1958), 338-354.

[14] A. Tayebi and E. Peyghan, *On a class of Riemannian metrics arising from Finsler structures*, C. R. Acad. Sci. Paris, Ser. I, **349** (2011), 319-322.

[15] S. Vacaru, *Finsler and Lagrange Geometries in Einstein and String Gravity*, Int. J. Geom. Methods. Mod. Phys. **5** (2008) 473-511.

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