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Common Fixed Points of Two Nonself Asymptotically Nonexpansive Mapping by a Simpler Iterative Process

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Abstract

In this paper, we use a new one-step iterative process to approximate the common fixed points of two nonself asymptotically nonexpansive mappings through some weak and strong convergence theorems.

1 Introduction

Throughout this paper $\mathbb{N}$ denotes the set of all positive integers. Let E be a real Banach space and C a nonempty subset of E. A subset C of E is called a retract of E if there exists a continuous map $P : E \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if P is a retraction, then $Py = y$ for all y in the range of P.

Chidume et al. [2] defined nonself asymptotically nonexpansive mappings as follows: Let $P : E \to C$ be a nonexpansive retraction of E into C. A nonself mapping $T : C \to E$ is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, we have $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x-y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$. Also T is called uniformly k-Lipschitzian if for some $k > 0$, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k\|x - y\|$ for all $n \in \mathbb{N}$ and $x, y \in C$. Note that every asymptotically nonexpansive mapping is uniformly k-Lipschitzian.

A point $x \in C$ is a fixed point of T provided $Tx = x$.

Key Words: one-step iterative process, nonself asymptotically nonexpansive mapping, common fixed point, condition $(\tilde{A})$, weak and strong convergence

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To approximate the common fixed points of two mappings, the following Ishikawa type two-steps iterative process is widely used (see, for example [5], [8], [11] and references cited therein):

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-a_n)x_n + a_n S^n y_n, \\ y_n = (1-b_n)x_n + b_n T^n x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{a_n\}$ and $\{b_n\}$ are in $[0,1]$ satisfying certain conditions. Note that approximating fixed points of two mappings has a direct link with the minimization problem, see for example [10].

Recently, Abbas et al. [1] introduced a new one-step iterative process to compute the common fixed points of two asymptotically nonexpansive mappings. Following is the modification of their process to the case of two nonself asymptotically nonexpansive mappings. Let E be a normed space and C its nonempty closed convex subset. Let $S, T : C \to E$ be nonself asymptotically nonexpansive mappings and $P : E \to C$ a nonexpansive retraction of E into C. We define $\{x_n\}$ in C as

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P\left(a_n S(PS)^{n-1}x_n + (1-a_n)T(PT)^{n-1}x_n\right), \quad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{a_n\}$ is in $[0,1]$ satisfying certain conditions.

In this paper, we will use this process to prove some weak and strong convergence theorems for approximating common fixed points of two nonself asymptotically nonexpansive mappings.

Let us recall the following definitions.

A Banach space E is said to satisfy Opial's condition [6] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying this condition are Hilbert spaces and all spaces $l^p (1 < p < \infty)$ where as $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition. A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

A Banach space E is said to satisfy the Kadec Klee property if for every sequence $\{x_n\}$ in E converging weakly to x together with $\|x_n\|$ converging strongly to $\|x\|$ imply $\{x_n\}$ converges strongly to x. Uniformly convex Banach spaces and Banach spaces of finite dimension are some of the examples of reflexive Banach spaces which satisfy the Kadec Klee property.

Next, we state the following useful lemmas.

Lemma 1.1. ([7]) Let $\{\delta_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that

$$\beta_n \geq 1 \quad \text{and} \quad \delta_{n+1} \leq \beta_n \delta_n + \gamma_n \quad \text{for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} (\beta_n - 1) < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Lemma 1.2. ([9]) Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.3. ([2]) Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero.

Lemma 1.4. ([3]) Let C be a convex subset of a uniformly convex Banach space. Then there is a strictly increasing and continuous convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for every Lipschitzian map $U : C \to C$ with Lipschitz constant $L \geq 1$, the following inequality holds:

$$\|U(tx + (1-t)y) - (tUx + (1-t)Uy\| \leq Lg^{-1}(\|x-y\| - L^{-1}\|Ux - Uy\|)$$

for all $x, y \in C$ and $t \in [0, 1]$.

Let $\omega_w(\{x_n\})$ denote the set of all weak subsequential limits of a bounded sequence $\{x_n\}$ in E. Then the following is actually Lemma 3.2 of Falset et al. [3]

Lemma 1.5. *Let E be a uniformly convex Banach space such that its dual E^* satisfies the Kadec Klee property. Assume that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0, 1]$ and $p_1, p_2 \in \omega_w(\{x_n\})$. Then $\omega_w(\{x_n\})$ is singleton.*

2 Some Preparatory Lemmas

In the sequel, we will write $F = F(S) \cap F(T)$ for the set of all common fixed points of the mappings S and T. If S and T are nonself asymptotically nonexpansive mappings with sequences $\{s_n\}, \{t_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty}(s_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(t_n - 1) < \infty$, then putting $k_n = \max\{s_n, t_n\}$, we have $\{k_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Thus in the sequel, we take the same sequence $\{k_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ for both the mappings S and T.

Lemma 2.1. *Let E be a normed space and C its nonempty closed convex subset. Let $S,T : C \to E$ be nonself asymptotically nonexpansive mappings satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence as defined in* (1.2) *where $\{a_n\}$ is a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $F \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$.*

Proof. Let $x^* \in F$. Then

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \left\|P\left(a_n S(PS)^{n-1}x_n + (1-a_n)T(PT)^{n-1}x_n\right) - Px^*\right\| \\
&\leq \left\|a_n S(PS)^{n-1}x_n + (1-a_n)T(PT)^{n-1}x_n - x^*\right\| \\
&= \left\|a_n\left(S(PS)^{n-1}x_n - x^*\right) + (1-a_n)(T(PT)^{n-1}x_n - x^*)\right\| \\
&\leq a_n\left\|S(PS)^{n-1}x_n - x^*\right\| + (1-a_n)\left\|(T(PT)^{n-1}x_n - x^*)\right\| \\
&\leq a_n k_n\|x_n - x^*\| + (1-a_n)k_n\|x_n - x^*\| \\
&= k_n\|x_n - x^*\|.
\end{aligned}
$$

Thus, by Lemma 1.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists for each $x^* \in F$. □

Lemma 2.2. *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $S,T : C \to E$ be nonself asymptotically nonexpansive mappings satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence as defined in* (1.2) *satisfying $\|x_n - S(PS)^{n-1}x_n\| \leq \|S(PS)^{n-1}x_n - T(PT)^{n-1}x_n\|$ for all $n \in \mathbb{N}$ and $\{a_n\}$ a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $F \neq \emptyset$, then $\lim_{n\to\infty}\|Sx_n - x_n\| = 0 = \lim_{n\to\infty}\|Tx_n - x_n\|$.*

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Suppose $\lim_{n\to\infty}\|x_n - x^*\| = c$ for some $c \geq 0$. Then $\left\|S(PS)^{n-1}x_n - x^*\right\| \leq k_n\|x_n - x^*\|$ implies that

$$\limsup_{n\to\infty}\left\|S(PS)^{n-1}x_n - x^*\right\| \leq c.$$

Similarly,

$$\limsup_{n\to\infty}\left\|T(PT)^{n-1}x_n - x^*\right\| \leq c.$$

Further, $\lim_{n\to\infty}\|x_{n+1}-x^*\| = c$ gives that

$$\begin{aligned}
c &= \lim_{n\to\infty}\|x_{n+1}-x^*\| \\
&= \lim_{n\to\infty}\left\|P\left((1-\alpha_n)T(PT)^{n-1}x_n+\alpha_n S(PS)^{n-1}x_n\right)-Px^*\right\| \\
&\le \lim_{n\to\infty}\|\,(1-\alpha_n)\left(T(PT)^{n-1}x_n-x^*\right)+\alpha_n\left(S(PS)^{n-1}x_n-x^*\right)\|\\
&\le \lim_{n\to\infty}\Big[(1-\alpha_n)\Big\|\limsup_{n\to\infty}\left(T(PT)^{n-1}x_n-x^*\right)\Big\| \\
&\quad +\alpha_n\Big\|\limsup_{n\to\infty}\left(S(PS)^{n-1}x_n-x^*\right)\Big\|\Big]\\
&= \lim_{n\to\infty}\left[(1-\alpha_n)\,c+\alpha_n c\right]\\
&= c
\end{aligned}$$

so that

$$\lim_{n\to\infty}\|\,(1-\alpha_n)\left(T(PT)^{n-1}x_n-x^*\right)+\alpha_n\left(S(PS)^{n-1}x_n-x^*\right)\|=c. \tag{2.1}$$

Applying Lemma 1.2, we obtain

$$\lim_{n\to\infty}\left\|S(PS)^{n-1}x_n-T(PT)^{n-1}x_n\right\|=0. \tag{2.2}$$

But then by the condition

$$\left\|x_n-S(PS)^{n-1}x_n\right\|\le\left\|S(PS)^{n-1}x_n-T(PT)^{n-1}x_n\right\|,$$

we get

$$\limsup_{n\to\infty}\|x_n-S(PS)^{n-1}x_n\|\le 0.$$

That is,

$$\lim_{n\to\infty}\|x_n-S(PS)^{n-1}x_n\|=0. \tag{2.3}$$

Then

$$\begin{aligned}
&\|x_n-T(PT)^{n-1}x_n\|\\
&\le\|x_n-S(PS)^{n-1}x_n\|+\left\|S(PS)^{n-1}x_n-T(PT)^{n-1}x_n\right\|
\end{aligned}$$

implies that

$$\lim_{n\to\infty}\|x_n-T(PT)^{n-1}x_n\|=0. \tag{2.4}$$

Now,

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&=\left\|P\left((1-\alpha_n)T(PT)^{n-1}x_n+\alpha_n S(PS)^{n-1}x_n\right)-Px_n\right\|\\
&\le\left\|(1-\alpha_n)T(PT)^{n-1}x_n+\alpha_n S(PS)^{n-1}x_n-x_n\right\|\\
&\le(1-\alpha_n)\left\|T(PT)^{n-1}x_n-x_n\right\|+\alpha_n\left\|S(PS)^{n-1}x_n-x_n\right\|
\end{aligned}$$

so that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.5}$$

Then

$$\|x_{n+1} - T(PT)^{n-1}x_n\| \le \|x_{n+1} - x_n\| + \|x_n - T(PT)^{n-1}x_n\|$$

implies by (2.4) and (2.5) that

$$\lim_{n\to\infty} \|x_{n+1} - T(PT)^{n-1}x_n\| = 0. \tag{2.6}$$

From (2.2), (2.6) and

$$\begin{aligned}
&\|x_{n+1} - S(PS)^{n-1}x_n\| \\
&\le \|x_{n+1} - T(PT)^{n-1}x_n\| + \left\|S(PS)^{n-1}x_n - T(PT)^{n-1}x_n\right\|,
\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_{n+1} - S(PS)^{n-1}x_n\| = 0. \tag{2.7}$$

Next, by making use of the fact that every asymptotically nonexpansive mapping is k-Lipschitzian, we have

$$\begin{aligned}
&\|x_{n+1} - Sx_{n+1}\| \\
&\le \|x_{n+1} - S(PS)^n x_{n+1}\| + \|S(PS)^n x_{n+1} - S(PS)^n x_n\| \\
&\quad + \|S(PS)^n x_n - Sx_{n+1}\| \\
&\le \|x_{n+1} - S(PS)^n x_{n+1}\| + \|S(PS)^n x_{n+1} - S(PS)^n x_n\| \\
&\quad + \|S(PS)^{1-1}(PS)^n x_n - S(PS)^{1-1}x_{n+1}\| \\
&\le \|x_{n+1} - S(PS)^n x_{n+1}\| + \|S(PS)^n x_{n+1} - S(PS)^n x_n\| \\
&\quad + k\|(PS)^n x_n - x_{n+1}\| \\
&= \|x_{n+1} - S(PS)^n x_{n+1}\| + \|S(PS)^n x_{n+1} - S(PS)^n x_n\| \\
&\quad + k\|PS(PS)^{n-1}x_n - Px_{n+1}\| \\
&\le \|x_{n+1} - S(PS)^n x_{n+1}\| + k_{n+1}\|x_{n+1} - x_n\| \\
&\quad + k\|S(PS)^{n-1}x_n - x_{n+1}\|
\end{aligned}$$

and so

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{2.8}$$

Moreover,

$$\begin{aligned}
&\|Sx_{n+1} - Tx_{n+1}\| \\
&\le \|Sx_{n+1} - S(PS)^n x_{n+1}\| + \|S(PS)^n x_{n+1} - T(PT)^n x_{n+1}\| \\
&\quad + \|T(PT)^n x_{n+1} - T(PT)^n x_n\| + \|T(PT)^n x_n - Tx_{n+1}\| \\
&\le k\|x_{n+1} - S(PS)^{n-1} x_{n+1}\| + \|S(PS)^n x_{n+1} - T(PT)^n x_{n+1}\| \\
&\quad + k_{n+1}\|x_{n+1} - x_n\| + k\|T(PT)^{n-1} x_n - x_{n+1}\| \\
&\le k\left(\|x_{n+1} - S(PS)^{n-1} x_n\| + \|S(PS)^{n-1} x_n - S(PS)^{n-1} x_{n+1}\|\right) \\
&\quad + \|S(PS)^n x_{n+1} - T(PT)^n x_{n+1}\| + k_{n+1}\|x_{n+1} - x_n\| \\
&\quad + k\|T(PT)^{n-1} x_n - x_{n+1}\| \\
&\le k\left(\|x_{n+1} - S(PS)^{n-1} x_n\| + k_n\|x_n - x_{n+1}\|\right) \\
&\quad + \|S(PS)^n x_{n+1} - T(PT)^n x_{n+1}\| + k_{n+1}\|x_{n+1} - x_n\| \\
&\quad + k\|T(PT)^{n-1} x_n - x_{n+1}\|
\end{aligned}$$

gives by (2.2), (2.5), (2.6) and (2.7) that

$$\lim_{n\to\infty} \|Sx_n - Tx_n\| = 0. \tag{2.9}$$

In turn, by (2.8) and (2.9) , we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof of the lemma. □

Lemma 2.3. *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $S, T : C \to E$ be nonself asymptotically nonexpansive mappings satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{x_n\}$ as defined in* (1.2). *Then, for any $p_1, p_2 \in F$, $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0, 1]$.*

Proof. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$ and so $\{x_n\}$ is bounded. Thus there exists a real number $r > 0$ such that $\{x_n\} \subseteq D \equiv \overline{B_r(0)} \cap C$, so that D is a closed convex bounded nonempty subset of C. Put $u_n(t) = \|tx_n + (1-t)p_1 - p_2\|$. Notice that $\lim_{n\to\infty} u_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} u_n(1) = \|x_n - p_2\|$ exist as proved in Lemma 2.1. Define $W_n : D \to D$ by:

$$W_n x = P(a_n S(PS)^{n-1} x + (1 - a_n)T(PT)^{n-1} x).$$

It is easy to verify that $W_n x_n = x_{n+1},$ $W_n p = p$ for all $p \in F$ and

$$\|W_n x - W_n y\| \le k_n \|x - y\| \quad \text{for all } x, y \in C,\ n \in \mathbb{N}.$$

Set

$$R_{n,m} = W_{n+m-1}W_{n+m-2}\cdots W_n, \quad m \in \mathbb{N}$$

and

$$v_{n,m} = \|R_{n,m}(tx_n + (1-t)p_1) - (tR_{n,m}x_n + (1-t)p_1)\|.$$

Then $\|R_{n,m}x - R_{n,m}y\| \leq \prod_{j=n}^{n+m-1} k_j \|x-y\|$, $R_{n,m}x_n = x_{n+m}$ and $R_{n,m}p = p$ for all $p \in F$. Applying Lemma 1.4 with $x = x_n,\ y = p_1, U = R_{n,m}$ and using the facts that $\sum_{k=1}^{\infty}(k_n - 1) < \infty$ and $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$, we obtain $v_{n,m} \to 0$ as $n \to \infty$ and for all $m \geq 1$.

Finally, from the inequality

$$\begin{aligned}
u_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\
&= \|tR_{n,m}x_n + (1-t)p_1 - p_2\| \\
&\leq v_{n,m} + \|R_{n,m}(tx_n + (1-t)p_1) - p_2\| \\
&\leq v_{n,m} + \prod_{j=n}^{n+m-1} k_j \|tx_n + (1-t)p_1 - p_2\| \\
&= v_{n,m} + \prod_{j=n}^{n+m-1} k_j u_n(t),
\end{aligned}$$

it follows that

$$\limsup_{n\to\infty} u_n(t) \leq \liminf_{n\to\infty} u_n(t).$$

Hence $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$. □

3 Common Fixed Point Approximations

3.1 Weak Convergence Results

Here we will approximate common fixed points of the mappings S and T through the weak convergence of the sequence $\{x_n\}$ defined in (1.2). Our first result in this direction uses the Opial's condition and the second one the Kadec Klee property.

Theorem 3.1. *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, S, T and $\{x_n\}$ be as taken in Lemma 2.2. If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $x^* \in F$. Then as proved in Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 2.2, $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $I - S$ is demiclosed with respect to zero by Lemma 1.3, therefore we obtain $Sz_1 = z_1$. Similarly, $Tz_1 = z_1$. Again in the same way, we can prove that $z_2 \in F$. Next, we prove the uniqueness. For this suppose that $z_1 \neq z_2$, then by the Opial's condition

$$
\begin{aligned}
\lim_{n\to\infty}\|x_n - z_1\| = \lim_{n_i\to\infty}\|x_{n_i} - z_1\| &< \lim_{n_i\to\infty}\|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty}\|x_n - z_2\| = \lim_{n_j\to\infty}\|x_{n_j} - z_2\| \\
&< \lim_{n_j\to\infty}\|x_{n_j} - z_1\| = \lim_{n\to\infty}\|x_n - z_1\|.
\end{aligned}
$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in F. □

Theorem 3.2. *Let E be a uniformly convex Banach space such that its dual E^* satisfies the Kadec Klee property . Let C, S, T and $\{x_n\}$ be as taken in Lemma 2.2. If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. By the boundedness of $\{x_n\}$ and reflexivity of E, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some p in C. By Lemma 2.2, we have $\lim_{i\to\infty}\|x_{n_i} - Sx_{n_i}\| = 0 = \lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\|$. This gives $p \in F$. To prove that $\{x_n\}$ converges weakly to p, suppose that $\{x_{n_k}\}$ is another subsequence of $\{x_n\}$ that converges weakly to some q in C. Then by Lemmas 2.2 and 1.3, $p, q \in W \cap F$ where $W = \omega_w(\{x_n\})$. Since $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$ by Lemma 2.3, therefore $p = q$ by Lemma 1.5. Consequently, $\{x_n\}$ converges weakly to $p \in F$ and this completes the proof. □

By putting $T = I$, the identity mapping, in the above two theorems, we have the following corollaries.

Corollary 3.3. *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, S be as taken in Lemma 2.1 and $\{x_n\}$ be defined as*

$$
\begin{cases}
x_1 = x \in C, \\
x_{n+1} = P\left(a_n S(PS)^{n-1}x_n + (1-a_n)x_n\right), \quad n \in \mathbb{N}.
\end{cases}
\tag{3.1}
$$

If $F(S) \neq \emptyset$, then $\{x_n\}$ converges weakly to a fixed point of S.

Corollary 3.4. *Let E be a uniformly convex Banach space such that its dual E^* satisfies the Kadec Klee property. Let C, S be as taken in Lemma 2.1 and $\{x_n\}$ as in (3.1). If $F(S) \neq \emptyset$, then $\{x_n\}$ converges weakly to a fixed point of S.*

3.2 Strong Convergence Results

Recall that for a nonempty set F, $d(x, F) = \inf\{\|x - x^*\| : x^* \in F\}$. We first prove a strong convergence thoerem in general real Banach spaces as follows.

Theorem 3.5. *Let E be a real Banach space and C, $\{x_n\}$, S, T be as taken in Lemma 2.1. If $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of S and T if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. As proved in Lemma 2.1, we have

$$\|x_{n+1} - p\| \leq k_n \|x_n - p\|$$

for all $p \in F$. This gives

$$d(x_{n+1}, F) \leq k_n d(x_n, F)$$

so that $\lim_{n\to\infty} d(x_n, F)$ exists. But by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} d(x_n, F) = 0$, therefore there exists a constant n_0 such that for all $n \geq n_0$, we have

$$d(x_n, F) < \frac{\epsilon}{4}.$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\epsilon}{4}$. There must exist $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{2}.$$

Now for $m, n \geq n_0$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\
&\leq 2\|x_{n_0} - p^*\| \\
&< 2\left(\frac{\epsilon}{2}\right) = \epsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a Banach space E, therefore it must converge in C. Let $\lim_{n\to\infty} x_n = q$. Now $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Hence $q \in F$. □

On lines similar to Fukhar-ud-din and Khan [4], we say that the two mappings $S, T : C \to E$ where C a subset of E, are said to satisfy Condition $(\tilde{A})$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that either $\|x - Tx\| \geq f(d(x, F))$ or $\|x - Sx\| \geq f(d(x, F))$ for all $x \in C$.

Our next theorem is an application of the above theorem and makes use of the Condition $(\tilde{A})$.

Theorem 3.6. *Let E be a uniformly convex Banach space and C, $\{x_n\}$ be as taken in Lemma 2.2. Let $S, T : C \to E$ be two nonself asymptotically nonexpansive mappings satisfying condition $(\tilde{A})$. If $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$. Let it be c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. Now $\|x_{n+1} - x^*\| \leq k_n \|x_n - x^*\|$ gives that $d(x_{n+1}, F) \leq k_n d(x_n, F)$ and so $\lim_{n\to\infty} d(x_n, F)$ exists by Lemma 1.1. By using condition $(\tilde{A})$, either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

In both the cases,

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Now applying above theorem, we get the result. □

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