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EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF POSITIVE SOLUTIONS FOR SOME NONLINEAR PARABOLIC SYSTEMS IN THE HALF-SPACE

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Abstract

We are concerned with the nonlinear parabolic system

$$\Delta u - au - \frac{\partial u}{\partial t} = \lambda p(x,t) f(v),$$

$$\Delta v - bv - \frac{\partial v}{\partial t} = \mu q(x,t) g(u),$$

in $\mathbb{R}^n_+ \times (0,\infty)$, subject to some Dirichlet boundary conditions, where the potentials p, q, a and b are allowed to satisfy some hypotheses related to the parabolic Kato class $P^\infty(\mathbb{R}^n_+)$, the functions f and g are nonnegative nondecreasing and continuous. More precisely, we shall prove the existence of positive continuous solutions with precise global behaviour. We will use some potential theory arguments.

1 Introduction

In this work, we deal with the existence of positive continuous solutions (in the sense of distributions) and their asymptotic behaviour for the following

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parabolic system

$$(\mathbf{P})\begin{cases}\Delta u-a(x,t)u-\frac{\partial u}{\partial t}=\lambda p(x,t)f\left(v\right), \text{ in } \mathbb{R}_{+}^{n}\times(0,\infty)\,,\\ \Delta v-b(x,t)v-\frac{\partial v}{\partial t}=\mu q(x,t)g\left(u\right), \text{ in } \mathbb{R}_{+}^{n}\times(0,\infty)\,,\\ u(x,0)=\varphi_{1}\left(x\right), \text{ in } \mathbb{R}_{+}^{n},\\ v(x,0)=\varphi_{2}\left(x\right), \text{ in } \mathbb{R}_{+}^{n},\\ u(x,t)=0;\ v(x,t)=0, \text{ on } \partial\mathbb{R}_{+}^{n}\times(0,\infty)\,,\end{cases}$$

where $\mathbb{R}_{+}^{n}=\left\{(x_{1},x_{2},...,x_{n})\in\mathbb{R}^{n},x_{n}>0\right\},\lambda,\mu\geq 0$, the initial conditions $\varphi_{1},\varphi_{2}:\mathbb{R}_{+}^{n}\rightarrow[0,\infty)$ are continuous.
As a motivation to our study, we give a short historic account. Both the parabolic problem

$$\begin{cases}\Delta u+V(x,t)f(u)-\frac{\partial u}{\partial t}=0 \text{ in } D\times(0,\infty)\,,\\ u(x,0)=u_{0}(x) \text{ in } \partial D,\end{cases}\tag{1.1}$$

and its elliptic counterpart

$$\Delta u+V(x)f(u)=0 \quad \text{in } \ D$$

have been widely studied.
In the case of the whole space $D=\mathbb{R}^{n}(n\geq 3)$, Zhang [14] established an essentially optimal condition on the potential $V=V(x,t)$ so that the problem (1.1) has global positive continuous solutions for $f\left(u\right)=u^{p}$ $(p>1)$. Indeed, the author gave a general integrability condition, which controls both the global growth and local singularity of V. More precisely, he introduced the parabolic Kato class $P^{\infty}\left(\mathbb{R}^{n}\right)$ for such potentials (see [14, 15]).
Inspired by the works of Zhang [14] and Zhang and Zhao [15], Mâatoug and Riahi [9] introduced for the case of the half space a parabolic Kato class $P^{\infty}\left(\mathbb{R}_{+}^{n}\right)$ and gave an existence result of the problem (1.1), where $f\left(u\right)=u^{p}$ $(p\geq 1)$ with bounded smooth initial condition u_{0}.
In [6], Mâagli et al treated the following problem

$$\begin{cases}\Delta u-u\varphi(.,u)-\dfrac{\partial u}{\partial t}=0 \text{ in } \mathbb{R}_{+}^{n}\times(0,\infty)\,,\\ u=0 \text{ on } \partial\mathbb{R}_{+}^{n}\times(0,\infty)\\ u(x,0)=u_{0}(x),x\in\mathbb{R}_{+}^{n},\end{cases}$$

where u_{0} was allowed to be not bounded. Then using arguments based on potential theory tools, they proved under some assumptions the existence of a positive continuous solution u in $\mathbb{R}_{+}^{n}\times(0,\infty)$ satisfying for each $t>0$ and $x\in\mathbb{R}_{+}^{n}$

$$cP_{t}u_{0}(x)\leq u(x,t)\leq P_{t}u_{0}(x),$$

where $c\in(0,1)$ and $P_{t}u_{0}$ is defined below by (1.3).

Similar results are given for various domains D, namely for $D = \mathbb{R}^n$ and for the case of unbounded domain of $\mathbb{R}^n$ with compact boundary, we refer the readers to [7, 8, 10, 14, 15] and references therein.
In this work, we are inspired by the elliptic counterpart of $(\mathbf{P})$ which was studied in [13] for $D = \mathbb{R}^n_+$ and in [5] for unbounded domain D of $\mathbb{R}^n (n \geq 3)$ with compact boundary. More precisely, Zeddini [13] considered the following system

$$(\mathbf{Q}) \begin{cases} \Delta u = \lambda p(x) g\left(v\right), \text{ in } \mathbb{R}^n_+, \\ \Delta v = \mu q(x) f\left(u\right), \text{ in } \mathbb{R}^n_+, \\ u_{/\partial \mathbb{R}^n_+} = a\varphi, \quad \lim\limits_{x_n \to +\infty} \dfrac{u(x)}{x_n} = \alpha \\ v_{/\partial \mathbb{R}^n_+} = b\psi, \quad \lim\limits_{x_n \to +\infty} \dfrac{v(x)}{x_n} = \beta, \end{cases}$$

where $\lambda, \mu \geq 0$, the functions $f, g : (0, \infty) \to [0, \infty)$ are continuous nondecreasing, the functions p, q are measurable and nonnegative belonging to the elliptic Kato class $K^\infty(\mathbb{R}^n_+)$ introduced and studied in [2] and [3]. We remark that the parabolic Kato class $P^\infty\left(\mathbb{R}^n_+\right)$ is a generalization of the elliptic one $K^\infty(\mathbb{R}^n_+)$. For a given $\lambda_0, \mu_0 > 0$, the author proved the following result:

Theorem 1. *For each $\lambda \in [0, \lambda_0)$ and each $\mu \in [0, \mu_0)$, the problem $(\mathbf{Q})$ has a positive continuous solution (u, v) satisfying*

$$\begin{cases} \left(1 - \frac{\lambda}{\lambda_0}\right)\left(\alpha x_n + aH\varphi\left(x\right)\right) \leq u\left(x\right) \leq \alpha x_n + aH\varphi\left(x\right) \\ \left(1 - \frac{\mu}{\mu_0}\right)\left(\beta x_n + bH\psi\left(x\right)\right) \leq v\left(x\right) \leq \beta x_n + bH\psi\left(x\right), \end{cases}$$

where $H\psi$ denotes the unique bounded harmonic function in $\mathbb{R}^n_+$ with boundary value the nonnegative bounded continuous function ψ.

We would like to mention that the difference between the counterpart of the problem $(\mathbf{P})$ and the problem $(\mathbf{Q})$ is essentially in the presence of the linear terms associated to the potentials a and b.
Hereinafter, the point $x \in \mathbb{R}^n_+$ is denoted by (x', x_n) with $x' \in \mathbb{R}^{n-1}$, $x_n > 0$. Note that $x \longrightarrow \partial\mathbb{R}^n_+$ means that $x = (x', x_n)$ tends to a point $(\xi, 0)$ of $\partial\mathbb{R}^n_+$. As done for the elliptic systems that is many results are claimed for elliptic systems by using the tools and techniques of the elliptic scalar equation (See [5, 13]). We will here treat the parabolic system $(\mathbf{P})$ by adopting similar techniques as in [6] based on potential theory arguments. So, let us recall briefly some notions related to the potential theory and we refer the reader to [1, 4, 11] for more details. We denote by $\Gamma\left(x, t, y, s\right)$ the heat kernel in $\mathbb{R}^n_+ \times (0, \infty)$ with Dirichlet boundary condition $u = 0$ on $\partial\mathbb{R}^n_+ \times (0, \infty)$ given

by

$$\Gamma\left(x,t,y,s\right)=\left(4\pi\right)^{-\frac{n}{2}}\left(1-\exp\left(-\frac{x_{n}y_{n}}{(t-s)}\right)\right)G_{\frac{1}{4}}\left(x,t,y,s\right),$$

where

$$G_{c}\left(x,t,y,s\right):=\frac{1}{\left(t-s\right)^{\frac{n}{2}}}\exp\left(-c\frac{\left|x-y\right|^{2}}{t-s}\right),\tag{1.2}$$

for $t>s$, $x,y\in\mathbb{R}_{+}^{n}$ and for each $c>0$.
For each nonnegative measurable function f on $\mathbb{R}_{+}^{n}$, we put

$$P_{t}f(x):=Pf(x,t)=\int_{\mathbb{R}_{+}^{n}}\Gamma\left(x,t,y,0\right)f(y)dy,\ t>0,\ x\in\mathbb{R}_{+}^{n}.\tag{1.3}$$

The family of kernels $\left(P_{t}\right)_{t>0}$ is a semigroup, that is $P_{t+s}=P_{t}P_{s}$ for $s,t>0$. We mention that for each nonnegative function f on $\mathbb{R}_{+}^{n}$, the map $(x,t)\longrightarrow P_{t}f(x)$ is lower semicontinuous on $\mathbb{R}_{+}^{n}\times(0,\infty)$ and it is continuous if f is further bounded. Moreover, let w be a nonnegative superharmonic function on $\mathbb{R}_{+}^{n}$, then for every $t>0, P_{t}w\leq w$ and consequently the mapping $t\longrightarrow P_{t}w$ is nonincreasing.
Now, let $(X_{t},\,t>0)$ be the Brownian motion in $\mathbb{R}_{+}^{n}$ and P^{x} be the probability measure on the Brownian continuous paths starting at x. For a nonnegative Borel measurable function q in $\mathbb{R}_{+}^{n}\times(0,\infty)$, we denote by V_{q} the kernel defined by

$$V_{q}f(x,t)=\int_{0}^{t}E^{x}\left(\exp\left(-\int_{0}^{s}q(X_{r},t-r)dr\right)f(X_{s},t-s)\right)ds,\tag{1.4}$$

where E^{x} is the expectation on P^{x} and f is a nonnegative measurable function on $\mathbb{R}_{+}^{n}\times(0,\infty)$. In particular, for $q=0$, $V_{0}=V$ is given by

$$Vf(x,t):=\int_{0}^{t}\int_{\mathbb{R}_{+}^{n}}\Gamma\left(x,t,y,s\right)f\left(y,s\right)dyds=\int_{0}^{t}P_{t-s}f\left(.,s\right)ds,$$

Note that $V=-\left(\Delta-\partial_{t}\right)^{-1}$.
Using Markov property, we have for each nonnegative Borel measurable function q such that $Vq<\infty$, the following resolvent equation

$$V=V_{q}+V_{q}(qV)=V_{q}+V(qV_{q}).\tag{1.5}$$

So for each measurable function u in $\mathbb{R}_{+}^{n}\times(0,\infty)$ such that V(q$|u|$)$<\infty$, we have

$$(I-V_{q}(q.))(I+V(q.))u=(I+V(q.))(I-V_{q}(q.))u=u.\tag{1.6}$$

Next, let us intoduce a function class of nonnegative superharmonic functions w in $\mathbb{R}_{+}^{n}$ which satisfy condition $(\mathbf{H}_{0})$.

Definition 1. *A nonnegative superharmonic function w satisfies condition* $(\mathbf{H}_0)$ *if ω is locally bounded in $\mathbb{R}^n_+$ such that the map $(x,t) \longrightarrow P_t\omega(x)$ is continuous in $\mathbb{R}^n_+ \times (0,\infty)$ and $\lim_{x\to\partial\mathbb{R}^n_+} P_t\omega(x) = 0$, for every $t > 0$.*

To clarify condition $(\mathbf{H}_0)$, we give some examples of functions satisfying $(\mathbf{H}_0)$ and for further examples see $[6, Sect.6]$.

Example 1. *Let w be a nonnegative bounded superharmonic function in $\mathbb{R}^n_+$, then w satisfies* $(\mathbf{H}_0)$.

Example 2. *The harmonic function defined on $\mathbb{R}^n_+$ by $\omega(x) := x_n^{\beta}, \beta \in (0,1]$ satisfies $(\mathbf{H}_0)$. In fact, a simple calculs yields $\Delta\omega(x) = \beta(\beta-1)x_n^{\beta-2}$ and then the function ω is superharmonic. Moreover using Tonelli Theorem and the semigroup's property we obtain*

$$\omega(x) - P_t\omega(x) = \beta(1-\beta)\int_0^t P_s\omega^{1-\frac{2}{\beta}}(x)\,ds.$$

Hence $P\omega \leq \omega$ and so $\lim_{x\to\partial\mathbb{R}^n_+} P_t\omega(x) = 0$. Furtheremore, the function $(x,t) \to \omega(x) - P_t\omega(x)$ is upper semicontinuous, which ensures the continuity of the function $(x,t) \to P_t\omega(x)$.

From now on, we fix a nonnegative superharmonic function ω satisfying condition $(\mathbf{H}_0)$, we suppose that $a, b \in P^{\infty}\left(\mathbb{R}^n_+\right)$ and we adopt the following hypotheses:

$(\mathbf{H}_1)$ The functions $f, g : (0,\infty) \longrightarrow [0,\infty)$ are nondecreasing and continuous.

$(\mathbf{H}_2)$ For $i = 1, 2$, there exists a constant $c_i > 1$ such that the function φ_i satisfies

$$\frac{1}{c_i}\omega(x) \leq \varphi_i(x) \leq c_i\omega(x) \tag{1.7}$$

and

$$\lim_{t\to 0} P_t\varphi_i(x) = \varphi_i(x) \tag{1.8}$$

for each $x \in \mathbb{R}^n_+$.

$(\mathbf{H}_3)$ The functions p and q are measurable nonnegative on $\mathbb{R}^n_+ \times (0,\infty)$ such that for each $c > 0$

$$p_c := \frac{pf(cP\omega)}{P\omega} \quad \text{and} \quad q_c := \frac{qg(cP\omega)}{P\omega}$$

belong to the parabolic Kato class $P^{\infty}\left(\mathbb{R}^n_+\right)$.

Before stating our main result let us give an example where the hypothesis $(\mathbf{H}_3)$ is satisfied.

Example 3. *Let p and q be nonnegative nontrivial functions in $P^{\infty}\left(\mathbb{R}_{+}^{n}\right)$. Moreover suppose that f and g are continuous functions such that there exists a constant $\delta > 0$ satisfying for each $t \in (0, \infty)$*

$$0 \leq f(t) \leq \delta t \text{ and } 0 \leq g(t) \leq \delta t \ .$$

Then for each $c > 0$,

$$0 \leq p_c := \frac{pf\left(cP\omega\right)}{P\omega} \leq c\delta p \in P^{\infty}\left(\mathbb{R}_{+}^{n}\right).$$

Similarly we obtain $q_c \in P^{\infty}\left(\mathbb{R}_{+}^{n}\right)$. Thus the hypothesis $(\mathbf{H}_3)$ is satisfied. The main result of this work is the following

Theorem 2. *Assume $(\mathbf{H}_1) - (\mathbf{H}_3)$. Then there exist two constants λ_0 and μ_0 such that for each $\lambda \in [0, \lambda_0)$ and each $\mu \in [0, \mu_0)$ the problem $(\mathbf{P})$ admits a positive continuous solution (u, v) on $\mathbb{R}_{+}^{n} \times (0, \infty)$ satisfying*

$$\begin{cases} 0 < (1 - \frac{\lambda}{\lambda_0})a_1 P\varphi_1 \leq u \leq P\varphi_1, \\ 0 < (1 - \frac{\mu}{\mu_0})a_2 P\varphi_2 \leq v \leq P\varphi_2, \end{cases}$$

where $a_1, a_2 \in (0, 1]$.

As consequence of the main Theorem we have the following

Corollary 1. *Assume $(\mathbf{H}_1) - (\mathbf{H}_3)$, then there exist two constants λ_0 and μ_0 such that for each $\lambda \in [0, \lambda_0)$ and each $\mu \in [0, \mu_0)$ the problem*

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = \lambda p(x,t) f\left(v\right), \ \ in \ \mathbb{R}_{+}^{n} \times (0, \infty), \\ \Delta v - \frac{\partial v}{\partial t} = \mu q(x,t) g\left(u\right), \ \ in \ \mathbb{R}_{+}^{n} \times (0, \infty), \\ u(x, 0) = \varphi_1\left(x\right) \ \ in \ \mathbb{R}_{+}^{n}, \\ v(x, 0) = \varphi_2\left(x\right) \ \ in \ \mathbb{R}_{+}^{n}, \\ u(x,t) = 0; \ v(x,t) = 0, \ \ on \ \partial\mathbb{R}_{+}^{n} \times (0, \infty), \end{cases}$$

admits a positive continuous solution (u, v) on $\mathbb{R}_{+}^{n} \times (0, \infty)$ satisfying

$$\begin{cases} \left(1 - \frac{\lambda}{\lambda_0}\right) P\varphi_1 \leq u \leq P\varphi_1, \\ \left(1 - \frac{\mu}{\mu_0}\right) P\varphi_2 \leq v \leq P\varphi_2. \end{cases}$$

The organization of this paper is as follows. In the next section we recall and we prove a number of basic results about the class $P^{\infty}\left(\mathbb{R}_{+}^{n}\right)$ and some continuity results. In section 3, we prove the existence result of the problem $(\mathbf{P})$. The last section of this work, is dedicated to some examples.

2 Preliminary results

In this section, we briefly describe some notations and results and we refer the readers to [6] for more details.
Given $c, h > 0$ and $q = q(x,t)$ a measurable function in $\mathbb{R}^n_+ \times (0,\infty)$, we put

$$N_{c,h}(q) :=$$
$$\sup_{(x,t)\in\mathbb{R}^n_+\times\mathbb{R}} \int_{t-h}^{t+h} \int_{B\left(x,\sqrt{h}\right)\cap\mathbb{R}^n_+} \min\left(1, \tfrac{y_n^2}{|t-s|}\right) G_c(x,|t-s|,y,0)\,|q(y,s)|\,dyds$$

and

$$N_{c,\infty}(q) := \lim_{h\to+\infty} N_{c,h}(q) =$$
$$\sup_{(x,t)\in\mathbb{R}^n_+\times\mathbb{R}} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n_+} \min\left(1, \tfrac{y_n^2}{|t-s|}\right) G_c(x,|t-s|,y,0)\,|q(y,s)|\,dyds,$$

where G_c is the function given by (1.2).
Next, we recall the definition of the functional class $P^\infty\left(\mathbb{R}^n_+\right)$.

Definition 2 (See [6])**.** *A Borel measurable function q in $\mathbb{R}^n_+ \times \mathbb{R}$ belongs to the parabolic Kato class $P^\infty\left(\mathbb{R}^n_+\right)$ if*

$$\lim_{h\longrightarrow 0} N_{c,h}(q) = 0$$

and

$$N_{c,\infty}(q) < +\infty,$$

for all $c > 0$ and $h > 0$.

Example 4. *As an example of functions belonging to $P^\infty\left(\mathbb{R}^n_+\right)$, the time independent Kato class $K^\infty\left(\mathbb{R}^n_+\right)$ used in the study of elliptic equations (See* [2, 3]*).*

Other examples of functions in $P^\infty\left(\mathbb{R}^n_+\right)$ are given by the following

Proposition 1 (See [6])**.** *The following assertions hold*
(i) $L^\infty\left(\mathbb{R}^n_+\right) \otimes L^1(\mathbb{R}) \subset P^\infty\left(\mathbb{R}^n_+\right)$.

(ii) $K^\infty\left(\mathbb{R}^n_+\right) \otimes L^\infty(\mathbb{R}) \subset P^\infty\left(\mathbb{R}^n_+\right)$.
(iii) For $1 < p < +\infty$ and $q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $s > \frac{np}{2}$ and $\delta < \frac{2}{p} - \frac{n}{s} < \nu$, we have

$$\frac{L^s\left(\mathbb{R}^n_+\right)}{\theta(.)^\delta\,(1+|.|)^{\nu-\delta}} \otimes L^q(\mathbb{R}) \subset P^\infty\left(\mathbb{R}^n_+\right),$$

where $\theta(x) = x_n$, $x \in \mathbb{R}^n_+$.

Proposition 2 (See [6])**.** *Let $q \in P^\infty\left(\mathbb{R}^n_+\right)$, then the function $(y,s) \to y_n^2 q(y,s)$ is in $L^1_{loc}\left(\overline{\mathbb{R}^n_+} \times \mathbb{R}\right)$. In particular, we have*

$$P^\infty\left(\mathbb{R}^n_+\right) \subset L^1_{loc}\left(\mathbb{R}^n_+ \times \mathbb{R}\right).$$

Proposition 3 (See [6])**.** *For each nonnegative function q in $P^\infty\left(\mathbb{R}^n_+\right)$, there exists a constant $\alpha_q > 0$ such that for each nonnegative superharmonic function ω in $\mathbb{R}^n_+$ we have*
$V(qP\omega)(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x,t,y,s)\, q(y,s)\, P_t\omega(y)\, dyds \leq \alpha_q P_t\omega(x)$, *for* $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$.

The following result will be useful to proving global existence and continuity of solutions.

Proposition 4 (See [6])**.** *Let w be a nonnegative superharmonic function in $\mathbb{R}^n_+$ satisfying $(\mathbf{H}_0)$ and q be a nonnegative function in $P^\infty\left(\mathbb{R}^n_+\right)$ then the family of functions*

$$\left\{(x,t) \longrightarrow \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x,t,y,s)\, f(y,s)\, dyds, |f| \leq qP\omega\right\}$$

is equicontinuous in $\mathbb{R}^n_+ \times (0,\infty)$. Moreover, for each $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$, we have $\lim\limits_{s \longrightarrow 0} Vf(x,s) = \lim\limits_{y \to \partial\mathbb{R}^n_+} Vf(y,t) = 0$, uniformly on f.

Now we claim the following result about continuity needed to achieve the proof of the main Theorem.

Proposition 5. *Let ω be a nonnegative superharmonic function satisfying the condition $(\mathbf{H}_0)$ and φ be a measurable function such that $0 \leq \varphi \leq \omega$ on $\mathbb{R}^n_+$, then the function $(x,t) \longrightarrow P_t\varphi(x)$ is continuous on $\mathbb{R}^n_+ \times (0,\infty)$.*

Proof. Let θ be a nonnegative Borel measurable function in $\mathbb{R}^n_+$ such that $\omega = \theta + \varphi$. Then the function $(x,t) \longrightarrow P_t\theta(x)$ is lower semi-continuous on $\mathbb{R}^n_+ \times (0,\infty)$. On the other hand, from $(\mathbf{H}_0)$, the function $(x,t) \longrightarrow P_t\omega(x)$ is continuous on $\mathbb{R}^n_+ \times (0,\infty)$. Therefore the function $(x,t) \longrightarrow P_t\varphi(x)$ is upper semi-continuous on $\mathbb{R}^n_+ \times (0,\infty)$. Using the fact that $(x,t) \longrightarrow P_t\varphi(x)$ is lower semi-continuous on $\mathbb{R}^n_+ \times (0,\infty)$, we deduce that $(x,t) \longrightarrow P_t\theta(x)$ is continuous on $\mathbb{R}^n_+ \times (0,\infty)$. $\square$

Proposition 6. *Let ω be a nonnegative superharmonic function satisfying condition* $(\mathbf{H}_0)$ *and let φ be a measurable function such that there exists a constant $c>0$ satisfying on $\mathbb{R}^n_+$*

$$\frac{1}{c}\omega \leq \varphi \leq c\omega. \tag{2.1}$$

Then for each nonnegative function $q \in P^{\infty}\left(\mathbb{R}^n_+\right)$, there exists a constant $\alpha_q > 0$ such that we have on $\mathbb{R}^n_+ \times (0,\infty)$

$$\exp(-c^2\alpha_q)P\varphi \leq P\varphi - V_q\left(qP\varphi\right) \leq P\varphi. \tag{2.2}$$

Proof. It is obviously seen that $P\varphi - V_q\left(qP\varphi\right) \leq P\varphi$. Now, we define the sequence $\left(f_k\right)_{k\in\mathbb{N}^*}$ on $\mathbb{R}^n_+ \times (0,\infty)$ by $f_k\left(x,t\right) = k\exp\left(-kt\right)P\varphi\left(x,t\right)$. Then by (1.5) we remark that for each $k \in \mathbb{N}^*$

$$V_q\left(qVf_k\right) \leq Vf_k. \tag{2.3}$$

Moreover, a simple calculus yields

$$Vf_k\left(x,t\right) = (1-\exp(-kt))P\varphi\left(x,t\right), k \in \mathbb{N}^*.$$

Consequently, we have

$$\sup_{k\in\mathbb{N}^*} Vf_k\left(x,t\right) = P\varphi\left(x,t\right). \tag{2.4}$$

Next, for each $k \in \mathbb{N}^*$, we consider the function

$$\gamma_k\left(\lambda\right) := Vf_k - \lambda V_{\lambda q}\left(qVf_k\right), \lambda \geq 0.$$

Then from (1.5) we obtain

$$\gamma_k\left(\lambda\right) = \left(V - V_{\lambda q}\left(\lambda qV\right)\right)f_k = V_{\lambda q}f_k.$$

Thus by (2.3) and (1.4), we deduce that γ_k is completely monotone on $[0,+\infty)$ to $(0,\infty)$. Therefore by [12, Theorem 12a], there exists a nonnegative measure μ on $[0,+\infty)$ such that

$$\gamma_k\left(\lambda\right) = \int_0^{\infty} \exp(-\lambda x)d\mu\left(x\right).$$

So using this fact and the Hölder inequality, we deduce that $Log\left(\gamma_k\right)$ is a convex function. Then we have

$$\gamma_k\left(0\right) \leq \gamma_k\left(1\right)\exp(-\frac{\gamma_k'\left(0\right)}{\gamma_k\left(0\right)}),$$

that is

$$Vf_k(x,t) \leq (Vf_k - V_q(qVf_k))(x,t)\exp(\frac{V(qVf_k)(x,t)}{Vf_k(x,t)}).$$

By letting k to infinity and using (2.4) we obtain on $\mathbb{R}^n_+ \times (0,\infty)$

$$P\varphi \leq (P\varphi - V_q(qP\varphi))\exp(\frac{V(qP\varphi)}{P\varphi}).$$

From Proposition 3 and (2.1) we deduce that

$$\exp(-c^2\alpha_q)P\varphi \leq (P\varphi - V_q(qP\varphi)).$$

□

3 Proof of the main result

Recall that for $i = 1, 2$, the function φ_i satisfies the hypothesis $(\mathbf{H}_2)$ and put $\theta_i := P_t\varphi_i$.

Proof of Theorem 2. Recall that $a, b \in P^\infty(\mathbb{R}^n_+)$. Put $a_1 = \exp(-c_1^2\alpha_a)$ and $a_2 = \exp(-c_2^2\alpha_b)$ where α_a and α_b are the constants given by Proposition 3 associated respectively to the functions a and b. Let $p_1 := p_{c_2}$ and $q_1 := q_{c_1}$ be the functions defined in the hypothesis $(\mathbf{H}_3)$ associated respectively to the constants c_2 and c_1 given in $(\mathbf{H}_2)$.
Put

$$\lambda_0 := \inf_{(x,t)\in\mathbb{R}^n_+\times(0,\infty)} \frac{(\theta_1 - V_a(a\theta_1))(x,t)}{V(pf(\theta_2))(x,t)} \tag{3.1}$$

and

$$\mu_0 := \inf_{(x,t)\in\mathbb{R}^n_+\times(0,\infty)} \frac{(\theta_2 - V_b(b\theta_2))(x,t)}{V(qg(\theta_1))(x,t)}. \tag{3.2}$$

Let us prove that λ_0 and μ_0 are tow positive constants.
By hypothesis $(\mathbf{H}_2)$ we have

$$\varphi_2 \leq c_2\omega.$$

So, the monotonicity of the function f yields

$$pf(\theta_2) \leq pf(c_2P\omega).$$

Therefore, by Proposition 3, there exists a positive constant $\alpha_{p_1} > 0$ such that, for each $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$, we have

$$V(pf(\theta_2))(x,t) \leq V(p_1P\omega)(x,t) \leq \alpha_{p_1}P\omega(x,t).$$

On the other hand, by using the hypothesis $(\mathbf{H}_1)$, it follows that

$$\frac{\theta_1(x,t)}{V(pf(\theta_2))(x,t)} \geq \frac{P\omega(x,t)}{c_1\alpha_{p_1}P\omega(x,t)} \geq \frac{1}{c_1\alpha_{p_1}},$$

which implies (by (2.2) in Proposition 6)

$$\frac{(\theta_1 - V_a(a\theta_1))(x,t)}{V(pf(\theta_2))(x,t)} \geq \frac{\theta_1(x,t)\,a_1}{V(pf(\theta_2))(x,t)} \geq \frac{a_1}{c_1\alpha_{p_1}} > 0.$$

Thus $\lambda_0 > 0$. Similarly we prove that $\mu_0 > 0$.
Now, let $\lambda \in [0,\lambda_0)$ and $\mu \in [0,\mu_0)$. We shall prove the existence of positive continuous solution of the problem $(\mathbf{P})$. To this aim we define the following sequences $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ as follows

$$\begin{cases} v_0 = \theta_2 - V_b(b\theta_2) \\ u_k = \theta_1 - V_a(a\theta_1 + \lambda pf(v_k)) \\ v_{k+1} = \theta_2 - V_b(b\theta_2 + \mu qg(u_k)). \end{cases}$$

We intend to prove by induction that for each $k \in \mathbb{N}$

$$\begin{cases} 0 < (1 - \frac{\lambda}{\lambda_0})a_1\theta_1 \leq u_k \leq u_{k+1} \leq \theta_1, \\ 0 < (1 - \frac{\mu}{\mu_0})a_2\theta_2 \leq v_{k+1} \leq v_k \leq \theta_2. \end{cases}$$

First, using (3.1), we have on $\mathbb{R}^n_+ \times (0,\infty)$

$$\lambda_0 V(pf(\theta_2)) \leq \theta_1 - V_a(a\theta_1). \tag{3.3}$$

Then, by the monotonicity of the function f and using the fact that $V_a \leq V$ and (3.3) we obtain

$$\begin{aligned} \theta_1 &\geq u_0 = \theta_1 - V_a(a\theta_1) - \lambda V_a(pf(\theta_2 - V_b(b\theta_2))) \\ &\geq \theta_1 - V_a(a\theta_1) - \lambda V(pf(\theta_2)). \end{aligned}$$

Thus, from Proposition 6, we obtain

$$\begin{aligned} \theta_1 &\geq \left(1 - \frac{\lambda}{\lambda_0}\right)(\theta_1 - V_a(a\theta_1)) \\ &\geq a_1\left(1 - \frac{\lambda}{\lambda_0}\right)\theta_1 > 0. \end{aligned}$$

Hence

$$v_1 - v_0 = -\mu V(qg(u_0)) \leq 0.$$

So the monotonicity of the function f yields

$$u_1 - u_0 = \lambda V_a \left(p\left[f\left(v_0\right) - f\left(v_1\right)\right]\right) \geq 0.$$

On the other hand, from (3.2), we have

$$\mu_0 V\left(qg\left(\theta_1\right)\right) \leq \theta_2 - V_b\left(b\theta_2\right). \tag{3.4}$$

So, since g is a nondecreasing function and using (3.4), it follows that

$$\begin{aligned} v_1 &\geq \theta_2 - V_b(b\theta_2) - \mu V_b\left(qg\left(\theta_1\right)\right) \\ &\geq \theta_2 - V_b(b\theta_2) - \mu V\left(qg\left(\theta_1\right)\right) \\ &\geq \left(1 - \frac{\mu}{\mu_0}\right)\left(\theta_2 - V_b(b\theta_2)\right). \end{aligned}$$

Then, by Proposition 6, it follows that

$$\begin{aligned} v_1 &\geq \left(1 - \frac{\mu}{\mu_0}\right)\left(\theta_2 - V_b(b\theta_2)\right) \\ &\geq a_2\left(1 - \frac{\mu}{\mu_0}\right)\theta_2. \end{aligned}$$

Therefore, we have

$$u_0 \leq u_1 \leq \theta_1$$

and

$$0 < a_2\left(1 - \frac{\mu}{\mu_0}\right)\theta_2 \leq v_1 \leq v_0.$$

Now, suppose that
$u_k \leq u_{k+1} \leq \theta_1$ and $0 < a_2\left(1 - \frac{\mu}{\mu_0}\right)\theta_2 \leq v_{k+1} \leq v_k$.
Then we have

$$v_{k+2} - v_{k+1} = -\mu V\left(q\left[g\left(u_{k+1}\right) - g\left(u_k\right)\right]\right) \leq 0$$

and

$$u_{k+2} - u_{k+1} = \lambda V\left(p\left[f\left(v_{k+1}\right) - f\left(v_{k+2}\right)\right]\right) \geq 0.$$

It is obvious that $u_{k+2} \leq \theta_1$. Now, since $u_{k+1} \leq \theta_1$, it follows from (3.4) and Proposition 6 that

$$\begin{aligned} v_{k+2} &\geq \theta_2 - V_b(b\theta_2) - \mu V\left(qg\left(\theta_1\right)\right) \\ &\geq \left(1 - \frac{\mu}{\mu_0}\right)\left(\theta_2 - V_b(b\theta_2)\right) \\ &\geq a_2\left(1 - \frac{\mu}{\mu_0}\right)\theta_2 > 0. \end{aligned}$$

Hence

$$u_{k+1} \leq u_{k+2} \leq \theta_1 - V_a\left(a\theta_1\right)$$

and

$$0 < \left(1 - \frac{\mu}{\mu_0}\right)\left(\theta_2 - V_b(b\theta_2)\right) \leq v_{k+2} \leq v_{k+1}.$$

Thus, the sequences $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ converge respectively to two functions u and v satisfying

$$\begin{cases} 0 < a_1\left(1 - \frac{\lambda}{\lambda_0}\right)\theta_1 \leq u \leq \theta_1, \\ 0 < a_2\left(1 - \frac{\mu}{\mu_0}\right)\theta_2 \leq v \leq \theta_2. \end{cases} \tag{3.5}$$

Furthermore, for each $k \in \mathbb{N}$, we have $f\left(v_k\right) \leq f\left(\theta_2\right)$ and $g\left(u_k\right) \leq g\left(\theta_1\right)$. Therefore, using hypothesis $(\mathbf{H}_3)$ we obtain for each $k \in \mathbb{N}$, $pf\left(v_k\right) \leq p_1 P\omega$ and $qg\left(u_k\right) \leq q_1 P\omega$.

So, by Proposition 3 and Lebesgue's theorem, we deduce that $V\left(pf\left(v_k\right)\right)$ and $V\left(qg\left(u_k\right)\right)$ converge respectively to $V\left(pf\left(v\right)\right)$ and $V\left(qg\left(u\right)\right)$ as k tends to infinity. Then (u,v) satisfies on $\mathbb{R}^n_+ \times (0,\infty)$

$$u = \theta_1 - V_a\left(a\theta_1 + \lambda pf\left(v\right)\right)$$

and

$$v = \theta_2 - V_b\left(b\theta_2 + \mu qg\left(u\right)\right).$$

or equivalently

$$u = (I - V_a(a.))\theta_1 - \lambda V_a(pf(v)) \tag{3.6}$$

and

$$v = (I - V_b(b.))\theta_2 - \mu V_b(qg(u)). \tag{3.7}$$

So applying the operators $(I + V(a.))$ and $(I + V(b.))$ respectively on both sides of the equations (3.6) and (3.7), we deduce by (1.5) and (1.6) that

$$u = \theta_1 - V(au + \lambda pf(v)) \tag{3.8}$$

and

$$v = \theta_2 - V(bu + \mu qf(u)). \tag{3.9}$$

Moreover, by $(\mathbf{H}_2)$ it follows that

$$au \leq a\theta_1 \leq c_1 aP\omega \tag{3.10}$$

and

$$bv \leq b\theta_2 \leq c_2 bP\omega. \tag{3.11}$$

Then from hypothesis $(\mathbf{H}_0)$ and Proposition 2, we obtain

$$au, bv \in L^1_{loc}\left(\mathbb{R}^n_+ \times (0,\infty)\right).$$

Moreover, we have

$$pf(v) \leq p_1 P\omega \tag{3.12}$$

and

$$qg(u) \leq q_1 P\omega. \tag{3.13}$$

So, from the hypothesis $(\mathbf{H}_0)$ and Proposition 2, we deduce that

$$pf(v), qg(u) \in L^1_{loc}\left(\mathbb{R}^n_+ \times (0,\infty)\right).$$

By $(3.10) - (3.13)$ and Proposition 4, we obtain

$$V(au), V(bv), V(pf(v)), V(qg(u)) \in \mathcal{C}\left(\mathbb{R}^n_+ \times (0,\infty)\right) \subset L^1_{loc}\left(\mathbb{R}^n_+ \times (0,\infty)\right).$$

In addition, using again hypothesis $(\mathbf{H}_2)$ and Proposition 4, we obtain

$$P\varphi_1, P\varphi_2 \in \mathcal{C}\left(\mathbb{R}^n_+ \times (0,\infty)\right).$$

Thus $u, v \in \mathcal{C}\left(\mathbb{R}^n_+ \times (0,\infty)\right)$.
Now applying the heat operator $(\Delta - \partial_t)$ in (3.8) and (3.9), we obtain clearly that (u,v) is a positive continuous solution (in the distributional sense) of

$$\begin{cases} \Delta u - a(x,t)u - \frac{\partial u}{\partial t} = \lambda p(x,t) f(v), & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ \Delta v - b(x,t)v - \frac{\partial v}{\partial t} = \mu q(x,t) g(u), & \text{in } \mathbb{R}^n_+ \times (0,\infty). \end{cases}$$

Next, using Proposition 4 and $(\mathbf{H}_2)$, we obtain

$$\lim_{t \longrightarrow 0} u(x,t) = \lim_{t \to 0} P_t \varphi_1(x) = \varphi_1(x)$$

and

$$\lim_{t \longrightarrow 0} v(x,t) = \lim_{t \to 0} P_t \varphi_2(x) = \varphi_2(x).$$

Finally, from the hypotheses $(\mathbf{H}_0)$ and $(\mathbf{H}_2)$, we conclude that

$$\lim\nolimits_{x \to \xi \in \partial \mathbb{R}^n_+} \theta_1(x,t) = 0$$

and

$$\lim\nolimits_{x \to \xi \in \partial \mathbb{R}^n_+} \theta_2(x,t) = 0.$$

Hence (u,v) is a positive continuous solution in $\mathbb{R}^n_+ \times (0,\infty)$ of the problem $(\mathbf{P})$. This completes the proof. □

4 Examples

In this section, we will give some examples as applications of Theorem 2.

Example 5. *Let σ be a nonnegative measure on $\partial\mathbb{R}^n_+$. It was shown in* [6], *that if there exists $0 < \alpha \leq \frac{n}{2}$ such that*

$$\sup_{x\in\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+}\frac{x_n}{|x-z|^{n-2\alpha}}\sigma(dz) < +\infty,$$

then the harmonic function defined on $\mathbb{R}^n_+$ by

$$K\sigma(x) := \Gamma(\frac{n}{2})\pi^{-\frac{n}{2}}\int_{\partial\mathbb{R}^n_+}\frac{x_n}{|x-z|^{n}}\sigma(dz),$$

satisfies the condition $(\mathbf{H}_0)$.
Moreover, it was proved in [6] *that there exists $c > 0$ such that*

$$P_t(K\sigma)(x) \leq c\frac{x_n}{t^\alpha}\int_{\partial\mathbb{R}^n_+}\frac{1}{|x-z|^{n-2\alpha}}\sigma(dz). \tag{4.1}$$

Now, let a, b be two functions in $P^\infty(\mathbb{R}^n_+)$ and $\omega(x) = K\sigma(x), x \in \mathbb{R}^n_+$. Let $\beta, \gamma \geq 1$ and consider two nonnegative functions h and g such that: $t \to \frac{h(t)}{t^{\alpha(\beta-1)}}, t \to \frac{g(t)}{t^{\alpha(\gamma-1)}} \in L^1(\mathbb{R})$. Suppose in addition that the functions φ_1 and φ_2 satisfy condition $(\mathbf{H}_2)$. Then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$ and each $\mu \in [0, \mu_0)$ the following problem

$$\begin{cases} \Delta u - au - \frac{\partial u}{\partial t} = \lambda h(t)v^\beta, \ \ in\ \mathbb{R}^n_+ \times (0,\infty), \\ \Delta v - bv - \frac{\partial v}{\partial t} = \mu g\left(t\right)u^\gamma, \ \ in\ \mathbb{R}^n_+ \times (0,\infty), \\ u(x,0) = \varphi_1(x);\ v(x,0) = \varphi_2(x),\ \ in\ \mathbb{R}^n_+, \\ u = v = 0,\ \ in\ \partial\mathbb{R}^n_+ \times (0,\infty), \end{cases}$$

admits a positive continuous solution (u, v) on $\mathbb{R}^n_+ \times (0,\infty)$. In fact, using (4.1) *we obtain*

$$\begin{aligned} p_1\left(x,t\right) \ \ : \ \ &= c_2^\beta p\left(x,t\right)\left(P\left(K\sigma\right)\right)^{\beta-1}\left(x\right) \\ &\leq \ c_2^\beta c^{\beta-1}\frac{h(t)}{t^{\alpha(\beta-1)}}\left(\int_{\partial\mathbb{R}^n_+}\frac{x_n}{|x-z|^{n-2\alpha}}\sigma(dz)\right)^{\beta-1}, \end{aligned}$$

for $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$. So, since

$$x \to \int_{\partial\mathbb{R}^n_+}\frac{x_n}{|x-z|^{n-2\alpha}}\sigma(dz) \in L^\infty(\mathbb{R}^n_+),$$

we conclude by Proposition $1\,(i)$ *that* $p_1 \in P^\infty\left(\mathbb{R}^n_+\right)$*, similarly we prove that* $q_1 \in P^\infty\left(\mathbb{R}^n_+\right)$*. So the hypothesis* $(\mathbf{H}_3)$ *is satisfied.*

Example 6. *Assume that the functions* a *and* b *belong to* $P^\infty(\mathbb{R}^n_+)$*. Let* $1 \le p < \infty$ *and* $q \ge 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Let* $r \ge \frac{np}{2}$ *and* $s < \frac{2}{p} - \frac{n}{r} < m$*. For* $\beta > 1, \tau \in (0,1]$*, we define the function* h *on* $\mathbb{R}^n_+ \times (0,\infty)$ *by*

$$h(x,t) := \frac{|f(x)|}{x_n^{s+(\beta-1)\tau}(1+|x|)^{m-s}}\,|g(t)|\,,$$

where $f \in L^r(\mathbb{R}^n_+)$, $g \in L^q(\mathbb{R})$.
Let k be a function on $\mathbb{R}^n_+ \times (0,\infty)$ such that $k \le \theta^{(1-\gamma)\tau} q_0$ for $\gamma > 1$ and $q_0 \in P^\infty\left(\mathbb{R}^n_+\right)$. Moreover, fix $\omega\,(x) = x_n^\tau$ and suppose that the functions $\varphi_1, \varphi_2 : \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfy $(\mathbf{H}_2)$. Then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that, for each $\lambda \in [0,\lambda_0)$ and each $\mu \in [0,\mu_0)$, the following problem

$$\begin{cases} \Delta u - au - \dfrac{\partial u}{\partial t} = \lambda h(x,t) v^\beta, \text{ in } \mathbb{R}^n_+ \times (0,\infty)\,, \\ \Delta v - bv - \dfrac{\partial v}{\partial t} = \mu k(x,t) u^\gamma, \text{ in } \mathbb{R}^n_+ \times (0,\infty)\,, \\ u(x,0) = \varphi_1(x);\ v(x,0) = \varphi_2(x), \text{ in } \mathbb{R}^n_+, \\ u = v = 0, \text{ in } \partial\mathbb{R}^n_+ \times (0,\infty)\,, \end{cases}$$

admits a positive continuous solution (u,v) on $\mathbb{R}^n_+ \times (0,\infty)$.
In fact, it is clear that $q_1 := kc_1^\gamma\,(P\omega)^{\gamma-1} \le c_1^\gamma q_0 \in P^\infty\left(\mathbb{R}^n_+\right)$. Furthermore, by Proposition $1(iii)$, we deduce that $p_1 \le c_2^\beta h \theta^{(\beta-1)\tau} \in P^\infty\left(\mathbb{R}^n_+\right)$. So the hypothesis $(\mathbf{H}_3)$ is satisfied.

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