An. Şt. Univ. Ovidius Constanţa Vol. 19(1), 2011, 67–74

THE EQUATIONS OF GENERALIZED COMPLEX STRUCTURES ON COMPLEX 2-TORI

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Abstract

We obtain the equations verified by any generalized complex structure on a complex 2-torus and we remark that the deformations in the sense of generalized complex structures, of the standard complex structure on a complex 2-torus, verify these equations.

1 Introduction

Nigel Hitchin defined in the paper [6] a generalized complex structure to be a complex structure, not on the tangent bundle T of a manifold, but on $T \oplus T^*$, unifying in this way the complex geometry and the symplectic geometry in some sense.

In this paper we obtain the equations verified by an arbitrary generalized complex structure on a complex 2-torus. Then, we remark that the generalized complex structures from the complete smooth family of deformations (in the sense of generalized complex structures) of the standard complex structure on a complex 2-torus, obtained in [3], verify these equations (for similar results on Kodaira surfaces see [2], [4]).

Key Words: Generalized complex manifolds, complex 2-torus.
Mathematics Subject Classification: 53C56, 32Q99, 53D99
Received: April, 2010
Accepted: December, 2011

2 Generalized complex structures on manifolds

A generalized complex structure on a manifold M (see [6], [5]) is defined to be a complex structure J ($J^2 = -1$) on the sum $T_M \oplus T_M^*$ of the tangent and cotangent bundles, which is required to be orthogonal with respect to the natural inner product on sections $X + \sigma$, $Y + \tau \in \mathcal{C}^\infty(T_M \oplus T_M^*)$ defined by

$$\langle X + \sigma, Y + \tau \rangle = \frac{1}{2}(\sigma(Y) + \tau(X)).$$

This is only possible if $\dim_{\mathbb{R}} M = 2n$, which we suppose.
In addition, the $(+\mathbf{i})$-eigenbundle

$$L \subset (T_M \oplus T_M^*) \otimes \mathbb{C}$$

of J is required to be involutive with respect to the Courant bracket, a skew bracket operation on smooth sections of $T_M \oplus T_M^*$ defined by

$$[X + \sigma, Y + \tau] = [X, Y] + \mathcal{L}_X \tau - \mathcal{L}_Y \sigma - \frac{1}{2} d(i_X \tau - i_Y \sigma),$$

where $\mathcal{L}_X$ and i_X denote the Lie derivative and interior product operations on forms.

Since J is orthogonal with respect to $\langle \cdot, \cdot \rangle$, the $(+\mathbf{i})$-eigenbundle L is a maximal isotropic subbundle of $(T_M \oplus T_M^*) \otimes \mathbb{C}$ of real index zero (i.e. $L \cap \bar{L} = \{0\}$). In fact, a generalized complex structure on M is completely determined by a maximal isotropic subbundle of $(T_M \oplus T_M^*) \otimes \mathbb{C}$ of real index zero, which is Courant involutive (see [5], [6]). For such a subbundle we have the decomposition

$$(T_M \oplus T_M^*) \otimes \mathbb{C} = L \oplus \bar{L},$$

and we may use the inner product $\langle \cdot, \cdot \rangle$ to identify $\bar{L} \equiv L^*$.

3 Generalized complex structures on 2-tori

Let $N = \mathbb{C}/\Lambda$ be a complex 2-torus, where $\mathbb{C}^2$ denotes the space of two complex variables (z, w) and $\Lambda \subset \mathbb{C}^2$ is an integral lattice of rank 4 (see, for example [1]).

We shall identify $\mathbb{C}^2$ with $\mathbb{R}^4$, the space of four real variables (x, y, u, v) by $z = x + \mathbf{i}y$, $w = u + \mathbf{i}v$. From the point of view of differential structure, a complex 2-torus is a parallelizable manifold, i.e. the tangent bundle T_N is globally generated by invariant vector fields $\{X, Y, U, V\}$ with all Poisson brackets zero. The complex structure endomorphism J is acting on T_N by

$$JX = Y \quad JY = -X, \quad JU = V, \quad JV = -U.$$

Let

$$T = \frac{1}{2}(X - \mathbf{i}Y), \quad W = \frac{1}{2}(U - \mathbf{i}V).$$

Then, the tangent bundle T_N is globally generated by $\{T, W, \bar{T}, \bar{W}\}$ and the cotangent bundle T_N^* is globally generated by the dual basis of 1-forms $\{\omega, \rho, \bar{\omega}, \bar{\rho}\}$.

We have

$$JT = \mathbf{i}T, \quad JW = \mathbf{i}W, \quad J\bar{T} = -\mathbf{i}\bar{T}, \quad J\bar{W} = -\mathbf{i}\bar{W}.$$

Now, we shall obtain the equations satisfied by any generalized complex structure on a complex 2-torus N. Recall (see [5]) that a generalized complex structure on N is completely determined by a maximal isotropic subbundle $L \subset (T_N \oplus T_N^*) \otimes \mathbb{C}$ of real index zero, which is Courant involutiv.

Let $L = \{v_1, v_2, v_3, v_4\}^{\sim}$ with $\dim_{\mathbb{R}} L = 4$, where

$$v_j = \gamma_{1j}T + \gamma_{2j}W + \gamma_{3j}\bar{T} + \gamma_{4j}\bar{W} + \delta_{1j}\omega + \delta_{2j}\rho + \delta_{3j}\bar{\omega} + \delta_{4j}\bar{\rho}, \; j = 1, 2, 3, 4,$$

and $\bar{L} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}^{\sim}$, where

$$\begin{aligned}\bar{v}_k = \bar{\gamma}_{1k}\bar{T} + \bar{\gamma}_{2k}\bar{W} + \bar{\gamma}_{3k}T + \bar{\gamma}_{4k}W + \bar{\delta}_{1k}\bar{\omega} + \bar{\delta}_{2k}\bar{\rho} + \\ + \bar{\delta}_{3k}\omega + \bar{\delta}_{4k}\rho, \quad k = 1, 2, 3, 4.\end{aligned}$$

We have γ_{ij}, $\delta_{ij} \in \mathcal{C}^{\infty}(N)$, $\forall\, i, j = 1, 2, 3, 4$.
The subbundle L is of real index zero (i.e. $L \cap \bar{L} = \{0\}$) if and only if the set $\{v_1, v_2, v_3, v_4, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ is a linearly independent system. A simple computation as in [4] gives us the following result:

Lemma 3.1. *The subbundle L is of real index zero if and only if the following condition holds:*

$$(*) \qquad \det \begin{pmatrix} \Gamma & \bar{\Gamma}_{3412} \\ \Delta & \bar{\Delta}_{3412} \end{pmatrix} \neq 0,$$

with entries the matrices

$$\Gamma = (\gamma_{ij})_{1 \le i,j \le 4} \quad \Delta = (\delta_{ij})_{1 \le i,j \le 4}$$

and Γ_{3412}, *and* $\bar{\Delta}_{3412}$ *their conjugate matrices with lines in the order* $(3, 4, 1, 2)$.

The next result is the following:

Lemma 3.2. *The subbundle L is isotropic if and only if the following conditions hold:*

$$(**) \qquad \sum_{k=1}^{4} (\gamma_{ki}\delta_{kj} + \gamma_{kj}\delta_{ki}) = 0, \quad i, j = 1, 2, 3, 4.$$

Proof. The subbundle L is isotropic if and only if $\langle v, w\rangle = 0 \ \forall \ v, w \in \mathcal{C}^\infty(L)$. Since the inner product $\langle \cdot, \cdot\rangle$ is bilinear, these conditions are equivalent to the conditions $\langle v_i, v_j\rangle = 0$, $\forall \ i, j = 1, 2, 3, 4$, i.e. to the conditions (**).

Now, we shall study the Courant involutivity of the subbundle L. We need to compute the Nijenhuis operator

$$Nij(A, B, C) = \frac{1}{3} \left(\langle [A, B], C\rangle + \langle [B, C], A\rangle + \langle [C, A], B\rangle\right),$$

where $A, B, C \in \mathcal{C}^\infty((T_N \oplus T_N^*) \otimes \mathbb{C})$. We have the following result:

Lemma 3.3. *For any $A, B, C \in \mathcal{C}^\infty((T_N \oplus T_N^*) \otimes \mathbb{C})$ and any $f \in \mathcal{C}^\infty(N)$ we have the formula:*

$$3Nij(A, fB, C) = 3fNij(A, B, C) + (\pi(A)f + \langle df, A\rangle)\langle B, C\rangle - (\pi(C)f + \langle df, C\rangle)\langle A, B\rangle.$$

where $\pi : T_N \oplus T_N^* \to T_N$ is the natural projection.

Proof. (see [4]).

We get the following result as in [4]:

Proposition 3.4. *The maximal isotropic subbundle $L \subset (T_N \oplus T_N^*) \otimes \mathbb{C}$ is Courant involutive if and only if*

$$Nij(v_i, v_j, v_k) = 0 \quad \forall \quad i, j, k = 1, 2, 3, 4.$$

Proof By Lemma 4.3 we get

$$3Nij(v_i, fv_j, v_k) = 3fNij(v_i, v_j, v_k) +$$

$$+ (\pi(v_i)f + \langle df, v_i\rangle)\langle v_j, v_k\rangle - (\pi(v_k)f + \langle df, v_k\rangle)\langle v_i, v_j\rangle, \ \forall \ f \in \mathcal{C}^\infty(N).$$

Since L is isotropic we obtain

$$3Nij(v_i, fv_j, v_k) = 3fNij(v_i, v_j, v_k).$$

Now, the result follows by using the additivity of the operator Nij and the Proposition 3.27 of [5]..

We shall use the following notation:

$$v_j = X_j + \xi_j \in \mathcal{C}^\infty((T_N \oplus T_N^*) \otimes \mathbb{C}),$$

where

$$X_j = \gamma_{1j}T + \gamma_{2j}W + \gamma_{3j}\bar{T} + \gamma_{4j}\bar{W} \in \mathcal{C}^\infty(T_N \otimes \mathbb{C})$$

and

$$\xi_j = \delta_{1j}\omega + \delta_{2j}\rho + \delta_{3j}\bar{\omega} + \delta_{4j}\bar{\rho} \in \mathcal{C}^\infty(T_N^* \otimes \mathbb{C}).$$

By definition, we have:

$$[v_i, v_j] = [X_i, X_j] + \mathcal{L}_{X_i}\xi_j - \mathcal{L}_{X_j}\xi_i - \frac{1}{2}(i_{X_i}\xi_j - i_{X_j}\xi_i).$$

By direct computation, we get (see [3], Lemma 3.2):

$$\begin{aligned} &[X_i, X_j] = (X_i(\gamma_{1j}) - X_j(\gamma_{1i}))T + (X_i(\gamma_{2j}) - X_j(\gamma_{2i}))W+ \\ &+(X_i(\gamma_{3j}) - X_j(\gamma_{3i}))\bar{T} + (X_i(\gamma_{4j}) - X_j(\gamma_{4i})) + \bar{W}. \end{aligned} \tag{1}$$

For any $Y = \alpha_1 T + \alpha_2 W + \alpha_3 \bar{T} + \alpha_4 \bar{W} \in \mathcal{C}^\infty(T_N \otimes \mathbb{C})$, we have

$$(\mathcal{L}_{X_i}\xi_j)(Y) = X_i(\xi_j(Y)) - \xi_j([X_i, Y])$$

and, after computation we get:

$$(\mathcal{L}_{X_i}\xi_j)(Y) = \sum_{k=1}^{4}\alpha_k X_i(\delta_{kj}) + \sum_{k=1}^{4}\delta_{kj}Y(\gamma_{ki}).$$

Analogously, we have

$$(\mathcal{L}_{X_j}\xi_i)(Y) = \sum_{k=1}^{4}\alpha_k X_j(\delta_{ki}) + \sum_{k=1}^{4}\delta_{ki}Y(\gamma_{kj}).$$

By direct computation, we get:

$$\begin{aligned} &(d(i_{X_i}\xi_j - i_{X_j}\xi_i))(Y) = \\ &= \textstyle\sum_{k=1}^{4}(\delta_{kj}Y(\gamma_{ki}) + \gamma_{ki}Y(\delta_{kj}) - \delta_{ki}Y(\gamma_{kj}) - \gamma_{kj}Y(\delta_{ki})). \end{aligned}$$

Now, a tedious but direct computation gives:

Lemma 3.5. *The Courant brackets are:*

$$[v_i, v_j] = [X_i, X_j] + v_{ij}, \quad [X_i, X_j] \in \mathcal{C}^\infty(T_N \otimes \mathbb{C}), \quad v_{ij} \in \mathcal{C}^\infty(T_n^* \otimes \mathbb{C}),$$

where

$$
\begin{aligned}
v_{ij} &= (X_i(\delta_{1j}) - X_j(\delta_{1i}) + \\
&+\frac{1}{2}\sum_{k=1}^{4}\delta_{kj}T(\gamma_{ki}) - \frac{1}{2}\sum_{k=1}^{4}\delta_{ki}T(\gamma_{kj}) - \frac{1}{2}\sum_{k=1}^{4}\gamma_{ki}T(\delta_{kj}) + \\
&+\frac{1}{2}\sum_{k=1}^{4}\gamma_{kj}T(\delta_{ki}))\omega + (X_i(\delta_{2j}) - X_j(\delta_{2i}) + \frac{1}{2}\sum_{k=1}^{4}\delta_{kj}W(\gamma_{ki}) - \\
&-\frac{1}{2}\sum_{k=1}^{4}\delta_{ki}W(\gamma_{kj}) - \frac{1}{2}\sum_{k=1}^{4}\gamma_{ki}W(\delta_{kj}) + \\
&+\frac{1}{2}\sum_{k=1}^{4}\gamma_{kj}W(\delta_{ki}))\rho + (X_i(\delta_{3j}) - X_j(\delta_{3i}) + \\
&+\frac{1}{2}\sum_{k=1}^{4}\delta_{kj}\bar{T}(\gamma_{ki}) - \frac{1}{2}\sum_{k=1}^{4}\delta_{ki}\bar{T}(\gamma_{kj}) - \\
&-\frac{1}{2}\sum_{k=1}^{4}\gamma_{ki}\bar{T}(\delta_{kj}) + \frac{1}{2}\sum_{k=1}^{4}\gamma_{kj}\bar{T}(\delta_{ki}))\bar{\omega} + \\
&+(X_i(\delta_{4j}) - X_j(\delta_{4i}) + \frac{1}{2}\sum_{k=1}^{4}\delta_{kj}\bar{W}(\gamma_{ki}) - \frac{1}{2}\sum_{k=1}^{4}\delta_{ki}\bar{W}(\gamma_{kj}) - \\
&-\frac{1}{2}\sum_{k=1}^{4}\gamma_{ki}\bar{W}(\delta_{kj}) + \frac{1}{2}\sum_{k=1}^{4}\gamma_{kj}\bar{W}(\delta_{ki}))\bar{\rho}.
\end{aligned} \tag{2}
$$

Now, we shall compute the Nijenhuis operator:

$$
6Nij(v_i, v_j, v_l) = 2\langle [v_i, v_j], v_l\rangle + 2\langle [v_j, v_l], v_i\rangle + 2\langle [v_l, v_i], v_j\rangle. \tag{3}
$$

We have;

$$
\begin{aligned}
2\langle [v_i, v_j], v_l\rangle &= 2\langle [X_i, X_j] + v_{ij}, X_l + \xi_l\rangle = \\
&= v_{ij}(X_l) + \xi_l([X_i, X_j]).
\end{aligned}
$$

We shall change the notation by

$$
T = e_1, \quad W = e_2, \quad \bar{T} = e_3, \quad \bar{W} = e_4
$$

and

$$
\omega = e^1, \quad \rho = e^2, \quad \bar{\omega} = e^3, \bar{\rho} = e^4.
$$

Then,

$$
X_j = \sum_{k=1}^{4}\gamma_{k_j}e_k, \quad \xi_j = \sum_{k=1}^{4}\delta_{kj}e^k,
$$

and we obtain by (1):

$$\xi_l[X_i, X_j]) = \sum_{s=1}^{4} \delta_{sl} \cdot \sum_{k=1}^{4} (\gamma_{ki} e_k(\gamma_{sj}) - \gamma_{kj} e_k(\gamma_{si}))$$

respectively, by (2):

$$v_{ij}(X_l) = \sum_{s,k=1}^{4} (\gamma_{sl}\gamma_{ki} e_k(\delta_{sj}) - \gamma_{sl}\gamma_{kj} e_k(\delta_{si})) +$$
$$\frac{1}{2} \sum_{s,k=1}^{4} (\gamma_{sl}\delta_{kj} e_s(\gamma_{ki}) - \gamma_{sl}\delta_{ki} e_s(\gamma_{kj}) - \gamma_{sl}\gamma_{ki} e_s(\delta_{kj}) + \gamma_{sl}\gamma_{kj} e_s(\delta_{ki})).$$

Now, a tedious but direct computation gives the following result:

Theorem 3.6. *The Courant involutive condition $Nij(v_i, v_j, v_l) = 0$ is equivalent to the condition:*

$$(***) \quad \begin{aligned} &\sum_{s,k=1}^{4} ((\gamma_{kl}\delta_{sj} - \gamma_{kj}\delta_{sl}) e_k(\gamma_{si}) + (\gamma_{ki}\delta_{sl} - \gamma_{kl}\delta_{si}) e_k(\gamma_{sj}) + \\ &+ (\gamma_{kj}\delta_{si} - \gamma_{ki}\delta_{sj}) e_k(\gamma_{sl})) + \\ &+ \sum_{s,k=1}^{4} ((\gamma_{sj}\gamma_{kl} - \gamma_{sl}\gamma_{kj}) e_k(\delta_{si}) + (\gamma_{sl}\gamma_{ki} - \gamma_{kl}\gamma_{si}) e_k(\delta_{sj}) + \\ &+ (\gamma_{si}\gamma_{kj} - \gamma_{ki}\gamma_{sj}) e_k(\delta_{sl})) = 0. \end{aligned}$$

Finally, we get the main result:

Corollary 3.7. *The equations of a generalized complex structure on a complex 2-torus are given by the conditions; (*), (**) and (***).*

Remark 3.1. *The generalized complex structures on a complex 2-torus N obtained by deformation theory in [3] are given by the subbundle*

$$L_\varepsilon = \{v_1, v_2, v_3, v_4\}^\sim$$

, where

$$v_1 = t_{11}T + t_{21}W + \bar{T} - t_{32}\bar{\rho}, v_2 = t_{12}T + t_{22}W + \bar{W} - t_{32}\bar{\omega},$$

$$v_3 = -t_{14}W + \omega - t_{11}\bar{\omega} - t_{12}\bar{\rho}, v_4 = t_{14}T + \rho - t_{21}\bar{\omega} - t_{22}\bar{\rho}.$$

By direct computation it follows that these generalized complex structures verify the equations (*), (**) and (***).

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