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A remark about fractional (f, n)-critical graphs

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Abstract

Let G be a graph of order p, and let a, b and n be nonnegative integers with $b \geq a \geq 2$, and let f be an integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for each $x \in V(G)$. A fractional f-factor is a function h that assigns to each edge of a graph G a number in [0,1], so that for each vertex x we have $d_G^h(x) = f(x)$, where $d_G^h(x) = \sum_{e \ni x} h(e)$ (the sum is taken over all edges incident to x) is a fractional degree of x in G. Then a graph G is called a fractional (f, n)-critical graph if after deleting any n vertices of G the remaining graph of G has a fractional f-factor. The binding number $bind(G)$ is defined as follows,

$$bind(G) = min\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

In this paper, it is proved that G is a fractional (f, n)-critical graph if $p \geq \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn}{a-1}$, $bind(G) \geq \frac{(a+b-1)(p-1)}{a(p-1)-bn}$ and $\delta(G) \neq \lfloor \frac{(b-1)p+a+b+bn-2}{a+b-1} \rfloor$.

1 Introduction

The graphs considered in this paper will be finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For any $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For

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$S \subseteq V(G)$, we write $N_G(S) = \bigcup_{x \in S} N_G(x)$, and denote by $G[S]$ the subgraph of G induced by S and by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. We use $\delta(G)$ to denote the minimum degree of G. The binding number of G is defined as

$$bind(G) = min\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

For a real number r, we use $\lfloor r \rfloor$ to denote the floor of r, which is the largest integer smaller than or equal to r, and also use $\lceil r \rceil$ to denote the ceiling of r, which is the least integer greater than or equal to r.

Let f be a nonnegative integer-valued function defined on $V(G)$. Then a spanning subgraph F of G is called an f-factor if $d_F(x) = f(x)$ for all $x \in V(G)$. If $f(x) = k$ for each $x \in V(G)$, then an f-factor is simply called a k-factor. A fractional f-factor is a function h that assigns to each edge of a graph G a number in [0,1], so that for each vertex x we have $d_G^h(x) = f(x)$, where $d_G^h(x) = \sum_{e \ni x} h(e)$ (the sum is taken over all edges incident to x) is a fractional degree of x in G. If $f(x) = k$ for each $x \in V(G)$, then a fractional f-factor is a fractional k-factor. A graph G is called a fractional (f, n)-critical graph if after deleting any n vertices of G the remaining graph of G has a fractional f-factor. If G is a fractional (f, n)-critical graph, then we also say that G is fractional (f, n)-critical. If $f(x) = k$ for each $x \in V(G)$, then a fractional (f, n)-critical graph is a fractional (k, n)-critical graph. A fractional (k, n)-critical graph is also called a fractional n-critical graph if $k = 1$. Some other terminologies and notations can be found in [1].

Zhou [5] obtained some sufficient conditions for graphs to have fractional k-factors. Yu and Liu [3] showed binding number and minimum degree conditions for graphs to have fractional k-factors. Cai and Liu [2] got a stability number condition for a graph to have a fractional f-factor. Zhou [4,6] gave two sufficient conditions for graphs to be fractional (f, n)-critical graphs. The following results on fractional (f, n)-critical graphs are known.

Theorem 1. [6] *Let G be a graph of order p, and let a, b and n be nonnegative integers such that $2 \leq a \leq b$, and let f be an integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for each $x \in V(G)$. If $bind(G) > \frac{(a+b-1)(p-1)}{ap-(a+b)-bn+2}$ and $p \geq \frac{(a+b)(a+b-3)}{a} + \frac{bn}{a-1}$, then G is a fractional (f, n)-critical graph.*

Theorem 2. [4] *Let a, b and n be nonnegative integers such that $1 \leq a \leq b$, and let G be a graph of order p with $p \geq \frac{(a+b-1)(a+b-2)+bn-2}{a}$, and let f be an integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for all*

$x \in V(G)$. *Suppose that*

$$|N_G(X)| > \frac{(b-1)p + |X| + bn - 1}{a+b-1}$$

for every non-empty independent subset X *of* $V(G)$, *and*

$$\delta(G) > \frac{(b-1)p + a + b + bn - 2}{a+b-1}.$$

Then G *is a fractional* (f, n)*-critical graph.*

In this paper, we proceed to study the fractional (f, n)-critical graphs, and obtain a binding number and minimum degree condition for a graph to be a fractional (f, n)-critical graph. Our main result is an improvement of Theorem 1 and will be given in the following section.

2 The Main Result and It's Proof

In the following, we give our main result.

Theorem 3. *Let* a, b *and* n *be nonnegative integers such that* $2 \le a \le b$, *and let* G *be a graph of order* p *with* $p \ge \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn}{a-1}$, *and let* f *be an integer-valued function defined on* $V(G)$ *such that* $a \le f(x) \le b$ *for each* $x \in V(G)$. *If* G *satisfies*

$$bind(G) \ge \frac{(a+b-1)(p-1)}{a(p-1) - bn}$$

and

$$\delta(G) \ne \lfloor \frac{(b-1)p + a + b + bn - 2}{a+b-1} \rfloor,$$

then G *is a fractional* (f, n)*-critical graph.*

The result of Theorem 3 is stronger than one of Theorem 1 if $\delta(G) \ne \lfloor \frac{(b-1)p+a+b+bn-2}{a+b-1} \rfloor$.

If $n = 0$ in Theorem 3, then we get the following corollary.

Corollary 1. *Let* a *and* b *be two integers such that* $2 \le a \le b$, *and let* G *be a graph of order* p *with* $p \ge \frac{(a+b-1)(a+b-2)-2}{a}$, *and let* f *be an integer-valued function defined on* $V(G)$ *such that* $a \le f(x) \le b$ *for each* $x \in V(G)$. *If* G *satisfies*

$$bind(G) \ge \frac{a+b-1}{a}$$

and

$$\delta(G) \neq \lfloor \frac{(b-1)p+a+b-2}{a+b-1} \rfloor,$$

then G has a fractional f-factor.

If $a = b = k$ in Theorem 3, then we obtain the following corollary.

Corollary 2. *Let k and n be nonnegative integers such that $k \geq 2$, and let G be a graph of order p with $p \geq 4k - 6 + \frac{kn}{k-1}$. If G satisfies*

$$bind(G) \geq \frac{(2k-1)(p-1)}{k(p-1)-kn}$$

and

$$\delta(G) \neq \lfloor \frac{(k-1)p+2k+kn-2}{2k-1} \rfloor,$$

then G is a fractional (k,n)-critical graph.

If $n = 0$ in Corollary 2, then we have the following corollary.

Corollary 3. *Let k be an integer such that $k \geq 2$, and let G be a graph of order p with $p \geq 4k - 6$. If G satisfies*

$$bind(G) \geq \frac{2k-1}{k}$$

and

$$\delta(G) \neq \lfloor \frac{(k-1)p+2k-2}{2k-1} \rfloor,$$

then G has a fractional k-factor.

Proof of Theorem 3. For any $X \subseteq V(G)$ with $X \neq \emptyset$ and $N_G(X) \neq V(G)$. Let $Y = V(G) \setminus N_G(X)$. Clearly, $\emptyset \neq Y \subseteq V(G)$. Now, we prove the following claims.

Claim 1. $X \cap N_G(Y) = \emptyset$.

Proof. We assume that $X \cap N_G(Y) \neq \emptyset$. Then there exists some vertex x such that $x \in X \cap N_G(Y)$. Since $x \in N_G(Y)$, we have $y \in Y$ such that $xy \in E(G)$. Thus, we obtain $y \in N_G(x) \subseteq N_G(X)$. Which contradicts $y \in Y = V(G) \setminus N_G(X)$. This completes the proof of Claim 1.

Claim 2. $|N_G(X)| > \frac{(b-1)p+|X|+bn-1}{a+b-1}$.

Proof. According to Claim 1, we get

$$|X| + |N_G(Y)| \leq p \tag{1}$$

and

$$N_G(Y) \neq V(G). \tag{2}$$

In terms of (1), (2) and the definition of $bind(G)$, we obtain

$$bind(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{p-|X|}{|V(G) \setminus N_G(X)|} = \frac{p-|X|}{p-|N_G(X)|}. \tag{3}$$

From (3), we have

$$|N_G(X)| \geq p - \frac{p-|X|}{bind(G)}. \tag{4}$$

Set $F(t) = p - \frac{p-|X|}{t}$. Then by $X \subseteq V(G)$ we get

$$F'(t) = \frac{p-|X|}{t^2} \geq 0.$$

Combining this with $bind(G) \geq \frac{(a+b-1)(p-1)}{a(p-1)-bn}$, we obtain

$$F(bind(G)) \geq F(\frac{(a+b-1)(p-1)}{a(p-1)-bn}),$$

that is,

$$p - \frac{p-|X|}{bind(G)} \geq p - \frac{p-|X|}{\frac{(a+b-1)(p-1)}{a(p-1)-bn}} = p - \frac{(p-|X|)(a(p-1)-bn)}{(a+b-1)(p-1)}. \tag{5}$$

Using (4), (5) and $p \geq \frac{(a+b-1)(a+b-2)-2}{a} + \frac{bn}{a-1}$, we have

$$\begin{aligned}
|N_G(X)| &\geq p - \frac{p-|X|}{bind(G)} \geq p - \frac{(p-|X|)(a(p-1)-bn)}{(a+b-1)(p-1)} \\
&= \frac{(b-1)(p-1)p + (a(p-1)-bn)|X| + bnp}{(a+b-1)(p-1)} \\
&= \frac{(b-1)(p-1)p + (p-1)|X| + ((a-1)(p-1)-bn)|X| + bnp}{(a+b-1)(p-1)} \\
&\geq \frac{(b-1)(p-1)p + (p-1)|X| + ((a-1)(p-1)-bn) + bnp}{(a+b-1)(p-1)} \\
&= \frac{(b-1)(p-1)p + (p-1)|X| + (a-1)(p-1) + bn(p-1)}{(a+b-1)(p-1)} \\
&= \frac{(b-1)p + |X| + bn + a - 1}{a+b-1} \\
&> \frac{(b-1)p + |X| + bn - 1}{a+b-1}.
\end{aligned}$$

The Proof of Claim 2 is complete.

By any $\emptyset \neq X \subseteq V(G)$ and $|N_G(X)| \geq \frac{(b-1)p+|X|+bn+a-1}{a+b-1}$, we obtain

$$\delta(G) \geq \frac{(b-1)p+a+bn}{a+b-1}. \tag{6}$$

Claim 3. $\delta(G) > \frac{(b-1)p+a+b+bn-2}{a+b-1}$.

Proof. Suppose that $\delta(G) \leq \frac{(b-1)p+a+b+bn-2}{a+b-1}$. Combining this inequality above with (6), we have

$$\lceil \frac{(b-1)p+a+bn}{a+b-1} \rceil \leq \delta(G) \leq \lfloor \frac{(b-1)p+a+b+bn-2}{a+b-1} \rfloor,$$

that is,

$$\delta(G) = \lceil \frac{(b-1)p+a+bn}{a+b-1} \rceil = \lfloor \frac{(b-1)p+a+b+bn-2}{a+b-1} \rfloor.$$

That contradicts the condition of Theorem 3. This completes the proof of Claim 3.

From Claim2, Claim 3 and Theorem 2, G is a fractional (f,n)-critical graph. This completes the proof of Theorem 3.

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