An. Şt. Univ. Ovidius Constanţa Vol. 19(1), 2011, 237–254

ON A GENERAL CLASS OF LINEAR AND POSITIVE OPERATORS

Ovidiu T. Pop, Mircea D. Fărcaş and Dan Bărbosu

Abstract

Suppose that $(L_m)_{m\geq 1}$ is a given sequence of linear and positive operators. Starting with the mentioned sequence, the new sequence $(K_m)_{m\geq 1}$ of linear and positive operators is constructed. For the operators $(K_m)_{m\geq 1}$ a convergence theorem and a Voronovskaja-type theorem are established. As particular cases of the general construction, we refined the Bernstein's operators, the Stancu's operators, the Mirakyan-Favard-Szasz operators, the Baskakov operators, the Bleimann-Butzer-Hahn operators, the Meyer-König-Zeller operators, the Ismail-May operators.

1 Introduction

In this section, we recall some notions and operators which will be used in the paper.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ be the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

Key Words: Linear and positive operators, convergence theorem, the first order modulus of smoothness, Voronovskaja-type theorem, Baskakov operators, Bernstein operators, Ismail-May operators, Mirakyan-Favard-Szasz operators, Stancu operators, Bleimann-Butzer-Hahn operators.

Mathematics Subject Classification: 41A10, 41A25, 41A35, 41A36

Received: January, 2010

Accepted: December, 2010

where $p_{m,k}(x)$ are the fundamental Bernstein polynomials defined by

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0,1,\ldots,m\}$ (see [5] or [25]).

For the following construction see [15]. Define the natural number m_0 by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1, 1-\beta\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.3}$$

For the real number β, we have that

$$m + \beta \geq \gamma_\beta \tag{1.4}$$

for any natural number m, $m \geq m_0$, where

$$\gamma_\beta = m_0 + \beta = \begin{cases} \max\big\{1+\beta, \{\beta\}\big\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta, 1\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.5}$$

For the real numbers α, β, $\alpha \geq 0$, we set

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if} \quad \alpha \leq \beta \\ 1 + \dfrac{\alpha-\beta}{\gamma_\beta}, & \text{if} \quad \alpha > \beta. \end{cases} \tag{1.6}$$

For the real numbers α and β, $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha,\beta)}$ and

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.7}$$

for any natural number $m \geq m_0$ and for any $k \in \{0,1,\ldots,m\}$.

For the real numbers α and β, $\alpha \geq 0$, m_0 and $\mu^{(\alpha,\beta)}$ defined by (1.3)-(1.6), let the operators $P_m^{(\alpha,\beta)} : C\big([0,\mu^{(\alpha,\beta)}]\big) \to C\big([0,1]\big)$ be defined for any function $f \in C\big([0,\mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.8}$$

for any natural number $m \geq m_0$ and for any $x \in [0,1]$. These operators are called Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [24]. Note that in [24], the domain of definition for the Stancu operators

is $C([0,1])$ and the numbers α and β verify the condition $0 \le \alpha \le \beta$.
In 1980 [4], G. Bleimann, P. L. Butzer and L. Hahn introduced the sequence of linear positive operators $(L_m)_{m\ge 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{1.9}$$

for any $x \in [0,\infty)$ and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f \mid f : [0,\infty) \to \mathbb{R},\ f$ bounded and continuous on $[0,\infty)\}$.

For $m \in \mathbb{N}$ were considered the operators $S_m : C_2([0,\infty)) \to C([0,\infty))$ defined for any function $f \in C_2([0,\infty))$ by

$$(S_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right), \tag{1.10}$$

for any $x \in [0,\infty)$, where $C_2([0,\infty)) = \left\{ f \in C([0,\infty)) : \lim\limits_{x\to\infty} \dfrac{f(x)}{1+x^2} \text{ exists and is finite} \right\}$. The operators $(S_m)_{m\ge 1}$ are called Mirakjan-Favard-Szász operators and were introduced in 1941 by G. M. Mirakjan in [12].

They were intensively studied by J. Favard in 1944 in [8] and O. Szász in 1950 in [26].

Let for $m \in \mathbb{N}$, the operators $V_m : C_2([0,\infty)) \to C([0,\infty))$ defined for any function $f \in C_2([0,\infty))$ by

$$(V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right), \tag{1.11}$$

for any $x \in [0,\infty)$. They are called the Baskakov operators and were introduced in 1957 by V. A. Baskakov in [2].

W. Meyer-König and K. Zeller introduced in [11] a sequence of linear and positive operators. After a slight adjustment given by E. W. Cheney and A. Sharma in [6], these operators $Z_m : B([0,1)) \to C([0,1))$, defined for any function $f \in B([0,1))$ by

$$(Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k}\right), \tag{1.12}$$

for any $m \in \mathbb{N}$ and for any $x \in [0,1)$. These operators are called the Meyer-König and Zeller operators. Observe that $Z_m : C([0,1]) \to C([0,1])$, $m \in \mathbb{N}$.

In the paper [10], M. Ismail and C. P. May consider the operators $(R_m)_{m\geq 1}$. For $m\in\mathbb{N}$, $R_m : C([0,\infty))\to C([0,\infty))$ is defined for any function $f\in C([0,\infty))$ by

$$(R_m f)(x) = e^{-\frac{mx}{1+x}}\sum_{k=0}^{\infty}\frac{m(m+k)^{k-1}}{k!}\left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{k}{m}\right) \tag{1.13}$$

for any $x\in[0,\infty)$.

In what follows, we consider $I\subset\mathbb{R}$, I an interval and we shall use the following sets of functions: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I)=\big\{f\,|\,f:I\to\mathbb{R},\ f \text{ bounded on } I\big\}$, $C(I)=\big\{f\,|\,f:I\to\mathbb{R},\ f \text{ continuous on } I\big\}$ and $C_B(I)=B(I)\cap C(I)$.

If $f\in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f;\cdot):[0,\infty)\to\mathbb{R}$ defined for any $\delta\geq 0$ by

$$\omega(f;\delta)=\sup\left\{|f(x')-f(x'')| : x',x''\in I, |x'-x''|\leq\delta\right\}. \tag{1.14}$$

2 Preliminaries

In the following, we consider the general construction and the results from [22], which we will use afterwards in the paper.

Let I, J be intervals with $I\subset[0,\infty)$ and $I\cap J\neq\emptyset$. For any $m\in\mathbb{N}$ and $k\in\mathbb{N}_0$ consider the the functions $\varphi_{m,k}:J\to\mathbb{R}$ with the property that $\varphi_{m,k}(x)\geq 0$ for any $x\in J$ and the linear and positive functionals $A_{m,k}:E(I)\to\mathbb{R}$. Let $E(I)$ and $F(J)$ be subsets of the set of real functions defined on I, respectively J such that the series

$$\sum_{k=0}^{\infty}\varphi_{m,k}(x)f(x_{m,k})$$

is convergent for any $f\in E(I)$ and $x\in J$. For any $x\in I$ consider the functions $\psi_x:I\to\mathbb{R}$, $\psi_x(t)=t-x$ for any $t\in I$ and we suppose that $\psi_x^i\in E(I)$, for any $x\in I\cap J$ and any $i\in\{0,1,2,\dots,s+2\}$. In what follows $s\in\mathbb{N}_0$ is even.

For $m\in\mathbb{N}$ define the operators $L_m:E(I)\to E(J)$ by

$$(L_m f)(x)=\sum_{k=0}^{\infty}\varphi_{m,k}(x)A_{m,k}(f) \tag{2.1}$$

for any $f\in E(I)$ and $x\in J$. It is immediately the following

Proposition 2.1. *The operators $(L_m)_{m\geq 1}$ are linear and positive on $E(I\cap J)$.*

For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define T_i by

$$(T_i L_m)(x) = m^i \left(L_m \psi_x^i\right)(x) = m^i \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}\left(\psi_x^i\right) \tag{2.2}$$

for any $x \in I \cap J$.

Theorem 2.1. [22] *If $f \in E(I)$ is a s times differentiable function in $x \in I \cap J$, with $f^{(s)}$ continuous in x, and if there exist α_s, $\alpha_{s+2} \in [0, \infty)$ and $m(s) \in \mathbb{N}$ such that*

$$\alpha_{s+2} < \alpha_s + 2 \tag{2.3}$$

and $\dfrac{(T_s L_m)(x)}{m^{\alpha_s}}$, $\dfrac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}}$ *are bounded for any $m \in \mathbb{N}$, $m \geq m(s)$, then*

$$\lim_{m \to \infty} m^{s-\alpha_s} \left[(L_m f)(x) - \sum_{i=0}^{s} \frac{1}{i! m^i} (T_i L_m)(x) f^{(i)}(x)\right] = 0. \tag{2.4}$$

Assume that f is a s times differentiable function on I with $f^{(s)}$ continuous on I and an interval $K \subset I \cap J$ exists such that there exist $m(s) \in \mathbb{N}$ and the constants $k_j(K) \in \mathbb{R}$ depending on K, so that for any $m \in \mathbb{N}$, $m \geq m(s)$ and $x \in K$ we have

$$\frac{(T_j L_m)(x)}{m^{\alpha_j}} \leq k_j(K) \tag{2.5}$$

where $j \in \{s, s+2\}$. Then the convergence given in (2.4) *is uniform on K and*

$$m^{s-\alpha_s} \left|(L_m f)(x) - \sum_{i=0}^{s} \frac{1}{i! m^i} (T_i L_m)(x) f^{(i)}(x)\right| \leq \tag{2.6}$$

$$\leq \frac{1}{s!} \left(k_s(K) + k_{s+2}(K)\right) \omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right)$$

for any $x \in K$ and $m \geq m(s)$.

Remark 2.1. In Theorem 2.1 we choose the smallest α_s and α_{s+2} if they exist.

Now, if $m \in \mathbb{N}$ and $\varphi_{m,k}(x) = 0$, $A_{m,k}(f) = 0$ for any $f \in E(I)$, any $x \in J$ and any $k \in \{m+1, m+2, \ldots\}$, then we obtain a class of operators defined by finite sums, so that the relation (2.1) becomes

$$(L_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f). \tag{2.7}$$

Remark 2.2. From above, it follows that the theorems from [22] hold for the operators defined by finite sums and for the operators defined by infinite sums.

3 Main results

Taking the above results into account, we can make the following construction (see [22] and [23]).

Let I, J be real intervals with $I \cap J \neq \emptyset$ and $p_m = m$ for any $m \in \mathbb{N}$ (the finite case) or $p_m = \infty$ for any $m \in \mathbb{N}$ (the infinite case). For any $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, consider the nodes $x_{m,k} \in I$ (in this construction we have $A_{m,k}(f) = f(x_{m,k})$) and the functions $\varphi_{m,k} : J \to \mathbb{R}$, with the property that $\varphi_{m,k}(x) \geq 0$, for any $x \in J$. We suppose that for any compact $K \subset I \cap J$ there exists the sequence $(u_m(K))_{m \geq 1}$, depending on K, such that

$$\lim_{m\to\infty} u_m(K) = 0 \tag{3.1}$$

and

$$\left| \sum_{k=0}^{p_m} \varphi_{m,k}(x) - 1 \right| \leq u_m(K) \tag{3.2}$$

for any $x \in K$, any $m \in \mathbb{N}$ and we note $u(K) = \sup\{u_m(K) : m \in \mathbb{N}\}$.

Remark 3.1. From (3.1) and (3.2) it follows that $\lim\limits_{m\to\infty} \sum\limits_{k=0}^{p_m} \varphi_{m,k}(x) = 1$ for any $x \in K$ and the convergence is uniform on K.

Let $w : I \to (0, \infty)$ be a fixed function, called the weight function, such there exists a positive constant M such that $M \leq w(x)$, for any $x \in I$ and the set functions

$$E_w(I) = \{f | f : I \to \mathbb{R} \text{ such that } wf \text{ is bounded on } I\}. \tag{3.3}$$

For $f \in E_w(I)$ there exists a positive constant $M(f)$, depending on f, such that $w(x)|f(x)| \leq M(f)$, for any $x \in I$.

Let $K \subset I \cap J$ compact set and $x \in K$. If $p_m = m$ for any $m \in \mathbb{N}$, then the sum $\sum\limits_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k})$ exists for any $m \in \mathbb{N}$.

If $p_m = \infty$ for any $m \in \mathbb{N}$, we consider the sequence $(s_n(m))_{n \geq 1}$ defined by $s_n(m) = \sum\limits_{k=0}^{n} \varphi_{m,k}(x) |f(x_{m,k})|$, for any $n \in \mathbb{N}$. Taking (3.2) into account, we get

$$\begin{aligned} s_n(m) &= \sum_{k=0}^{n} \varphi_{m,k}(x) \frac{1}{w(x_{m,k})} w(x_{m,k}) |f(x_{m,k})| \leq \frac{M(f)}{M} \sum_{k=0}^{n} \varphi_{m,k}(x) \leq \\ &\leq \frac{M(f)}{M} (1 + u_m(K)) \leq \frac{M(f)}{M} (1 + u(K)), \end{aligned}$$

from where it follows that the sum $\sum\limits_{k=0}^{\infty} \varphi_{m,k}(x)|f(x_{m,k})|$ exists for any $m \in \mathbb{N}$. It follows that the sum $\sum\limits_{k=0}^{\infty} \varphi_{m,k}(x) f(x_{m,k})$ exists and then from the above results, we get that the sum $\sum\limits_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k})$ exists for any $m \in \mathbb{N}$.

For $m \in \mathbb{N}$ let the given operator $L_m : E_w(I) \to F(J)$ defined by

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k}) \tag{3.4}$$

for any $x \in J$ and any $f \in E_w(I)$, with the property that for any $f \in E_w(I) \cap C(I)$, we have

$$\lim_{m\to\infty} (L_m f)(x) = f(x) \tag{3.5}$$

uniformly on any compact $K \subset I \cap J$.

Remark 3.2. We suppose that the functions $\psi_x, e_i \in E_w(I)$, $x \in I$, where $e_i : I \to \mathbb{R}$, $e_i(t) = t^i$ for any $t \in I$, $i \in \{0, 1, 2, 3, 4\}$.

Remark 3.3. Taking the Bohman-Korovkin Theorem into account, from (3.5) it follows that for the operators $(L_m)_{m\geq 1}$ we have

$$\lim_{m\to\infty} (L_m e_i)(x) = e_i(x) \tag{3.6}$$

uniformly on any compact $K \subset I \cap J$, $i \in \{0, 1, 2\}$ and

$$\lim_{m\to\infty} (L_m \psi_x^2)(x) = 0 \tag{3.7}$$

uniformly on any compact $K \subset I \cap J$, where $x \in I$.

Remark 3.4. From Remark 3.3 it follows that for any compact $K \subset I \cap J$ there exist the sequences $(v_m(K))_{m\geq 1}$, $(w_m(K))_{m\geq 1}$ depending on K, such that

$$\lim_{m\to\infty} v_m(K) = \lim_{m\to\infty} w_m(K) = 0 \tag{3.8}$$

and

$$|(L_m e_1)(x) - x| \leq v_m(K), \tag{3.9}$$

$$(L_m \psi_x^2)(x) \leq w_m(K), \tag{3.10}$$

for any $x \in K$ and any $m \in \mathbb{N}$. We suppose in the following that there exists $0 < \alpha_2 < 2$, α_2 not depending on K, such that the sequence

$(m^{2-\alpha_2} w_m(K))_{m\geq 1}$ is bounded and $\lim\limits_{m\to\infty} m^{2-\alpha_2} w_m(K) = 0$. So, there exists $k_2(K) > 0$, depending on K such that

$$m^{2-\alpha_2} w_m(K) \leq k_2(K) \tag{3.11}$$

for any $m \in \mathbb{N}$.

Lemma 3.1. *For any $K \subset I \cap J$ there exists the constants $k_0(K)$ and $k_2(K)$, depending on K, such that*

$$(T_0 L_m)(x) \leq k_0(K) \tag{3.12}$$

and

$$\frac{(T_2 L_m)(x)}{m^{\alpha_2}} \leq k_2(K) \tag{3.13}$$

for any $x \in K$ and any $m \in \mathbb{N}$.

Proof. Let $m \in \mathbb{N}$ and $x \in K$. Then taking (3.2) into account, we obtain that

$$(T_0 L_m)(x) = (L_m e_0)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) \leq 1 + u_m(K) \leq 1 + u(K) = k_0(K)$$

Further, we have

$$\frac{(T_2 L_m)(x)}{m^{\alpha_2}} = \frac{m^2 (L_m \psi_x^2)(x)}{m^{\alpha_2}} = m^{2-\alpha_2} (L_m \psi_x^2)(x)$$

and taking (3.10), (3.11) into account we obtain (3.13). □

In the following, for $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$ we consider the nodes $y_{m,k} \in I$ such that

$$\beta_m = \sup_{k \in \{0,1,\cdots,p_m\} \cap \mathbb{N}_0} |x_{m,k} - y_{m,k}| < \infty \tag{3.14}$$

for any $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} m^{2-\alpha_2} \beta_m = 0, \tag{3.15}$$

so there exists $l > 0$ such that

$$m^{2-\alpha_2} \beta_m \leq l \tag{3.16}$$

for any $m \in \mathbb{N}$. For $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, we note $\beta_{m,k} = x_{m,k} - y_{m,k}$, and then $|\beta_{m,k}| \leq \beta_m$ for any $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$ and any $m \in \mathbb{N}$.

For $m \in \mathbb{N}$ define the operator $K_m : E_w(I) \to F(J)$ by

$$(K_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(y_{m,k}), \tag{3.17}$$

for any $x \in I$ and any $f \in E_w(I)$.

Lemma 3.2. *For any $K = [0, b] \subset I \cap J$ there exist the constants $k_0'(K)$ and $k_2'(K)$, depending on K, such that*

$$(T_0 K_m)(x) \le k_0'(K) \tag{3.18}$$

and

$$\frac{(T_2 K_m)(x)}{m^{\alpha_2}} \le k_2'(K) \tag{3.19}$$

for any $x \in K$ and any $m \in \mathbb{N}$.

Proof. We have $(T_0 K_m)(x) = (K_m e_0)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) = (L_m e_0)(x) = (T_0 L_m)(x) \le 1 + u_m(K) \le 1 + u(K)$ and we can take $k_0'(K) = k_0(K)$. Further,we have

$$\frac{(T_2 K_m)(x)}{m^{\alpha_2}} = \frac{m^2 (K_m \psi_x^2)(x)}{m^{\alpha_2}} = m^{2-\alpha_2} (K_m \psi_x^2)(x)$$

and

$$\begin{aligned}
&(K_m \psi_x^2)(x) = (K_m e_2)(x) - 2x(K_m e_1)(x) + x^2 (K_m e_0)(x) = \\
&= \sum_{k=0}^{p_m} \varphi_{m,k}(x) y_{m,k}^2 - 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x) y_{m,k} + x^2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) = \\
&= \sum_{k=0}^{p_m} \varphi_{m,k}(x)(x_{m,k} - \beta_{m,k})^2 - 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x)(x_{m,k} - \beta_{m,k}) + \\
&+ x^2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) x_{m,k}^2 - 2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) x_{m,k} \beta_{m,k} + \\
&+ \sum_{k=0}^{p_m} \varphi_{m,k}(x) \beta_{m,k}^2 - 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x) x_{m,k} + 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x) \beta_{m,k} + \\
&+ x^2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) \le (L_m \psi_x^2)(x) + 2\beta_m (L_m e_1)(x) + (\beta_m^2 + 2x\beta_m)(L_m e_0)(x)
\end{aligned}$$

so that

$$\begin{aligned}&m^{2-\alpha_2}(K_m\psi_x^2)(x)\leq m^{2-\alpha_2}(L_m\psi_x^2)(x)+2m^{2-\alpha_2}\beta_m(L_me_1)(x)+\\&+m^{2-\alpha_2}\beta_m(\beta_m+2x)(L_me_0)(x)\leq k_2(K)+2l(b+v(K))+\\&+l(\beta+2b)(1+u(K))=k_2'(K)\end{aligned}$$

where $v(K)=sup\{v_m(K):m\in\mathbb{N}\}$ and $\beta=sup\{\beta_m:m\in\mathbb{N}\}$. □

Lemma 3.3. *If* $\alpha_4>3\alpha_2-2$ *then*
i) If $x\in I\cap J$ *and* $\frac{(T_4L_m)(x)}{m^{\alpha_4}}$ *is bounded for any* $m\in\mathbb{N}$*, then* $\frac{(T_4K_m)(x)}{m^{\alpha_4}}$ *is bounded for any* $m\in\mathbb{N}$*.*
ii) If $K=[0,b]\subset I\cap J$ *and* $\frac{(T_4L_m)(x)}{m^{\alpha_4}}$ *is bounded on* K *for any* $m\in\mathbb{N}$*, then* $\frac{(T_4K_m)(x)}{m^{\alpha_4}}$ *is bounded on* K *for any* $m\in\mathbb{N}$*.*

Proof. We have

$$\begin{aligned}&(K_m\psi_x^4)(x)=\sum_{k=0}^{p_m}\varphi_{m,k}(x)(x_{m,k}-x)^4-4\sum_{k=0}^{p_m}\varphi_{m,k}(x)\beta_{m,k}(x_{m,k}-x)^3+\\&+6\sum_{k=0}^{p_m}\varphi_{m,k}(x)\beta_{m,k}^2(x_{m,k}-x)^2-4\sum_{k=0}^{p_m}\varphi_{m,k}(x)\beta_{m,k}^3(x_{m,k}-x)+\\&+\sum_{k=0}^{p_m}\varphi_{m,k}(x)\beta_{m,k}^4\leq(L_m\psi_x^4)(x)+4\beta_m|(L_m\psi_x^3)(x)|+6\beta_m^2(L_m\psi_x^2)(x)+\\&+4\beta_m^3|(L_m\psi_x)(x)|+\beta_m^4(L_me_0)(x)\end{aligned}$$

so that we can write

$$\begin{aligned}&\frac{(T_4K_m)(x)}{m^{\alpha_4}}=m^{4-\alpha_4}(K_m\psi_x^4)(x)\leq m^{4-\alpha_4}(L_m\psi_x^4)(x)+\\&+4m^{4-\alpha_4}\beta_m|(L_m\psi_x^3)(x)|+6m^{4-\alpha_4}\beta_m^2(L_m\psi_x^2)(x)+\\&+4m^{4-\alpha_4}\beta_m^3|(L_m\psi_x)(x)|+m^{4-\alpha_4}\beta_m^4(L_me_0)(x)=m^{4-\alpha_4}(L_m\psi_x^4)(x)+\\&+4\beta_mm^{4-\alpha_4}|(L_m\psi_x^3)(x)|+6(m^{2-\alpha_2}\beta_m)^2m^{2-\alpha_2}(L_m\psi_x^2)(x)m^{-2+3\alpha_2-\alpha_4}+\\&+4(m^{2-\alpha_2}\beta_m)^3|(L_m\psi_x)(x)|m^{-2+3\alpha_2-\alpha_4}+\\&+(m^{2-\alpha_2}\beta_m)^4(L_me_0)(x)m^{-4+4\alpha_2-\alpha_4}.\end{aligned}$$

Further, applying the Cauchy's inequality for linear and positive operators (see [17]), we get

$$(L_m\psi_x^3)^2(x)\leq(L_m\psi_x^2)(x)(L_m\psi_x^4)(x)$$

and

$$\left[m^{\frac{6-\alpha_2-\alpha_4}{2}}|(L_m\psi_x^3)(x)|\right]^2 \le \frac{(T_2L_m)(x)}{m^{\alpha_2}}\frac{(T_4L_m)(x)}{m^{\alpha_4}}$$

so that we have

$$m^{4-\alpha_4}\beta_m(L_m\psi_x^3)(x) = m^{2-\alpha_2}\beta_m m^{\frac{6-\alpha_2-\alpha_4}{2}}(L_m\psi_x^3)(x)m^{\frac{-2+3\alpha_2-\alpha_4}{2}}.$$

Taking into account the conditions $0 < \alpha_2 < 2$, $0 < \alpha_4 < 4$, $\alpha_4 < \alpha_2 + 2$ and $\alpha_4 > 3\alpha_2 - 2$, we obtain $-2+3\alpha_2-\alpha_4 < 0$ so it follows that $\beta_m m^{4-\alpha_4}|(L_m\psi_x^3)(x)|$ is bounded. On the other hand, we have that $|(L_m\psi_x)(x)| \le \sqrt{(L_me_0)(x)(L_m\psi_x^2)(x)}$ and $-4+4\alpha_2-\alpha_4 = (-2+3\alpha_2-\alpha_4)+(-2+\alpha_2) < 0$. From (3.9), (3.16), the above remarks and the inequality verified by $\frac{(T_4K_m)(x)}{m^{\alpha_4}}$, it follows the conclusion of the lemma. □

Theorem 3.1. *If $f \in E_w(I)$ is continuous at $x \in I \cap J$, then*

$$\lim_{m\to\infty}(K_mf)(x) = f(x). \tag{3.20}$$

If f is continuous on I, $K \subset I \cap J$ is a compact, then the convergence given in (3.20) *is uniform on K and*

$$\left|(K_mf)(x) - \left(\sum_{k=0}^{p_m}\varphi_{m,k}(x)\right)f(x)\right| \le (k_0'(K)+k_2'(K))\omega\left(f;\frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{3.21}$$

for any $x \in K$ and any $m \in \mathbb{N}$.

Proof. One applies Theorem 2.1 for $s = 0$ and Lemma 3.2. □

Corollary 3.1. *If $f \in E_w(I)$ is continuous on I,*

$$\sum_{k=0}^{p_m}\varphi_{m,k}(x) = 1$$

for any $x \in J$ and $m \in \mathbb{N}$, $K \subset I \cap J$ is a compact, then

$$|(K_mf)(x) - f(x)| \le (k_0'(K)+k_2'(K))\omega\left(f;\frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{3.22}$$

for any $x \in K$ and any $m \in \mathbb{N}$.

Proof. Directly from Theorem 3.1. □

Lemma 3.4. *We have*

$$\lim_{m\to\infty} m^{2-\alpha_2} \sum_{k=0}^{p_m} \varphi_{m,k}(x)\beta_{m,k} = 0$$

$$\lim_{m\to\infty} m^{2-\alpha_2} \sum_{k=0}^{p_m} \varphi_{m,k}(x)x_{m,k}\beta_{m,k} = 0$$

$$\lim_{m\to\infty} m^{2-\alpha_2} \sum_{k=0}^{p_m} \varphi_{m,k}(x)\beta_{m,k}^2 = 0$$

Proof. For the first relation, we have $-\beta_m \le \beta_{m,k} \le \beta_m$ for any $m \in \mathbb{N}$, $k \in \{0, 1, \dots, p_m\} \cap \mathbb{N}_0$ so that

$$\begin{aligned} &- m^{2-\alpha_2}\beta_m \sum_{k=0}^{p_m} \varphi_{m,k}(x) \le m^{2-\alpha_2} \sum_{k=0}^{p_m} \varphi_{m,k}(x)\beta_{m,k} \le \\ &\le m^{2-\alpha_2}\beta_m \sum_{k=0}^{p_m} \varphi_{m,k}(x)\beta_{m,k} \end{aligned}$$

and we take into account that $\lim\limits_{m\to\infty} m^{2-\alpha_2}\beta_m = 0$. The other relations can be proved analogously. $\square$

Theorem 3.2. *If $f \in E_w(I)$ is a two times differentiable function at $x \in I \cap J$, with $f^{(2)}$ continuous at x and $\frac{(T_4L_m)(x)}{m^{\alpha_4}}$ is bounded for any $m \in \mathbb{N}$, $m \ge m(2)$, then*

$$\begin{aligned} &\lim_{m\to\infty} m^{2-\alpha_2} \Bigg[(K_m f)(x) - (T_0 L_m)(x) f(x) - \frac{1}{m}(T_1 L_m)(x) f^{(1)}(x) - \\ &- \frac{1}{2m^2}(T_2 L_m)(x) f^{(2)}(x) \Bigg] = 0. \end{aligned} \tag{3.23}$$

Proof. From Theorem 2.1, we have

$$\begin{aligned} &\lim_{m\to\infty} m^{2-\alpha_2} \Bigg[(K_m f)(x) - (T_0 K_m)(x) f(x) - \frac{1}{m}(T_1 K_m)(x) f^{(1)}(x) - \\ &- \frac{1}{2m^2}(T_2 K_m)(x) f^{(2)}(x) \Bigg] = 0. \end{aligned}$$

But

$$\begin{aligned}
(T_0K_m)(x) &= (T_0L_m)(x),\\
(T_1K_m)(x) &= (T_1L_m)(x) - m\sum_{k=0}^{p_m}\varphi_{m,k}(x)\beta_{m,k},\\
(T_2K_m)(x) &= (T_2L_m)(x) - 2m^2\sum_{k=0}^{p_m}\varphi_{m,k}(x)x_{m,k}\beta_{m,k}+\\
&\quad + m^2\sum_{k=0}^{p_m}\varphi_{m,k}(x)\beta_{m,k}^2 + 2m^2x\sum_{k=0}^{p_m}\varphi_{m,k}(x)\beta_{m,k}
\end{aligned}$$

and taking Lemma 3.4 into account, the relation (3.23) results. □

Remark 3.5. The relation (3.23) is a Voronovskaja-type theorem.

In the following, in every application, we have $\sum\limits_{k=0}^{p_m}\varphi_{m,k}(x) = 1$, so $(T_0L_m)(x) = 1$ for any $x \in J$ and $m \in \mathbb{N}$ and $u_m(K) = 0$ for any $K \subset I \cap J$ and $m \in \mathbb{N}$.

We consider the applications from [23]. In the following, by particularization of the sequence $y_{m,k}$, $m \in \mathbb{N}$, $k \in \{0, 1, \dots, p_m\} \cap \mathbb{N}_0$ and applying Corollary 3.1, Theorem 3.1 and Theorem 3.2 from this paper we can obtain convergence theorem, approximation theorems and Voronovskaja-type theorems for the new operators. Because every application is a simple substitute in the theorems of this section, we won't replace anything. In the Applications 3.1, 3.2, 3.5, 3.6 and 3.7, we take $w(x) = 1$, $x \in I$. In the Applications 3.3 and 3.4, we take $w(x) = \frac{1}{1+x^2}$, $x \in I$.

Application 3.1. If $I = J = [0, 1]$, $E(I) = F(J) = C([0, 1])$, $x_{m,k} = \frac{k}{m}$, $m \in \mathbb{N}$, $k \in \{0, 1, \dots, m\}$, we get the Bernstein operators. We have $u_m([0, 1]) = 0$, $v_m([0, 1]) = 0$ and $w_m([0, 1]) = \frac{1}{4m}$, $m \in \mathbb{N}$. We consider the nodes $y_{m,k} = \frac{\sqrt{k(k+1)}}{m}$, $m \in \mathbb{N}$, $k \in \{0, 1, \dots, m\}$. Then on verify immediately that $\beta_m = \frac{1}{m+\sqrt{m(m+1)}}$, $m \in \mathbb{N}$ and $\lim\limits_{m\to\infty}\beta_m = 0$. In this case, the operators $(K_m)_{m\geq 1}$ have the form

$$(K_mf)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{\sqrt{k(k+1)}}{m}\right),$$

$f \in C([0, 1])$, $x \in [0, 1]$, $m \in \mathbb{N}$ and we get $(T_1B_m)(x) = 0$, $(T_2B_m)(x) = mx(1-x)$, $(T_4B_m)(x) = (3m^2-6m)x^2(1-x)^2+mx(1-x)$, $k_0(K) = k_0'(K) = 1$, $k_2(K) = \frac{5}{4}$, $k_4(K) = \frac{19}{16}$, $k_2'(K) = \frac{11+2\sqrt{2}}{4}$.

Application 3.2. We study a particular case of the Stancu operators. Let $\alpha = 10$ and $\beta = -\frac{1}{2}$. We obtain $I = [0, 22]$, $K = [0, 1]$ and for any $f \in C([0, 22])$, $x \in [0, 1]$ and $m \in \mathbb{N}$

$$(P_m^{(10,-1/2)} f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{2k+20}{2m-1}\right).$$

We consider the nodes $y_{m,k} = \frac{(4k+40)m}{(2m-1)^2}$. In this case, the operators $(K_m)_{m\geq 1}$ have the form

$$(K_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{m(4k+40)}{(2m-1)^2}\right),$$

where $f \in C([0, 22])$, $x \in [0, 1]$, $m \in \mathbb{N}$. We get $(T_1 P_m^{(10,-1/2)})(x) = \frac{m(20+x)}{2m-1}$, $(T_2 P_m^{(10,-1/2)})(x) = m^2 \cdot \frac{4mx(1-x)+(20+x)^2}{(2m-1)^2}$, $(T_4 P_m^{(10,-1/2)})(x) = \frac{m^4}{(2m-1)^4}[48m^2x^2(1-x)^2 + 16mx(1-x) - 96mx^2(1-x)^2 + 32(20+x)mx(1-x) + 24(20+x)^2 mx(1-x) + (20+x)^4]$, $k_0(K) = 1$, $k_0'(K) = 1$, $\alpha_2 = 1$, $\alpha_4 = 2$; because $\lim\limits_{m\to\infty} \dfrac{(T_2 P_m^{(10,-1/2)})(x)}{m} = x(1-x)$ and $x(1-x) \leq \frac{1}{4}$ for any $x \in [0, 1]$, it follows that $k_2(K) = \frac{5}{4}$ and similarly $k_4(K) = \frac{19}{16}$. Further, we have $k_2'(K) = 100$, taking into account that $u_m(K) = 0$ and $v_m(K) = \frac{42}{2m-1}$.

Application 3.3. If $I = J = [0, \infty)$, $E(I) = C_2([0, \infty))$, $F(J) = C([0, \infty))$, $K = [0, b]$, $b > 0$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = e^{-mx}\frac{(mx)^k}{k!}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, we obtain the Mirakjan-Favard-Szász operators and we have $u_m(K) = 0$, $v_m(K) = 0$ and $w_m(K) = \frac{b}{m}$, $m \in \mathbb{N}$. We consider the nodes $y_{m,k} = \frac{2k(k+1)}{m(2k+1)}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and we have $\beta_m = \frac{1}{2m}$, $m \in \mathbb{N}$. In this case, the operators $(K_m)_{m\geq 1}$ are

$$(K_m f)(x) = e^{-mx} \sum_{k=o}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{2k(k+1)}{m(2k+1)}\right),$$

where $f \in C_2([0, \infty))$, $x \in [0, \infty)$, $m \in \mathbb{N}$. We get $(T_1 S_m)(x) = 0$, $(T_2 S_m)(x) = mx$, $(T_4 S_m)(x) = 3m^2x^2 + mx$, $k_0(K) = 1 = k_0'(K)$, $k_2(K) = b$, $k_4(K) = 3b^2 + b$, $k_2'(K) = 2b + \frac{1}{4}$ (see [18]).

Application 3.4. Let $I = J = [0, \infty)$, $E(I) = C_2([0, \infty))$, $F(J) = C([0, \infty))$, $K = [0, b]$, $b > 0$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = (1+x)^{-m}\binom{m+k-1}{k}\left(\frac{x}{1+x}\right)^k$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$. In this case we get the Baskakov operators and we have

$u_m(K)=0$, $v_m(K)=0$ and $w_m(K)=\frac{b(1+b)}{2m}$, $m\in\mathbb{N}$. We consider the nodes $y_{m,k}=\frac{\sqrt{4k^2+4k+2}}{2m}$, $m\in\mathbb{N}$, $k\in\mathbb{N}_0$ and we have $\beta_m=\frac{1}{m\sqrt{2}}$. The operators $(K_m)_{m\geq 1}$ have the form

$$(K_mf)(x)=(1+x)^{-m}\sum_{k=0}^{\infty}\binom{m+k-1}{k}\left(\frac{x}{1+x}\right)^k f\left(\frac{\sqrt{4k^2+4k+2}}{2m}\right),$$

where $f\in C_2([0,\infty))$, $x\in[0,\infty)$, $m\in\mathbb{N}$. We get (see [18]) $(T_1V_m)(x)=0$, $(T_2V_m)(x)=mx(1+x)$, $(T_4V_m)(x)=3m(m+2)x^4+6m(m+2)x^3+m(3m+7)x^2+mx$, $k_2(K)=b(1+b)$, $k_4(K)=9b^4+18b^3+10b^2+b$ and $k_2'(K)=b^2+2(1+2\sqrt{2})+\frac{1}{2}$.

Application 3.5. If $I=J=[0,\infty)$, $E(I)=F(J)=C([0,\infty))$, $K=[0,b]$, $b>0$, $p_m=\infty$, $x_{m,k}=\frac{k}{m}$, $\varphi_{m,k}(x)=\frac{m(m+k)^{k-1}}{k!}\left(\frac{x}{1+x}\right)^k e^{-\frac{(k+m)x}{1+x}}$, $m\in\mathbb{N}$, $k\in\mathbb{N}_0$, we get the Ismail-May operators and we have $u_m(K)=0$, $v_m(K)=0$ and $w_m(K)=\frac{b(1+b)^2}{m}$, $m\in\mathbb{N}$. We consider the nodes $y_{m,k}=\frac{\sqrt[3]{k^2(k+1)}}{m}$, $m\in\mathbb{N}$, $k\in\mathbb{N}_0$ and we have $\beta_m=\frac{1}{3m}$. In this case, the operators $(K_m)_{m\geq 1}$ are

$$(K_mf)(x)=e^{\frac{-mx}{1+x}}\sum_{k=0}^{\infty}\frac{m(m+k)^{k-1}}{k!}\left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{\sqrt[3]{k^2(k+1)}}{m}\right),$$

where $f\in C([0,\infty))$, $m\in\mathbb{N}$. We obtain $(T_1R_m)(x)=0$, $(T_2R_m)(x)=mx(1+x)^2$, $(T_4R_m)(x)=3m^2x^2(1+x)^4+m(6x+4)x^2(1+x)^4+mx(1+x)^4(1+3x)^2$, $k_2(K)=1+b(1+b)^2$, $k_4(K)=1+b^2(1+b)^4$ and $k_2'(K)=b^3+2b^2+\frac{7}{3}b+\frac{10}{9}$ (see [20]).

For the Bleimann-Butzer-Hahn operators and for the Meyer-König and Zeller operators we only give the convergence and approximation theorems.

Application 3.6. We consider $I=J=[0,\infty)$, $E(I)=F(J)=C_B([0,\infty))$, $K=[0,b]$, $b>0$, $p_m=m$, $x_{m,k}=\frac{k}{m+1-k}$, $\varphi_{m,k}(x)=\frac{1}{(1+x)^m}\binom{m}{k}x^k$, $m\in\mathbb{N}$, $k\in\{0,1,\dots,m\}$. In this case we get the Bleimann-Butzer-Hahn operators and we have $u_m(K)=0$, $v_m(K)=b\left(\frac{b}{1+b}\right)^m$ and $w_m(K)=\frac{4b(1+b)^2}{m+2}$, $m\in\mathbb{N}$ (see [19]). We consider the nodes $y_{m,k}=\frac{\gamma_m k}{m+1-k}$, $m\in\mathbb{N}$, $k\in\{0,1,\dots,m\}$, where $(\gamma_m)_{m\geq 1}$ is a sequence of real numbers with the property that $\lim\limits_{m\to\infty}m(1-\gamma_m)=0$ and we have $\beta_m=m|1-\gamma_m|$, $m\in\mathbb{N}$. The operators $(K_m)_{m\geq 1}$ have the form

$$(K_mf)(x)=(1+x)^{-m}\sum_{k=0}^{\infty}\binom{m}{k}x^k f\left(\frac{\gamma_m k}{m+1-k}\right),$$

where $x \in [0,\infty)$, $m \in \mathbb{N}$, $f \in C_B([0,\infty))$. We obtain $(T_0 L_m)(x) = 1$, $(T_1 L_m)(x) = -mx\left(\frac{x}{1+x}\right)^m$, $k_2(K) = 4b(1+b)^2$, for $m \geq 24(1+b)$ and for $\beta_m = 1 - \frac{1}{m^2}$, $m \in \mathbb{N}$, we obtain $k_2'(K) = 4b(1+b)^2 + \frac{(1+2b)(1+3b)}{1+b}$.

Application 3.7. If $I = J = [0,1]$, $E(I) = B([0,1])$, $E(J) = C([0,1])$, $K = [0,1]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m+k}$, $(\varphi_{m,k})(x) = \binom{m+k}{k}(1-x)^{m+1}x^k$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, we get the Meyer-König and Zeller operators and we have $u_m([0,1]) = 0$, $v_m([0,1]) = 0$ and $w_m([0,1]) = \frac{1}{4(m+1)}$, $m \in \mathbb{N}$. We consider the nodes $y_{m,k} = \frac{k+\gamma_m}{m+k+\gamma_m}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, where $(\gamma_m)_{m\geq 1}$ is a sequence of real numbers such that

$$\lim_{m\to\infty} \frac{\gamma_m}{m+\gamma_m} = 0.$$

Then on verify immediately that $\beta_m = \frac{\gamma_m}{m+\gamma_m}$, $m \in \mathbb{N}$ and the operator $(K_m)_{m\geq 1}$ have the form

$$(K_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k}(1-x)^{m+1}x^k f\left(\frac{k+\gamma_m}{m+k+\gamma_m}\right),$$

where $f \in B([0,1])$, $x \in [0,1]$, $m \in \mathbb{N}$. For $\gamma_m = \frac{1}{m}$, we obtain $(T_0 Z_m)(x) = 1$, $k_0(K) = 1$, $k_2(K) = 2$, $(T_1 Z_m)(x) = 0$ (see [18]) and $k_2'(K) = \frac{13}{2}$.

References

[1] Agratini, O., *Approximation by linear operators*, Cluj University Press, Cluj-Napoca, 2000 (Romanian).

[2] Baskakov, V. A., *An example of a sequence of linear positive operators in the space of continuous functions*, Dokl. Acad. Nauk, SSSR, **113** (1957), 249-251.

[3] Becker, M., Nessel, R. J., *A global approximation theorem for Meyer-König and Zeller operators*, Math. Zeitschr., **160** (1978), 195-206.

[4] Bleimann, G., Butzer, P. L., Hahn, L. A., *Bernstein-type operator approximating continuous functions on the semi-axis*, Indag. Math., **42** (1980), 255-262.

[5] Bernstein, S. N., *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow **13**(2) (1912-1913), 1-2.

[6] Cheney, E. W., Sharma, A., *Bernstein power series*, Canadian J. Math., **16** (1964), 2, 241-252.

[7] Ditzian, Z., Totik, V., *Moduli of Smoothness*, Springer Verlag, Berlin, 1987.

[8] Favard, J., *Sur les multiplicateurs d'interpolation*, J. Math. Pures Appl., **23** (9) (1944), 219-247.

[9] Fărcaş, M. D., *An extension for the Bernstein-Stancu operators*, An Univ. Oradea, Fasc. Mat., Tom **15** (2008), 23-27.

[10] Ismail, M., May, C. P., *On a family of approximation operators*, J. Math. Anal. Appl., **63** (1978), 446-462.

[11] Meyer-König, W., Zeller, K., *Bernsteinsche Potenzreihen*, Studia Math., **19** (1960), 89-94.

[12] Mirakjan, G. M., *Approximation of continuous functions with the aid of polynomials*, Dokl. Acad. Nauk SSSR, **31** (1941), 201-205 (Russian).

[13] Mortici, C., Oancea, I., *A nonsmooth extension for the Bernstein-Stancu operators and an application*, Studia Univ. "Babeş-Bolyai", Mathematica, **51**(2) 2006, 69-81.

[14] Müller, M. W., *Die Folge der Gammaoperatoren*, Dissertation, Stuttgart, 1967.

[15] Pop, O. T., *New properties of the Bernstein-Stancu operators*, Anal. Univ. Oradea, Fasc. Matematica, **11** (2004), 51-60.

[16] Pop, O. T., *The generalization of Voronovskaja's theorem for a class of linear and positive operators*, Rev. Anal. Num. Théor. Approx., **34** (2005), No. 1, 79-91.

[17] Pop, O. T., *About the first order modulus of smoothness*, Carpathian J. Math., **20** (2004), No. 1, 101-108.

[18] Pop, O. T., *About some linear and positive operators defined by infinite sum*, Dem. Math., **39** (2006), No. 2, 377-388.

[19] Pop, O. T., *On operators of the type Bleimann, Butzer and Hahn*, Anal. Univ. Timişoara, **43** (2005), Fasc. 1, 115-124.

[20] Pop, O. T., *The generalization of Voronovskaja's theorem for exponential operators*, Creative Math & Inf., **16** (2007), 54-62.

[21] Pop, O. T., *About a general property for a class of linear positive operators and applications*, Rev. Anal. Num. Théor. Approx., **34** (2005), No. 2, 175-180.

[22] Pop, O. T., *About some linear operators*, International Journal of Mathematics and Mathematical Sciences, vol. 2007, Article ID 91781, 13 pages, 2007.

[23] Pop, O. T., Fărcaş, M., *About a class of linear positive operators obtained by choosing the nodes*, J. Inequal. Pure Math. **10**(1) (2009), Art. 30, 9pp.

[24] Stancu, D. D., *About a generalization of Bernstein's polynomial*, Studia Univ. Babeş-Bolyai, Ser. Math.-Phys., **14** (1969), 31-45 (Romanian).

[25] Stancu, D. D., Coman, Gh., Agratini, O., Trîmbiţaş, R., *Numerical analysis and approximation theory*, **I**, Cluj University Press, Cluj-Napoca, 2001 (Romanian).

[26] Szász, O., *Generalization of. S. N. Bernstein's polynomials to the infinite interval*, J. Research, National Bureau of Standards, **45** (1950), 239-245.

[27] Timan, A. F., *Theory of Approximation of Functions of Real Variable*, New York: Macmillan Co. 1963, MR22#8257.

[28] Voronovskaja, E., *Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de S. N. Bernstein*, C. R. Acad. Sci. URSS (1932), 79-85.

National College "Mihai Eminescu"
5 Mihai Eminescu, 440014 Satu Mare, Romania
ovidiutiberiu@yahoo.com

National College "Mihai Eminescu"
5 Mihai Eminescu, 440014 Satu Mare, Romania
mirceafarcas2005@yahoo.com

North University of Baia Mare
Department of Mathematics and Computer Science
76 Victoriei, 430122 Baia Mare, Romania
barbosudan@yahoo.com