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CO PRIME PATH DECOMPOSITION NUMBER OF A GRAPH

K. NAGARAJAN and A. NAGARAJAN

Abstract

A *decomposition* of a graph G is a collection ψ of edge-disjoint subgraphs $H_1, H_2, \ldots, H_n$ of G such that every edge of G belongs to exactly one H_i. If each H_i is a path in G, then ψ is called a *path partition* or *path cover* or *path decomposition* of G. A *co prime path decomposition* of a (p, q)-graph G is a path cover ψ of G such that the length of all the paths in ψ are co prime with q. The minimum cardinality of a co prime path decomposition of G is called the *co prime path decomposition number* of G and is denoted by $\pi_\phi(G)$. In this paper, a study of the parameter π_ϕ is initiated and the value of π_ϕ for some standard graphs is determined. Further, bounds for π_ϕ are obtained and the graphs attaining the bounds are characterized.

1 Introduction

By a graph, it means that a finite, undirected, non-trivial, connected graph without loops and multiple edges. The order and size of a graph are denoted by p and q respectively. For terms not defined here Harary [5] is referred to.

Let $P = (v_1, v_2, \ldots, v_n)$ be a path in a graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$. The vertices $v_2, v_3, \ldots, v_{n-1}$ are called *internal vertices* of P and v_1 and v_n are called *external vertices* of P. The length of a path is denoted by $l(P)$. A cycle with exactly one chord is called a θ*-graph*. A *spider tree* is a tree in which it has a unique vertex of degree 3.

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An *odd tree* is a tree in which all the vertices have odd degree. For vertices x and y in a connected graph G, the *detour distance* $D(x,y)$ is the length of a longest x-y path in G. The *detour diameter* D of G is defined to be $D = max\{D(x,y) : x, y \in V(G)\}$.

The following number theoretic concepts and results [4, 7] will be useful in proving the theorems in this paper.

Let a and b be positive integers. If a *divides* b, it means that there is a positive integer k such that $b = ka$ and it is denoted by $a \mid b$. If a *does not divide* b, then it is denoted by $a \nmid b$. The *greatest common divisor(gcd)* of a and b is denoted
by (a,b). If $(a,b) = 1$, then it is said that a and b are *co prime* or *relatively prime*.

Result 1.1. *If $d \mid a$ and $d \mid b$, then $d \mid a \pm b$.*

Result 1.2. *1 is co prime with any positive integer.*

Result 1.3. *2 is co prime with any odd positive integer.*

Result 1.4. *Any two consecutive positive integers are co prime.*

Result 1.5. *Any two consecutive odd positive integers are co prime.*

Result 1.6. *Any prime number p is co prime with any positive integer a, if $p \nmid a$.*

Result 1.7. *If $(a,b) = 1$, then $(b - a, b) = 1$.*

A *decomposition* of a graph G is a collection of edge-disjoint subgraphs H_1, H_2,
$\ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. If each $H_i \cong H$, then we say that G has a H-*decomposition* and it is denoted by $H \mid G$. In this paper, this definition is extended to non-isomorphic decomposition. If each H_i is a path, then it is called a *path partition* or *path cover* or *path decomposition* of G. The minimum cardinality of a path partition of G is called the path partition number of G and is denoted by $\pi(G)$ and any path partition ψ of G for which $|\psi| = \pi(G)$ is called a *minimum path partition* or π-*cover* of G. The parameter π was studied by Harary and Schwenk [6], Peroche [12], Stanton *et.al.,* [13] and Arumugam and Suresh Suseela [2].

Various types of path decompositions and corresponding parameters have been studied by several authors by imposing conditions on the paths in the decomposition. Some such path decomposition parameters are acyclic graphoidal covering number [2], simple path covering number [1], 2-graphoidal path covering number [9] and m-graphoidal path covering number [10]. Another such decomposition is equiparity path decomposition (EQPPD) which was defined by K. Nagarajan, A. Nagarajan and I. Sahul Hamid [11].

Definition 1.8. *[11] An equiparity path decomposition(EQPPD) of a graph G is a path cover ψ of G such that the lengths of all the paths in ψ have the same parity.*

Since for any graph G, the edge set $E(G)$ is an equiparity path decomposition, the collection $\mathcal{P}_P$ of all equiparity path decompositions of G is non-empty. Let $\pi_P(G) = min\{|\Psi| : \Psi \in \mathcal{P}_P\}$. Then $\pi_P(G)$ is called the *equiparity path decomposition number* of G and any equiparity path decomposition ψ of G for which $|\psi| = \pi_P(G)$ is called a *minimum equiparity path decomposition* of G or π_P*-cover* of G. The parameter π_P was studied in [11].

If the lengths of all the paths in ψ are even(odd) then we say that ψ is an *even (odd) parity path decomposition*, shortly *EPPD (OPPD).*

Theorem 1.9. *[11] For any $n \geq 1$, $\pi_P(K_{2n}) = n$.*

Theorem 1.10. *[3] For any connected (p, q)-graph G, if q is even, then G has a P_3-decomposition.*

Theorem 1.11. *[8] If a graph G is neither a 3-cycle nor an odd tree, then G admits a $\{P_3, P_4\}$-decomposition that consists several copies of P_3 and exactly one*
copy of P_4.

Now, a new path called co prime path will be defined as follows.

Definition 1.12. *Let G be a (p, q)-graph and let P be a path in G. If $(l(P), q) = 1$, then P is called a co prime path in G.*

Note that the edges of a graph are co prime paths. The co prime path of length $l > 1$ is called *proper co prime path*, otherwise it is called *improper co prime path*.

Example 1.13. *Consider the following graph G.*

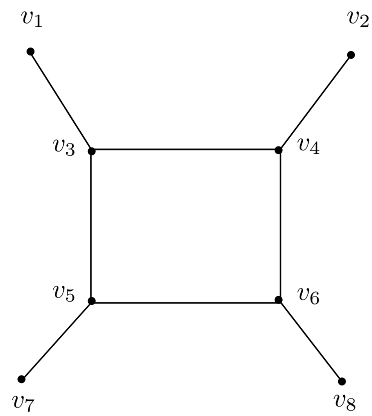

Fig 1.1

Here $q = 8$. The path $(v_7, v_5, v_3, v_4, v_6, v_8)$ is a co prime path, but the path $(v_7, v_5, v_3, v_4, v_6)$ is not a co prime path. Also the path (v_1, v_3) is an improper co prime path.

Next, a co prime path decomposition of a graph G is defined.

Definition 1.14. *A co prime path decomposition(CPPD) of a (p, q)-graph G is a path cover Ψ of G such that the lengths of all the paths in Ψ are co prime with q.*

Since the edge set $E(G)$ is a co prime path decomposition for any graph G, the collection $\mathcal{P}_\phi$ of all co prime path decompositions Ψ of G is non-empty. Let $\pi_\phi(G) = min\{|\Psi| : \Psi \in \mathcal{P}_\phi\}$. Then $\pi_\phi(G)$ is called the *co prime path decomposition number* of G. Any co prime path decomposition Ψ of G for which $|\Psi| = \pi_\phi(G)$ is called a *minimum co prime path decomposition* of G or π_ϕ*-cover* of G. Here the symbol ϕ is used as a subscript for π, because in Number Theory, $\phi(n)$ denotes the number of positive integers which are less than n and co prime with n.

Example 1.15. *Consider the following spider tree.*

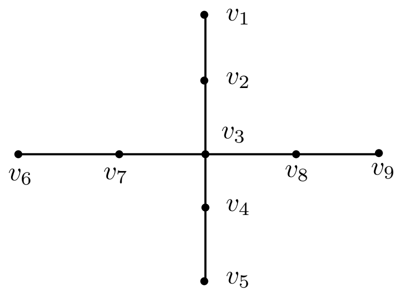

Fig 1.2

Here $q = 8$ *and* $\{(v_1, v_2, v_3, v_4), (v_4, v_5), (v_6, v_7, v_3, v_8), (v_8, v_9)\}$ *forms a* π_ϕ*-cover so that* $\pi_\phi(G) = 4$. *Note that* $\{(v_1, v_2, v_3, v_4, v_5), (v_6, v_7, v_3, v_8, v_9)\}$ *forms a* π*-cover so that* $\pi(G) = 2$.

Remark 1.16. *Let* $\Psi = \{P_1, P_2, \ldots, P_n\}$ *be a CPPD of a* (p, q)*-graph* G *such that* $l(P_1) \leq l(P_2) \leq \ldots \leq l(P_n)$. *Since every edge of* G *is exactly in one path* P_i, $\sum_{i=1}^{n} l(P_i) = q$. *Hence every CPPD of* G *gives rise to a partition of a positive integer* q *into the positive integers (not necessarily distinct) which are co prime with* q.

In this paper, a study of the parameter π_ϕ is initiated and the value of π_ϕ for some standard graphs is determined. Further, bounds for π_ϕ are obtained and the graphs attaining the bounds are characterized.

2 Main Results

Hereafter, it is considered that G as a graph, which is not a path. First, a general result which is useful in determining the value of π_ϕ is presented.

Theorem 2.1. *For any CPPD* Ψ *of a graph* G, *let* $t_\Psi = \sum_{P \in \Psi} t(P)$, *where* $t(P)$ *denotes the number of internal vertices of* P *and let* $t = \max t_\Psi$, *where the maximum is taken over all co prime path decompositions* Ψ *of* G. *Then* $\pi_\phi(G) = q - t$.

Proof. Let Ψ be any CPPD of G.

$$\begin{aligned} \text{Then } q &= \sum_{P \in \Psi} |E(P)| \\ &= \sum_{P \in \Psi} (t(P) + 1) \end{aligned}$$

$$= \sum_{P \in \Psi} t(P) + |\Psi|$$

$$= t_\Psi + |\Psi| .$$

Hence $|\Psi| = q - t_\Psi$ so that $\pi_\phi = q - t$. □

Next, some bounds for $\pi_\phi$will be found. First, a simple bound for π_ϕ in terms of the size of G is found.

Theorem 2.2. *For any graph G of odd size, $\pi_\phi(G) \leq \frac{q+1}{2}$.*

Proof. If there exists an edge e in G which is not a bridge, let $H = G - e$. If not, G is a tree and in this case, let $H = G - v$, where v is a pendant vertex and let e be the edge incident at v.

Now, in either of the cases, H is connected with even number of edges and hence by Theorem 1.10, H has a P_3-decomposition, say Ψ. Hence $\Psi \cup \{e\}$ is a co prime path cover of G so that $\pi_\phi(G) \leq |\Psi| = \frac{q-1}{2} + 1 = \frac{q+1}{2}$. □

Remark 2.3. *The bound given in Theorem 2.2 is sharp. For the star graph $K_{1,q}$, where q is odd, $\pi_\phi = \frac{q+1}{2}$.*

The following problem naturally arises.

Problem 2.4. *Characterize the graphs of an odd size for which $\pi_\phi = \frac{q+1}{2}$.*

Next, upper bound for the parameter π_ϕ will be found.

Theorem 2.5. *If a (p, q)-graph G is neither a 3-cycle nor an odd tree, with q odd and $q \not\equiv 0(mod3)$, then $\pi_\phi(G) \leq \frac{q-1}{2}$.*

Proof. From Theorem 1.11, it is clear that G admits a $\{P_3, P_4\}$-decomposition that consists of $\frac{q-3}{2}$ copies of P_3 and exactly one copy of P_4. Since q is odd and $q \not\equiv 0(mod3)$, then by the definition of CPPD, it follows that $\pi_\phi(G) \leq \frac{q-3}{2} + 1 = \frac{q-1}{2}$. □

Remark 2.6. *The bound in Theorem 2.5 is sharp. For example, consider the following spider tree.*

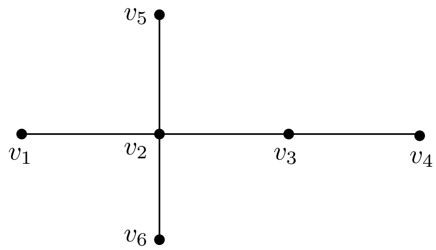

Fig 2.1

Here $q = 5 \not\equiv 0(mod3)$ *and* $\{(v_1, v_2, v_3, v_4), (v_5, v_2, v_6)\}$ *forms a* π_ϕ*-cover so that* $\pi_\phi(G) = 2 = \frac{q-1}{2}$.

Then, a following problem can be made.

Problem 2.7. *Characterize the graphs of an odd size with* $q \not\equiv 0(mod3)$ *for which* $\pi_\phi = \frac{q-1}{2}$.

Next, the value of π_ϕ for the paths is found.

Theorem 2.8. *For a path* P_p $(p \geq 3)$, $\pi_\phi(P_p) = 2$.

Proof. Let $P_p = (v_1, v_2, \ldots, v_p)$. Then clearly $\Psi = \{(v_1, v_2, \ldots, v_{p-1}), (v_{p-1}, v_p)\}$ is a CPPD of P_p so that $\pi_\phi(P_p) \leq 2$. Since $(l(P_p), q) = q > 1$, the path itself is not a CPPD and clearly $\pi_\phi(P_p) \geq 2$. Thus, $\pi_\phi(P_p) = 2$. □

Now, the graphs attaining the extreme bounds are characterized.

Theorem 2.9. *For a graph* G, $1 \leq \pi_\phi(G) \leq q$. *Then* $\pi_\phi(G) = 1$ *if and only if* $G \cong K_2$ *and* $\pi_\phi(G) = q > 1$ *if and only if* q *is even and* G *has no proper co prime paths.*

Proof. The inequalities are trivial. Now, suppose $\pi_\phi(G) = 1$. Assume that $G \not\cong K_2$. If G is a path of length ≥ 2, then by Theorem 2.8, a contradiction is obtained. If G is not a path, then any path decomposition of G contains at least two paths so that $\pi_\phi(G) \geq 2$, which is a contradiction. Thus, $G \cong K_2$. Converse is obvious.

Now, suppose that $\pi_\phi(G) = q > 1$. Then it follows from Theorem 2.2 that q is even. Then G has no proper co prime path of length 2. Suppose G has a proper co prime path P such that $l(P) \geq 3$. Then the path P together with the remaining edges form a CPPD Ψ of G so that $\pi_\phi(G) \leq |\Psi| = q - l(P) + 1 < q$, which is a contradiction. Thus, G has no proper co prime paths. Converse is obvious. □

Remark 2.10. *From Theorem 2.9 and Theorem 2.8, it is observed that any CPPD of a graph* $G \ncong K_2$ *contains at least two paths and hence* $\pi_\phi(G) \geq 2$.

Theorem 2.11. *For any* (p, q)*-graph* G *with* $q \geq 3$, $\pi_\phi(G) = q - 1$ *if and only if* G *is isomorphic to either* P_4 *or* K_3 *or* $K_{1,3}$.

Proof. Suppose $\pi_\phi(G) = q - 1$. If G is a path of length ≥ 3, then from Theorem 2.8 and by hypothesis, it follows that $q = 3$ and hence $G \cong P_4$. Now, let G be a graph which is not a path. If G has a co prime path P with $l(P) \geq 3$, then the path P together with the remaining edges form a CPPD Ψ of G so that $\pi_\phi(G) \leq |\Psi| = 1 + (q - l(P)) < q - 1$, which is a contradiction. Thus, every co prime path in G is of length 1 or 2. Hence any two edges in G are adjacent, so that G is either a triangle K_3 or a star $K_{1,q}$. If G has co prime paths of length 1 only, then $\pi_\phi(G) = q$, which is a contradiction. So G has at least one co prime path of length 2. Then q is odd and $\pi_\phi(G) \geq \frac{q+1}{2}$. From Theorem 2.2 , it is clear that $\pi_\phi(G) = \frac{q+1}{2}$. By hypothesis, $q = 3$ and hence G is isomorphic to either K_3 or $K_{1,3}$. Converse is obvious. □

Next, the value of π_ϕ for the cycles is found.

Theorem 2.12. *For a cycle* C_p, $\pi_\phi(C_p) = 2$.

Proof. Let $C_p = (v_1, v_2, \ldots, v_p, v_1)$. Then clearly $\Psi = \{(v_1, v_2, \ldots, v_p), (v_p, v_1)\}$ is a CPPD of C_p so that $\pi_\phi(C_p) \leq 2$. Since any path decomposition of C_p contains at least two paths, $\pi_\phi(C_p) \geq 2$. Thus, $\pi_\phi(C_p) = 2$. □

The following observation gives the lower bound for π_ϕ in terms of π.

Observation 2.13. *Since every co prime path decomposition of a graph* G *is a path cover,* $\pi(G) \leq \pi_\phi(G)$.

Remark 2.14. *The inequality in Observation 2.13 is strict for a path of length* ≥ 2 *in which* $\pi = 1 \neq 2 = \pi_\phi$

The equality holds in Observation 2.13 if q is prime, which will be proved in the following theorem.

Theorem 2.15. *For any* (p, q)*-graph* G *which is not a path,* $\pi(G) = \pi_\phi(G)$ *if* q *is prime.*

Proof. Since q is prime, the lengths of all the paths in any path cover of G are co prime with q. Thus, any path cover is a CPPD and hence $\pi_\phi(G) \leq \pi(G)$. From Observation 2.13, it is clear that $\pi(G) = \pi_\phi(G)$. □

Remark 2.16. *The converse of the Theorem 2.15 is not true. For the cycle of composite size, it is seen that* $\pi = 2 = \pi_\phi$.

Next, the value of π_ϕ for any graph of 6 edges is found.

Theorem 2.17. *Let G be a graph with 6 edges. Then*

$$\pi_\phi(G) = \begin{cases} 2, & \text{if } D(G) \geqslant 5 \\ 6, & \text{otherwise} \end{cases}$$

where D is the detour diameter of G.

Proof. Suppose $D(G) \geq 5$. If G is isomorphic to a path of length 6, then from Theorem 2.8, it follows that $\pi_\phi(G) = 2$. Let P be a path of length 5 in G. Then, clearly the path P and the remaining one edge form a CPPD of G so that $\pi_\phi(G) \leq 2$. By Remark 2.10, it follows that $\pi_\phi(G) = 2$. If $D(G) \leq 4$, then $l(P) \leq 4$ for all paths P in G. Since $q = 6$, it is seen that any path of length greater than 1 is not a co prime path in G. From Theorem 2.9, it is clear that $\pi_\phi(G) = 6$. □

The next theorem shows that the relationship between $\pi_P(G)$ and $\pi_\phi(G)$, if q is even.

Theorem 2.18. *For any (p, q)-graph G, $\pi_P(G) \leq \pi_\phi(G)$ if q is even.*

Proof. Since q is even, the positive integers which are co prime to q are odd. If Ψ is any CPPD of G, then the lengths of all the paths in Ψ are odd. Hence Ψ is an OPPD of G. Thus, every CPPD is an OPPD so that $\pi_P(G) \leq \pi_\phi(G)$. □

Remark 2.19. *The inequality in Theorem 2.18 is strict. Consider the complete graph K_4 in which $q = 6$. From Theorem 1.9 and Theorem 2.17, it is clear that $\pi_P(K_4) = 2 < 6 = \pi_\phi(K_4)$. Further, the bound in Theorem 2.18 is sharp. For the even cycle, $\pi_P = 2 = \pi_\phi$.*

Remark 2.20. *The converse of the Theorem 2.18 is not true. Consider the following spider tree G.*

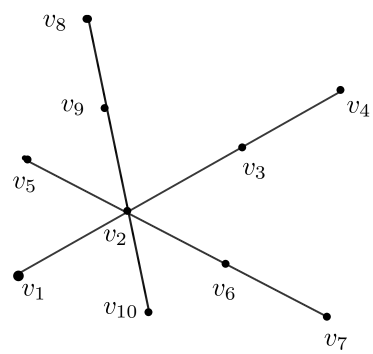

Fig 2.2

Here $\{(v_1, v_2, v_3, v_4), (v_5, v_2, v_6, v_7), (v_8, v_9, v_2, v_{10})\}$ *forms a* π_P*-cover and* $\{(v_8, v_9, v_2, v_3, v_4), (v_2, v_6, v_7), (v_1, v_2, v_5), (v_2, v_{10})\}$ *forms a* π_ϕ*-cover so that* $\pi_P(G) = 3 < 4 = \pi_\phi(G)$*, but* $q = 9$ *which is odd.*

Now the following problem naturally arises.

Problem 2.21. *Characterize the graphs of even size for which* $\pi_P(G) = \pi_\phi(G)$.

Next, the value of π_ϕ for the stars will be found.

Theorem 2.22. *For a star* $K_{1,n}$, $\pi_\phi(K_{1,n}) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd} \end{cases}$.

Proof. Let $V(K_{1,n}) = V_1 \cup V_2$, where $V_1 = \{x_1\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$.
Case(i): n is odd.

Let $\Psi = \bigcup_{i=1}^{i=\frac{n-1}{2}} \{(y_{2i-1}, x_1, y_{2i})\} \bigcup \{(x_1, y_n)\}$. Then Ψ contains paths of lengths 1 or 2, which are co prime paths and hence $\pi_\phi(K_{1,n}) \leq |\Psi| = \frac{n+1}{2}$. Further, since every vertex of $K_{1,n}$ is of odd degree, they are the end vertices of paths in any path cover of $K_{1,n}$. Thus, $\pi_\phi(K_{1,n}) \geq \frac{n+1}{2}$ and hence $\pi_\phi(K_{1,n}) = \frac{n+1}{2}$.
Case(ii): n is even.

It is observed that every path in $K_{1,n}$ is of length 1 or 2. Since n is even, all the paths of length greater than 1 are not co prime paths. Hence $\pi_\phi(K_{1,n}) = n$. □

The following theorem is useful in proving Theorem 2.25.

Theorem 2.23. *For a* θ*-graph* G, $\pi(G) = 2$.

Proof. Let $(v_1, v_2, \ldots, v_k, \ldots v_p, v_1)$ be a cycle in G and let v_1v_k be a chord of the cycle. Let $P = (v_1, v_2, \ldots, v_k, \ldots, v_p)$ and $Q = (v_p, v_1, v_k)$. Then $\{P, Q\}$ forms a path cover of G so that $\pi(G) \leq 2$. Clearly, G is not a path, any path cover of G contains at least two members so that $\pi(G) \geq 2$. Thus, $\pi(G) = 2$. □

The next lemma will be useful to find the value π_ϕ for θ-graph.

Lemma 2.24. *For an even integer $q \geq 4$, $(\frac{q}{2} - 1, q) = 1$ if $\frac{q}{2}$ is even and $(\frac{q}{2} - 2, q) = 1$ if $\frac{q}{2}$ is odd.*

Proof. **Case(i):** $\frac{q}{2}$ is even.

Suppose $(\frac{q}{2}-1, q) = d$. Then $d \mid q$ and $d \mid \frac{q}{2}-1$. This implies that $d \mid q-(\frac{q}{2}-1)$. That is, $d \mid \frac{q}{2}+1$. Then $d \mid (\frac{q}{2}+1)-(\frac{q}{2}-1)$. Thus, $d \mid 2$. This implies $d = 1$ or 2. Since q is even and $\frac{q}{2} - 1$ is odd, $d = 1$.

Case(ii): $\frac{q}{2}$ is odd.

Suppose $(\frac{q}{2} - 2, q) = d$. Then $d \mid q$ and $d \mid \frac{q}{2} - 2$. This implies that $d \mid q - (\frac{q}{2} - 2)$. That is $d \mid \frac{q}{2} + 2$. Then $d \mid (\frac{q}{2} + 2) - (\frac{q}{2} - 2)$. Thus, $d \mid 4$. Since q is even and $\frac{q}{2} - 2$ is odd, $d = 1$ or 3. Suppose $d = 3$. Then $3 \mid q$ and $3 \mid \frac{q}{2} - 2$. Since q is even, $3 \mid q \Rightarrow 3 \mid \frac{q}{2} \Rightarrow 3 \nmid \frac{q}{2} - 2$, which is a contradiction. Hence $d = 1$. □

Theorem 2.25. *For a $\theta-$graph with G, $\pi_\phi(G) = \begin{cases} 6, & \text{if } q = 6 \\ 2, & \text{otherwise} \end{cases}$.*

Proof. Note that $q = p + 1$. If $p = 4$, then $q = 5$ which is a prime. From Theorem 2.15 and Theorem 2.23, it is clear that $\pi_\phi(G) = 2$. If $p = 5$, then $q = 6$ and $D(G) = 4$. From Theorem 2.17, it is clear that $\pi_\phi(G) = 6$. Suppose $p \geq 6$. Let $C = (v_1, v_2, \ldots, v_k, \ldots v_p, v_1)$ be a cycle in G and let v_1v_k be the chord of it (Fig 2.3). If q is prime, then from Theorem 2.15 and Theorem 2.23, it follows that $\pi_\phi(G) = 2$. If q is composite, then there are the following two cases.

Case (i): q is odd.

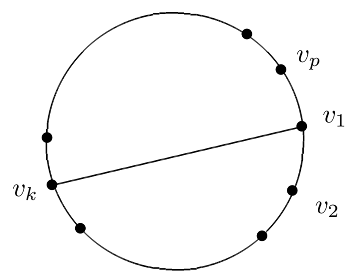

Fig 2.3

Let $P = (v_1, v_2, \dots, v_k, \dots v_p)$ and $Q = (v_p, v_1, v_k)$ (Fig 2.3). Then $l(P) = q - 2$ and $l(Q) = 2$. From the Result 1.5 and the Result 1.3, it is clear that
$(l(P), q) = (l(Q), q) = 1$. Then $\{P, Q\}$ forms a CPPD of G so that $\pi_\phi(G) \le 2$. By Remark 2.10, it follows that $\pi_\phi(G) = 2$.
Case (ii): q is even.

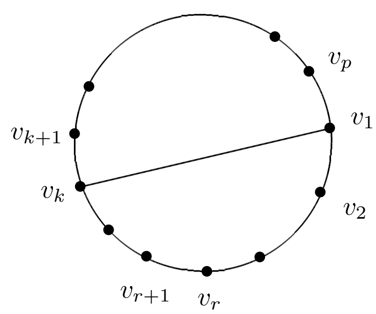

Fig 2.4

Note that $l(C) = q-1$. Let $P = (v_1, v_2, \dots, v_k)$ and $Q = (v_k, v_{k+1}, \dots, v_p, v_1)$ be the (v_1, v_k)-sections of C. (Fig 2.4). Since $l(C)$ is odd, either $l(P) > l(Q)$ or $l(Q) > l(P)$. Without loss of generality, it is assumed that $l(P) > l(Q)$. Also since $l(C) = q - 1$, $l(P) > \frac{q}{2}$. Then there is a vertex v_r in P such that the length of the (v_1, v_r)-section of P is $\frac{q}{2} - 2$. Now, consider the path $R = (v_k, v_1, v_2, \dots, v_r)$ which is of length $\frac{q}{2} - 1$. From the Lemma 2.24, it follows that $(l(R), q) = 1$. Now, let $S = (v_r, v_{r+1}, \dots, v_k, \dots, v_p, v_1)$. Note that $l(S) = q - l(R)$. By the Result 1.7, it is seen that $(l(S), q) = 1$. Thus, $\{R, S\}$ forms a CPPD of G so that $\pi_\phi(G) \le 2$. By Remark 2.10, it follows that $\pi_\phi(G) = 2$. □

The case (ii) of the Theorem 2.25 is illustrated in the following example.

Example 2.26. *Consider the θ-graph G of size $q = 16$. Let $C = (v_1, v_2, \dots, v_{10}, \dots v_{15}, v_1)$ be a cycle in G and let $v_1 v_{10}$ be the chord of it (Fig 2.5).*

Note that $l(C) = 15 = q - 1$. Let $P = (v_1, v_2, \dots, v_{10})$ and $Q = (v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_1)$ be the (v_1, v_{10})-sections of C (Fig 2.5). It is seen that $l(P) > l(Q)$ and $l(P) = 9 > \frac{q}{2}$. Then there is a vertex v_r in P such that the length of the (v_1, v_r)-section of P is $\frac{q}{2} - 2 = 6$ and so $r = 7$. Now, consider the path $R = (v_{10}, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ which is of length $7 = \frac{q}{2} - 1$. Then $(7, 16) = (l(R), q) = 1$. Let $S = (v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_1)$ and note that $l(S) = 16 - 7 = 9 = q - l(R)$. Then $(9, 16) = (l(S), q) = 1$. Thus, $\{R, S\}$ forms a CPPD of G so that $\pi_\phi(G) = 2$.

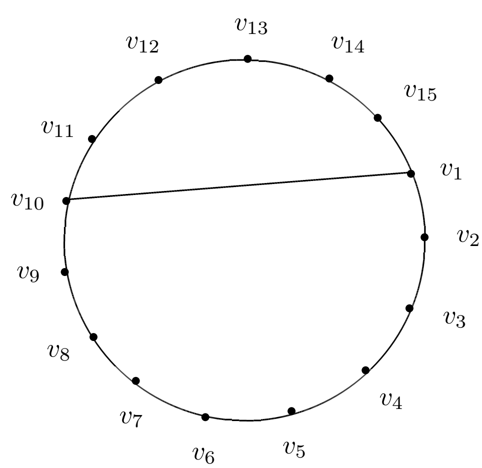

Fig 2.5

References

[1] S. Arumugam and I. Sahul Hamid, *Simple path covers in graphs*, Int. J. of Math. Combinatorics, **3** (2008), 94–104.

[2] S. Arumugam and J. Suresh Suseela, *Acyclic graphoidal covers and path partitions in a graph*, Discrete Math., **190**(1998), 67–77.

[3] G. Chartrand and L. Lesniak, *Graphs and Digraphs,* Fourth Edition, CRC Press, Boca Raton, 2004.

[4] David M. Burton, *Elementary Number Theory*, Universal Book Stall, 1990.

[5] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, 1972.

[6] F. Harary and A. J. Schwenk, *Evolution of the path number of a graph, covering and packing in graphs II*, Graph Theory and Computing, Ed. R. C. Road, Academic Press, New York, (1972), pp. 39–45.

[7] Ivan Niven and Herbert S.Zuckerman, *An Introduction to the Theory of Numbers,* Wiley Eastern Limited, 1960.

[8] Mingqing Zhai and Changhong Lii, *Path decomposition of graphs with given path length*, Acta Math. Appl. Sinica, English Series, **22**(4) (2006), 633–638.

[9] K. Nagarajan, A. Nagarajan and S. Somasundram, 2-*graphoidal Path Covers*, Int. J. Appl. Math., **21**(4) (2008), 615–628.

[10] K. Nagarajan, A. Nagarajan and S. Somasundram, m-*graphoidal Path Covers of a Graph*, Proceedings of the Fifth International Conference on Number Theory and Smarandache Notions, (2009), 58–67.

[11] K. Nagarajan, A. Nagarajan and I. Sahul Hamid, *Equiparity Path Decomposition Number of a Graph*, Int. J. Math. Combinatorics, **1** (2009), 61–76.

[12] B. Peroche, *The path number of some multipartite graphs*, Annals of Discrete Math., **9** (1982), 193–197.

[13] R. G. Stanton, D. D. Cowan and L. O. James, *Some results on path numbers*, Proc. Louisiana Conf. on Combinatorics, Graph Theory and computing, (1970), 112–135.

K. Nagarajan
Department of Mathematics, Sri S.R.N.M.College
Sattur - 626 203, Tamil Nadu, India
k_nagarajan_srnmc@yahoo.co.in

A. Nagarajan
Department of Mathematics, V.O.C.College
Tuticorin - 628 008, Tamil Nadu, India
nagarajan.voc@gmail.com