An. Şt. Univ. Ovidius Constanţa Vol. 19(1), 2011, 5–22

HEXAGONAL 2-COMPLEXES HAVE A STRONGLY CONVEX METRIC

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Abstract

We give two distinct proofs for the fact that any finite simply connected hexagonal 2-complex has a strongly convex metric. In our first proof we show that these complexes are CAT(0) spaces, while the second proof makes use of the fact that finite, simply connected hexagonal 2-complexes are collapsible. Both proofs rely on the fact that hexagonal 2-complexes have the 12-property.

Introduction

We investigate in this paper, in two distinct manners, whether finite simply connected hexagonal 2-complexes have a strongly convex metric ([13], [16], [14], [15]). The main observation which permits this study is that any hexagonal 2-complex has the 12-property (see [8], [1]).

Our first proof relies on the following important fact. In dimension 2, the 12-property (6-property, 8-property) coincides with the CAT(0) property of the standard piecewise Euclidean metric on a simply connected hexagonal (simplicial, cubical) complex (see [5], chapter II.5, page 207). We will prove that hexagonal 2-complexes have the 12-property. Hence, since hexagonal 2-complexes are, according to their definition, endowed with the standard piecewise Euclidean metric, they are non-positively curved. We show further

Key Words: hexagonal 2-complex, 12-property, CAT(0) metric, collapsibility, concave collection, strongly convex metric.
Mathematics Subject Classification: 05C99, 05C75
Received: November, 2009
Accepted: December, 2010
The first author is partially supported by the King Saud University D.S.F.P.Program

that the curvature at the exterior vertices of such spaces is bounded above by a strictly negative real number. Our proof of this is similar to the one given by I.-C. Lazăr for the fact that any simplicial 2-complex obtained by performing an elementary collapse on a CAT(0) simplicial 2-complex, remains a CAT(0) space (see [9], Proposition 2; [12], Proposition 3.1.3.). Simply connected hexagonal 2-complexes are therefore CAT(0) spaces (see [2], [5], [4], [6]) and hence strongly convex (see [5], chapter II.1, page 160). Similarly, simply connected simplicial 2-complexes with the 6-property also have a strongly convex metric, when endowed with the standard piecewise Euclidean metric.

Our second proof uses results proven in [10] on finite simply connected hexagon 2-complexes with the 12-property. Besides, it relies on the fact that collapsible hexagonal 2-complexes are strongly convex. The proof of this is one of the paper's goals. Similarly in [16] ([3]) it is proven that any collapsible simplicial 2-complex (cubical 2-complex) admits a strongly convex metric. Hence, since finite, simply connected, simplicial 2-complexes (square 2-complexes) with the 6-property (8-property), collapse to a point (see [7], [11]), one may conclude that finite, simply connected simplicial 2-complexes (cubical 2-complexes) with the 6-property (8-property) admit a strongly convex metric (see [3]). In this paper we obtain a similar result on finite, simply connected hexagonal 2-complexes. Namely, we prove that, due to the fact that simply connected hexagonal 2-complexes have the $12-$property and are therefore collapsible (see [10]), they have a strongly convex metric. We note that, although the 12-property on the more general hexagon 2-complexes also ensures their collapsibility, if their fundamental group vanishes (see [10]), a similar result does not hold on finite, simply connected hexagon 2-complexes with the 12-property. We emphasize that, although the intersection of any two 2-cells of a hexagonal 2-complex is either the empty set, or a single common face of the two intersecting cells, in a hexagon 2-complex such intersection may be a union of faces. The paper's main result is included in the second author's Ph.D. thesis (see [12]).

1 Preliminaries

We present in this section the notions we shall work with and the results we shall refer to.

Let (X, d) be a metric space. Given a path $\gamma : [a, b] \to X$ in X, its *length* is defined by

$$L(\gamma) = \sup\{\textstyle\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i))\},$$

where the supremum is taken over all possible subdivisions of $[a, b]$, $a = t_0 < t_1 < ... < t_n = b$. (X, d) is a *length space* if for any two points x, y in X

$$d(x,y) = \inf\{L(\gamma)|\gamma \text{ is a path from } x \text{ to } y\}.$$

We call d a *length*, or an *intrinsic metric*, and we allow ∞ as a possible value of d.

A path $\gamma : [a,b] \to X$ in a metric space (X,d) is called a *segment* if its length is minimal among the paths with the same endpoints. It follows that, if (X,d) is a length space, a segment is defined as follows: a path $\gamma : [a,b] \to X$ is a segment if and only if its length is equal to the distance between its endpoints $L(\gamma) = d(\gamma(a),\gamma(b))$.

If there exists a (a unique) segment between any two points in a length space (X,d), then d is called a *convex* (*strongly convex*) metric.

A point z in a metric space (X,d) is called a *midpoint* between points $x,y \in X$ if $d(x,z) = d(z,y) = \frac{1}{2}d(x,y)$.

Let (X,d) be a metric space. If d is a convex (strongly convex) metric, then for every $x,y \in X$ there exists a (a unique) midpoint z. In case X is complete, the converse implication holds as well (see [6], chapter 2.4.4, page 42).

In a compact metric space (X,d), there exists a segment between any two points x,y that can be connected by at least one rectifiable curve (see [6], chapter 2.5.2, page 49).

Let (X,d) be a compact strongly convex metric space. The *concave collection* T for d is a finite set of segments in X which satisfy the following condition: $\forall \rho,\tau \in T$, $\forall x_1,x_2 \in \rho$, $\forall y_1,y_2 \in \tau$, we have

$$d(x_m,y_m) \leq \tfrac{1}{2}[d(x_1,y_1) + d(x_2,y_2)],$$

where x_m and y_m are the midpoints of the segments $[x_1,x_2]$ and $[y_1,y_2]$.

Let X be a length space. A curve $\gamma : I \to X$ is called a *geodesic segment* (or a *geodesic*) if for every $t \in I$ there exists an interval J containing a neighborhood of t in I such that $\gamma|_J$ is a segment. In other words, a geodesic segment is a curve which is locally a segment.

We call a length space X a *geodesic space* if every pair of points in X can be joined by a segment.

Let $k \leq 0$ be a real number. Let X_k^2 denote a simply connected complete Riemannian $2-$manifold of constant curvature k. So X_0^2 is the Euclidean plane $\mathbb{R}^2$. If $k < 0$, X_k^2 is the hyperbolic plane.

A geodesic triangle $\triangle = \triangle(p,q,r)$ in a geodesic space X is a configuration of three segments (edges) connecting three points (vertices) in pairs. A comparison triangle for $\triangle$ is a geodesic triangle $\overline{\triangle} = \overline{\triangle}(\overline{p},\overline{q},\overline{r})$ in X_k^2 with the same edge lengths. For any $x \in \triangle$, say $x \in [p,q]$, there exists a *comparison point* $\overline{x}$, i.e. a point $\overline{x} \in [\overline{p},\overline{q}]$ such that $d(p,x) = d_{X_k^2}(\overline{p},\overline{x})$.

A metric space X is a CAT(k)-space if it is a geodesic space all of whose geodesic triangles satisfy the so called *CAT(k)-inequality*. Namely, for any geodesic triangle $\triangle(p,q,r) \subset X$, and any two points $x, y \in \triangle$, we have

$$d(x,y) \leq d_{X_k^2}(\overline{x}, \overline{y}),$$

where $\overline{x}, \overline{y}$ are the corresponding points in the comparison triangle $\overline{\triangle}$.

A geodesic space X *has curvature* $\leq k$ if the CAT(k)- inequality holds locally in X. If X has curvature ≤ 0, we say X is *nonpositively curved.*

If γ_1, γ_2 are two segments with the same initial point $x = \gamma_1(0) = \gamma_2(0)$ in a geodesic space X, the *Aleksandrov angle* between γ_1 and γ_2 at x is defined as

$$\angle_x(\gamma_1(s), \gamma_2(t)) = \limsup_{s,t \to 0} \overline{\angle}_x(\gamma_1(s), \gamma_2(t)),$$

where $\overline{\angle}_x(\gamma_1(s), \gamma_2(t))$ denotes the angle at the vertex corresponding to x in a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle in X with vertices at $x, \gamma_1(s), \gamma_2(t)$.

A metric space X is a CAT(k) space if and only if it is a geodesic space and if, for any geodesic triangle $\triangle$ in X, the Aleksandrov angle at any vertex is not greater than the corresponding angle in a comparison triangle $\overline{\triangle} \subset X_k^2$ (see [5], chapter II.1, page 161). If X is CAT(k), then it is also CAT(k') for every $k' > k$ (see [5], chapter II.1, page 165).

There is a unique segment between any two points of a CAT(k) space (see [5], chapter II.1, page 161). Hence, since strongly convex metric spaces are contractible and locally contractible (see [14]), so are CAT(k) spaces.

The first proof given to the paper's main result will make frequent use of Aleksandrov's lemma which is given below (for the proof see [5], chapter I.2, page 25).

Lemma 1.1. *Let a, b, c, d be points in the Euclidean plane $\mathbb{R}^2$ such that a and c are in different half-planes with respect to the line bd. Consider a triangle $\triangle(a', b', c')$ in $\mathbb{R}^2$ such that $\overline{d}(a,b) = \overline{d}(a',b')$, $\overline{d}(b,c) = \overline{d}(b',c')$, $\overline{d}(a,d) + \overline{d}(d,c) = \overline{d}(a',c')$ and let d' be a point on the segment $[a'c']$ such that $\overline{d}(a,d) = \overline{d}(a',d')$.*

Then $\angle_d(a,b) + \angle_d(b,c) < \pi$ if and only if $\overline{d}(b',d') < \overline{d}(b,d)$. In this case, one also has $\angle_{a'}(b',d') < \angle_a(b,d)$ and $\angle_{c'}(b',d') < \angle_c(b,d)$.

And $\angle_d(a,b) + \angle_d(b,c) > \pi$ if and only if $\overline{d}(b',d') > \overline{d}(b,d)$. In this case, one also has $\angle_{a'}(b',d') > \angle_a(b,d)$ and $\angle_{c'}(b',d') > \angle_c(b,d)$.

A *triangle* is a 2-simplex isometric to a 2-simplex in $\mathbb{R}^2$. The unit 2-hexagon J is isometric to a regular hexagon in $\mathbb{R}^2$ with edges of length one. We call a unit 2-hexagon simply a *hexagon.*

We define a hexagonal 2-complex by mimicking the definition of a simplicial 2-complex, using hexagons instead of simplices.

A 2-*dimensional hexagonal complex* K is the quotient of a disjoint union of hexagons $L = \bigcup_\Lambda J_\lambda$ by an equivalence relation $\sim$. The restrictions $p_\lambda : J_\lambda \to K$ of the natural projection $p : L \to K = L|_\sim$ are required to satisfy:

1. for every $\lambda \in \Lambda$, the map p_λ is injective;

2. if $p_\lambda(J_\lambda) \bigcap p_{\lambda'}(J_{\lambda'}) \neq \emptyset$, then there is an isometry $h_{\lambda,\lambda'}$ from a face $T_\lambda \subset J_\lambda$ onto a face $T_{\lambda'} \subset J_{\lambda'}$ such that $p_\lambda(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.

We note that the intersection of any two cells in a hexagonal 2-complex is either the empty set, or a single common vertex, or a single common edge.

There are many interesting examples of cell 2-complexes all of whose 2-cells also have six 1-dimensional faces, but which do not satisfy all the conditions of the above definition. We use the term hexagon 2-complex to describe this larger class of complexes and introduce it below.

A *convex X_k^n-polyhedral cell* C is the convex hull of a finite set of points in X_k^n. The *support* of a point $x \in C$, denoted *supp*(x), is the unique face of C containing x in its interior.

Let $(C_\lambda : \lambda \in \Lambda)$ be a family of convex X_k^n-polyhedral cells and let $L = \cup_{\lambda\in\Lambda}(C_\lambda x\{\lambda\})$ denote their disjoint union. Let $\sim$ be an equivalence relation on L and let $K = L|_\sim$. Let $p : L \to K$ be the natural projection and define $p_\lambda : C_\lambda \to K$ by $p_\lambda(x) := p(x, \lambda)$. K is called an n-dimensional *X_k^n-polyhedral complex* if:

1. for all $\lambda \in \Lambda$, the restriction of p_λ to the interior of each face of C_λ is injective;

2. for all $\lambda_1, \lambda_2 \in \Lambda$ and $x_1 \in C_{\lambda_1}$, $x_2 \in C_{\lambda_2}$, if $p_{\lambda_1}(x_1) = p_{\lambda_2}(x_2)$ then there is an isometry $h : supp(x_1) \to supp(x_2)$ such that $p_{\lambda_1}(y) = p_{\lambda_2}(h(y))$ for all $y \in supp(x_1)$.

A 2-*dimensional hexagon complex* is a 2-dimensional X_k^2-polyhedral complex whose 2-cells have six 1-dimensional faces. We note that the intersection of any two cells in a hexagon 2-complex is either the empty set, or at most six common vertices, or / and at most six common edges. So in a hexagon 2-complex such intersection may be a union of faces.

Let K be a cell complex. $|K|$ denotes the underlying space of K, and $K^{(k)}$ denotes the k-skeleton of K.

Let α be an i-cell of K. If β is a k-dimensional face of α but not of any other cell in K, then we say there is an *elementary collapse* from K to $K' = K \setminus \{\alpha, \beta\}$. We denote an elementary collapse by $K \searrow K'$. If

$K = K_0 \supseteq K_1 \supseteq ... \supseteq K_n = L$ are cell complexes such that there is an elementary collapse from K_{j-1} to K_j, $1 \leq j \leq n$, then we say that K *collapses* to L.

A *closed edge* is an edge together with its endpoints. An *oriented edge* of K is an oriented 1-cell of K, $e = [v_0, v_1]$. We denote by $i(e) = v_0$, the *initial* vertex of e, by $t(e) = v_1$, the *terminus* of e, and by $e^{-1} = [v_1, v_0]$, the *inverse* of e. A finite sequence $\alpha = e_1 e_2 ... e_n$ of oriented closed edges in K such that $t(e_i) = i(e_{i+1})$ for all $1 \leq i \leq n-1$, is called an *edge-path* in K. If $t(e_n) = i(e_0)$, then we call α a *closed edge-path* or *cycle*. We denote by $|\alpha|$ the number of 1-cells contained in α and we call $|\alpha|$ the *length* of α.

Let σ be a cell of K. The *star* of σ in K, denoted $St(\sigma, K)$, is the union of all cells that contain σ. The *link* of σ in K, denoted $Lk(\sigma, K)$, consists of all cells in the star of σ in K which are disjoint from σ and which, together with σ, span a cell of K.

A subcomplex L in K is called *full* (in K) if any cell of K spanned by a set of vertices in L, is a cell of L. A *full cycle* in K is a cycle that is full as subcomplex of K. The *systole* of K is given by

$$sys(K) = \min\{|\alpha| : \alpha \text{ is a full cycle in } K\}.$$

A cell 2-complex has the *k-property* if the link of each of its vertices is a graph of systole at least k, $k \in \{6, 8, 12\}$.

2 Hexagonal 2-complexes are CAT(0) spaces

In this section we give a first proof for the fact that simply connected hexagonal 2-complexes are strongly convex. Namely, we will study the existence of a CAT(0) metric on a hexagonal 2-complex by showing that such complex has the 12-property. It is therefore non-positively curved at any of its points except for its exterior vertices. We investigate further the curvature of the complex at these vertices and show that it is strictly bounded above by zero. A similar proof was given in [9] ([12]) for the fact that any CAT(0) simplicial 2-complex remains, after performing an elementary collapse on it, non-positively curved. Any simply connected hexagonal $2-$complex is hence a CAT(0) space and therefore strongly convex.

Lemma 2.1. *Any hexagonal 2-complex K has the $12-$property.*

Proof. Let v be an interior vertex of K. Since any two 2-cells in a hexagonal 2-complex can intersect each other along at most one 1-cell of the complex, there must exist at least three 1-cells e_1, e_2 and e_3 adjacent to v. So there must exist at least three 2-cells σ_1, σ_2 and σ_3 such that σ_1 and σ_2 intersect each other along e_1, σ_2 and σ_3 intersect each other along e_2, and σ_3 and σ_1 intersect each other along e_3. So the link of v in K contains at least 12 edges. Thus K has the 12-property.

□

Because the intersection of any two 2-cells of a hexagon 2-complex may be a union of faces, hexagon 2-complexes do not necessarily have the 12-property. Take, for instance, two 2-cells and glue them along three of their six 1-dimensional faces. Because the resulting complex has interior vertices whose links in the complex contain less than 12 edges, it does not have the 12-property.

Since hexagonal 2-complexes are, according to their definition, endowed with the standard piecewise Euclidean metric, the above lemma implies that, except for their exterior vertices, these spaces are everywhere non-positively curved. We investigate further the curvature of a hexagonal 2-complex at its exterior vertices.

Lemma 2.2. *Let K be a hexagonal 2-complex. Let e be an edge of K such that exactly two 2-cells σ_1 and σ_2 of K intersect each other along e and nowhere else. Let r be a point that belongs to σ_1, and let p and q be two distinct points that belong to σ_2. The points p, q and r are chosen such that at most one of them coincides with one of the endpoints of e. Then the geodesic triangle $\triangle(p, q, r)$ in $|K|$ satisfies the CAT(0) inequality.*

Proof. We denote by s the intersection point of e and $[p, r]$, and by t the intersection point of e and $[q, r]$.

Let $\triangle(\overline{r}, \overline{s}, \overline{t})$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, s, t)$ in $|K|$. Let $\triangle(\overline{s}, \overline{t}, \overline{q})$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(s, t, q)$ in $|K|$. The comparison triangles $\triangle(\overline{r}, \overline{s}, \overline{t})$ and $\triangle(\overline{s}, \overline{t}, \overline{q})$ are placed such that the points $\overline{r}$ and $\overline{q}$ lie in different half-planes with respect to the line through $\overline{s}$ and $\overline{t}$. Let $\triangle(\overline{\overline{r}}, \overline{\overline{s}}, \overline{\overline{q}})$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, s, q)$ in $|K|$. Let $\triangle(\overline{\overline{p}}, \overline{\overline{s}}, \overline{\overline{q}})$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, s, q)$ in $|K|$. The comparison triangles $\triangle(\overline{\overline{r}}, \overline{\overline{s}}, \overline{\overline{q}})$ and $\triangle(\overline{\overline{p}}, \overline{\overline{s}}, \overline{\overline{q}})$ are placed such that the points $\overline{\overline{r}}$ and $\overline{\overline{p}}$ lie in different half-planes with respect to the line through $\overline{\overline{s}}$ and $\overline{\overline{q}}$. Let $\overline{\overline{t}} \in [\overline{\overline{r}}, \overline{\overline{q}}]$ be a comparison point for $t \in [r, q]$.

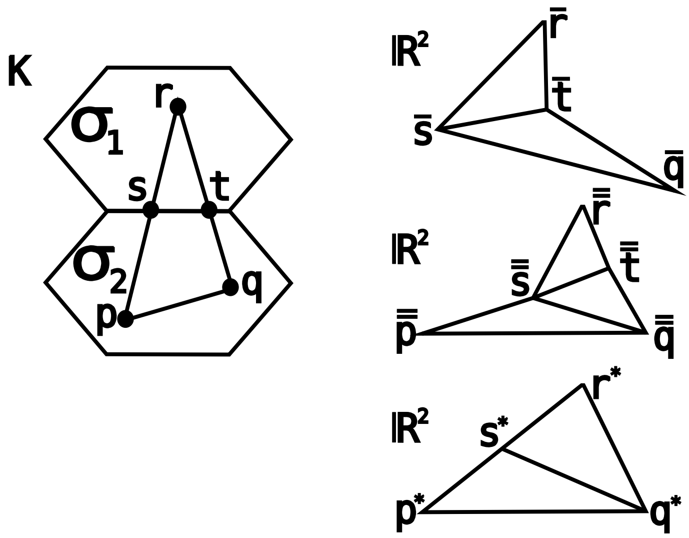

Figure 1

Because σ_1 is a CAT(0) space, the geodesic triangle $\triangle(r, s, t)$ in $|K|$ fulfills the CAT(0) inequality.

Because $t \in [r, q]$, the CAT(0) inequality implies $\pi = \angle_t(r, q) \leq \angle_t(r, s) + \angle_t(s, q) \leq \angle_{\overline{t}}(\overline{r}, \overline{s}) + \angle_{\overline{t}}(\overline{s}, \overline{q})$. Hence, since $\angle_{\overline{t}}(\overline{r}, \overline{s}) + \angle_{\overline{t}}(\overline{s}, \overline{q}) \geq \pi$, according to Aleksandrov's lemma, we have $d_{\mathbb{R}^2}(\overline{s}, \overline{t}) \leq d_{\mathbb{R}^2}(\overline{\overline{s}}, \overline{\overline{t}})$. Hence $\angle_{\overline{r}}(\overline{s}, \overline{t}) \leq \angle_{\overline{\overline{r}}}(\overline{\overline{s}}, \overline{\overline{t}})$. The CAT(0) inequality implies that $\angle_r(s, t) \leq \angle_{\overline{r}}(\overline{s}, \overline{t})$. So it follows that

$$\angle_r(s, t) \leq \angle_{\overline{\overline{r}}}(\overline{\overline{s}}, \overline{\overline{t}}). \tag{1}$$

Let $\triangle(r^*, p^*, q^*)$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(r, p, q)$ in $|K|$. Let $s^* \in [r^*, p^*]$ be a comparison point for $s \in [r, p]$.

Because $s \in [r, p]$, the CAT(0) inequality implies $\pi = \angle_s(r, p) \leq \angle_s(r, q) + \angle_s(q, p) \leq \angle_{\overline{\overline{s}}}(\overline{\overline{r}}, \overline{\overline{q}}) + \angle_{\overline{\overline{s}}}(\overline{\overline{q}}, \overline{\overline{p}})$. Aleksandrov's lemma further implies $\angle_{\overline{\overline{r}}}(\overline{\overline{s}}, \overline{\overline{q}}) \leq \angle_{r^*}(s^*, q^*)$ and hence, according to (1), we have

$$\angle_r(s, t) \leq \angle_{r^*}(s^*, q^*). \tag{2}$$

Aleksandrov's lemma also implies that $d_{\mathbb{R}^2}(\overline{\overline{s}}, \overline{\overline{q}}) \leq d_{\mathbb{R}^2}(s^*, q^*)$. Hence, since $d_{\mathbb{R}^2}(\overline{\overline{s}}, \overline{\overline{q}}) = d(s, q)$, we have

$$\angle_p(s, q) \leq \angle_{p^*}(s^*, q^*). \tag{3}$$

One can similarly show that

$$\angle_q(p, r) \leq \angle_{q^*}(p^*, r^*). \tag{4}$$

The inequalities (2), (3) and (4) guarantee that the geodesic triangle $\triangle(p, q, r)$ in $|K|$ satisfies the CAT(0) inequality.

$\square$

Lemma 2.3. *Let K be a hexagonal 2-complex. Let v be an exterior vertex of K such that exactly two 2-cells σ_1 and σ_2 of K intersect each other at v and nowhere else. Let r be a point that belongs to σ_1, and let p and q be two distinct points that belong to σ_2. The points p, q and r are chosen such that none of them coincides with v. Then the geodesic triangle $\triangle(p, q, r)$ in $|K|$ satisfies the CAT(k) inequality for any real number $k < 0$.*

Proof. We note that the segment $[r, p]$ ($[r, q]$) is the concatenation of the segments $[r, v]$ and $[v, p]$ ($[r, v]$ and $[v, q]$).

Let $\triangle(\overline{p}, \overline{q}, \overline{r})$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, q, r)$ in $|K|$. Let $\overline{v_1} \in [\overline{r}, \overline{q}]$ be a comparison point for $v \in [r, q]$, and let $\overline{v_2} \in [\overline{r}, \overline{p}]$ be a comparison point for $v \in [r, p]$. Let $\triangle(\overline{\overline{v}}, \overline{\overline{p}}, \overline{\overline{q}})$ in $\mathbb{R}^2$ be a comparison triangle for the geodesic triangle $\triangle(v, p, q)$ in $|K|$.

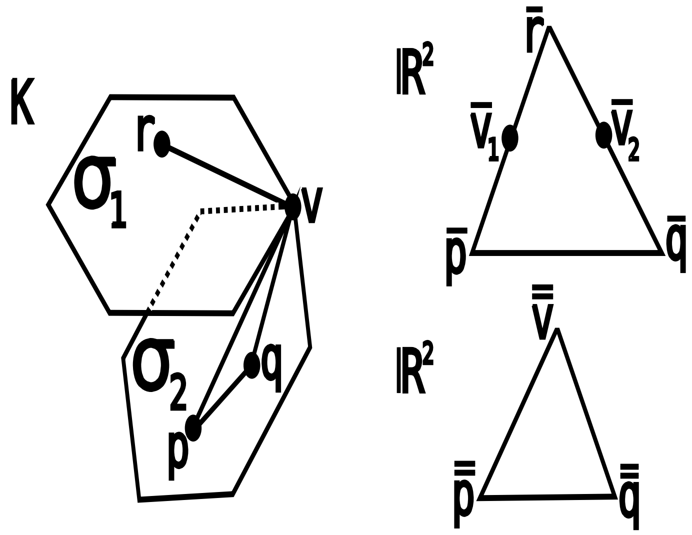

Figure 2

We note that $\angle_r(p, q) = 0$ and hence, since the points p, q and r differ from the point v, the following strict inequality holds $\angle_r(p, q) < \angle_{\overline{r}}(\overline{p}, \overline{q})$. Because the geodesic triangle $\triangle(v, p, q)$ in $|K|$ satisfies the CAT(0) inequality, we have $\angle_q(v, p) \leq \angle_{\overline{\overline{q}}}(\overline{\overline{v}}, \overline{\overline{p}})$. Since the point r differs from the point v, elementary Euclidean geometry guarantees that $\angle_{\overline{\overline{q}}}(\overline{\overline{v}}, \overline{\overline{p}}) < \angle_{\overline{q}}(\overline{v_2}, \overline{p})$. So $\angle_q(r, p) < \angle_{\overline{q}}(\overline{r}, \overline{p})$.

One can similarly show that $\angle_p(r,q) < \angle_{\overline{p}}(\overline{r},\overline{q})$. Hence the geodesic triangle $\triangle(p,q,r)$ in $|K|$ satisfies the CAT(0) inequality. Furthermore, since all comparison inequalities are strict, the geodesic triangle $\triangle(p,q,r)$ in $|K|$ satisfies the CAT(k) inequality for any real number $k < 0$.

□

Lemma 2.4. *Let K be a hexagonal 2-complex. Let v be an exterior vertex of K such that exactly three 2-cells σ_1, σ_2 and σ_3 of K intersect each other at v and nowhere else. Let p be a point that belongs to σ_1, let q be a point that belongs to σ_2, and let r be a point that belongs to σ_3. The points p, q and r are chosen such that none of them coincides with v. Then the geodesic triangle $\triangle(p,q,r)$ in $|K|$ satisfies the CAT(k) inequality for any real number $k < 0$.*

Proof. We note that the segment $[r,p]$ ($[r,q]$; $[p,q]$) is the concatenation of the segments $[r,v]$ and $[v,p]$ ($[r,v]$ and $[v,q]$; $[p,v]$ and $[v,q]$).

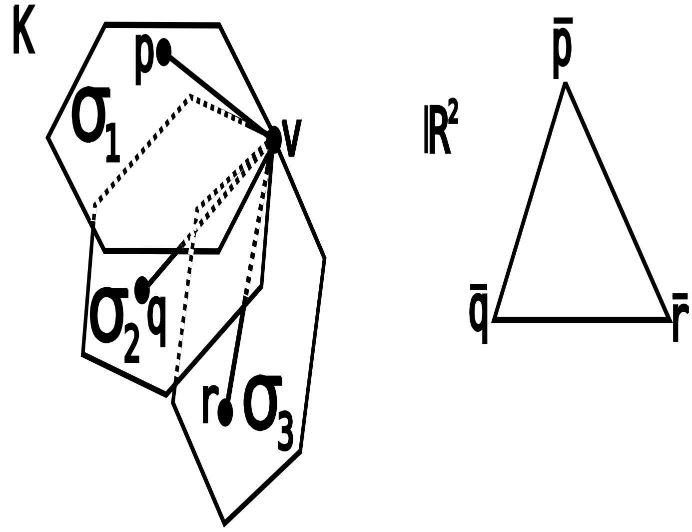

Figure 3

Let $\triangle(\overline{p},\overline{q},\overline{r})$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p,q,r)$ in $|K|$. We note that $\angle_r(p,q) = 0$, $\angle_p(r,q) = 0$ and $\angle_q(p,r) = 0$. Hence $\angle_r(p,q) < \angle_{\overline{r}}(\overline{p},\overline{q})$, $\angle_p(r,q) < \angle_{\overline{p}}(\overline{r},\overline{q})$ and $\angle_q(p,r) < \angle_{\overline{q}}(\overline{p},\overline{r})$. So the geodesic triangle $\triangle(p,q,r)$ in $|K|$ satisfies the CAT(k) inequality for any real number $k < 0$.

□

Lemma 2.5. *Let K be a hexagonal 2-complex. Let v be an exterior vertex of K and let e be an edge of K such that v is one of its faces, such that exactly two 2-cells σ_1 and σ_2 of K intersect each other along e and nowhere else, and such that exactly two pairs of 2-cells (σ_1 and σ_3; σ_2 and σ_3) of K intersect each other at v and nowhere else. Let p be a point that belongs to σ_1, let r be a point that belongs to σ_2, and let q be a point that belongs to σ_3. The points p, q and r are chosen such that none of them coincides with v. Then the geodesic triangle $\triangle(p, q, r)$ in $|K|$ satisfies the CAT(k) inequality for any real number $k < 0$.*

Proof. We note that the segment $[r, q]$ ($[p, q]$) is the concatenation of the segments $[r, v]$ and $[v, q]$ ($[p, v]$ and $[v, q]$).

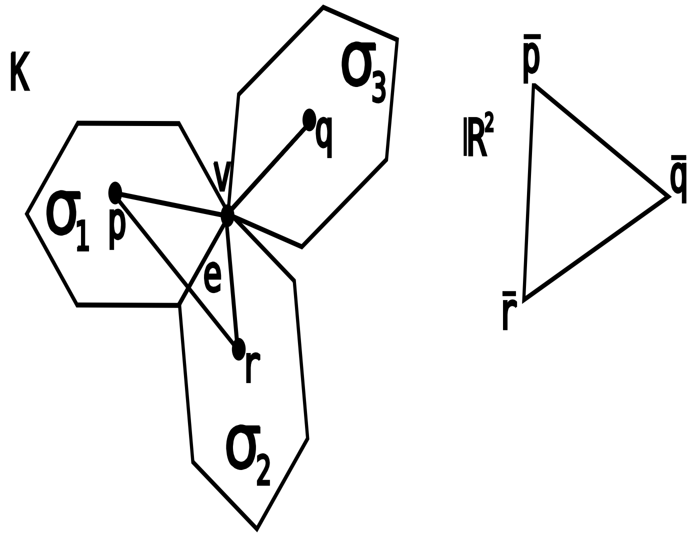

Figure 4

Let $\triangle(\overline{p}, \overline{q}, \overline{r})$ be a comparison triangle in $\mathbb{R}^2$ for the geodesic triangle $\triangle(p, q, r)$ in $|K|$.

We note that $\angle_q(p, r) = 0$. Hence, since the point q differs from the points v, $\angle_q(p, r) < \angle_{\overline{q}}(\overline{p}, \overline{r})$. Lemma 2.2 implies that the geodesic triangle $\triangle(p, v, r)$ in $|K|$ satisfies the CAT(0) inequality. Because the points p, q and r differ from the point v, arguing as in the proof of Lemma 2.3, we get $\angle_r(p, q) < \angle_{\overline{r}}(\overline{p}, \overline{q})$, $\angle_p(r, q) < \angle_{\overline{p}}(\overline{r}, \overline{q})$. So the geodesic triangle $\triangle(p, q, r)$ in $|K|$ satisfies the CAT(k) inequality for any real number $k < 0$.

□

The Lemmas 2.3, 2.4 and 2.5 imply the following theorem.

Theorem 2.6. *Let K be a hexagonal 2-complex. Then K has curvature strictly bounded above by zero at any of its exterior vertices.*

The above theorem guarantees that a hexagonal 2-complexes is non-positively curved at any of its exterior vertices. So, because a hexagonal 2-complex is, according to its definition, endowed with the standard piecewise Euclidean metric, Lemma 2.1 ensures that a hexagonal 2-complex is everywhere non-positively curved. Hence, since finite, simply connected, non-positively curved spaces are CAT(0) spaces (see [5], chapter II.4, page 194), the main result of the paper follows.

Corollary 2.7. *Any simply connected hexagonal 2-complex is a CAT(0) space. In particular, it has a strongly convex metric.*

3 Hexagonal 2-complexes have a strongly convex metric

In this section we give a second proof for the fact that finite, simply connected hexagonal 2-complexes have a strongly convex metric. Because hexagonal 2-complexes have the 12-property, finite, simply connected hexagonal 2-complexes are, according to [10], collapsible. The essential step of the proof will be therefore to show that collapsible hexagonal 2-complexes are strongly convex. The proof of this step is based on a lemma proven by W. White in [16]. We start by presenting this lemma.

Lemma 3.1. *Suppose that $X \cup \sigma$ is a metric space and that $X \cap \sigma = \tau$ is a segment. Let d be a strongly convex metric for X and let T be a concave collection for d that contains τ. Suppose $abcde$ is a triangle with vertices at a, d, and e, and let $\varphi : abcde \to \sigma$ be a homeomorphism such that $\varphi(bc) = \tau$ and $d(\varphi(x), \varphi(y)) = d_{\mathbb{R}^2}(x, y)$ for every $x, y \in bc$. Then there is a strongly convex metric d' for $X \cup \sigma$ such that:*

$$d'(x,y) = \begin{cases} d(x,y) \text{ for all } x,y \in X, \\ d_{\mathbb{R}^2}(\varphi^{-1}(x), \varphi^{-1}(y)) \text{ for all } x,y \in \sigma, \\ \min_{z\in\tau}\{d'(x,z) + d'(z,y)\} \text{ for all } x \in \sigma, y \in X \text{ or } x \in X, y \in \sigma, \end{cases}$$

and $T \cup \{\varphi(ab), \varphi(cd), \varphi(de), \varphi(ea)\}$ is a concave collection for d'.

The above lemma implies the following result.

Theorem 3.2. *Let X be a finite hexagonal 2-complex and let d be a strongly convex metric on X. Let T be a concave collection for d which covers*

$|X^{(1)}|$. Let $\sigma^{(2)}$ and $\tau^{(1)}$ be two cells such that τ is a free face of the hexagon σ. We consider the hexagonal 2-complex $X' = X \cup \{\sigma, \tau\}$ such that $X' \searrow X$ is an elementary collapse. Then $|X'|$ has a strongly convex metric d' such that $d'(x, y) = d(x, y)$ for all $x, y \in |X|$, and there exists a concave collection T' for d' which covers $|X'^{(1)}|$.

Proof. Let $X \cap \sigma^{(2)} = \{\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}, \tau_4^{(1)}, \tau_5^{(1)}\}$. Because the concave collection T for d covers $|X^{(1)}|$, the segments $\tau_1, \tau_2, \tau_3, \tau_4$ and τ_5 belong to T. We consider a subtriangulation $[u_0, u_1], [u_1, u_2], ..., [u_{k-1}, u_k]$ of $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ and note that the segment $[u_{i-1}, u_i]$ belongs to an element of T, $1 \le i \le k$. On τ we choose the points v_i, m_i, w_i, $1 \le i \le k$, and n_i, o_i, q_i, $1 \le i \le k-1$, ordered as follows:

$$u_0, v_1, m_1, w_1, n_1, o_1, q_1, v_2, m_2, w_2, n_2, o_2, q_2, ...,$$

$$v_{k-1}, m_{k-1}, w_{k-1}, n_{k-1}, o_{k-1}, q_{k-1}, v_k, m_k, w_k, u_k$$

.

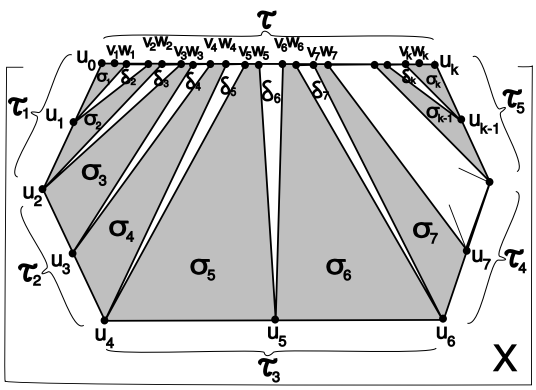

Figure 5

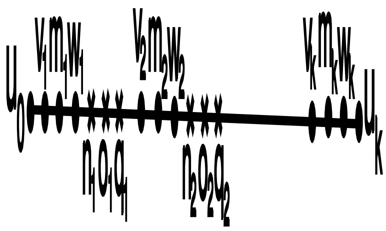

Figure 6

We denote by σ_i the quadrilateral with vertices at u_{i-1}, u_i, w_i and v_i that is contained in σ, $1 \le i \le k$. We note that σ_i intersects X along the segment $[u_{i-1}, u_i]$ that belongs to an element of T, $1 \le i \le k$. Lemma 3.1 therefore implies that $X_1 = X \cup (\cup_{i=1}^{k} \sigma_i)$ has a strongly convex metric d_1 such that $T_1 = T \cup \{[v_1, u_0], [u_1, w_1], [w_1, m_1], [m_1, v_1], ..., [v_k, u_{k-1}], [u_k, w_k], [w_k, m_k], [m_k, v_k]\}$ is a concave collection for d_1 which covers $|X_1^{(1)}|$. We note that the segment $[v_i, w_i]$ with respect to the metric d_1 is the concatenation of the segments $[v_i, u_{i-1}], [u_{i-1}, u_i]$ and $[u_i, w_i]$, $1 \le i \le k$.

We will show that the metric d_1 can be extended by induction to a strongly convex metric d' on the rest of $|X'| = |X \cup \sigma|$. For $2 \le i \le k$, let δ_i denote the triangle whose boundary contains the points w_{i-1}, u_{i-1} and v_i. For $2 \le i \le k$, let $X_i = X_1 \cup (\cup_{i=2}^{k} \delta_i)$ and let

$$T_i = T_1 \cup (\cup_{i=2}^{k} \{[w_{i-1}, n_{i-1}], [n_{i-1}, o_{i-1}], [o_{i-1}, q_{i-1}], [q_{i-1}, v_i]\}).$$

Suppose that for some $j \in \{1, ..., k\}$, there exists a strongly convex metric d_j on X_j such that $d_j(x, y) = d_1(x, y)$ for all $x, y \in X_1$, and such that T_j is a concave collection for d_j which covers $|X_j^{(1)}|$.

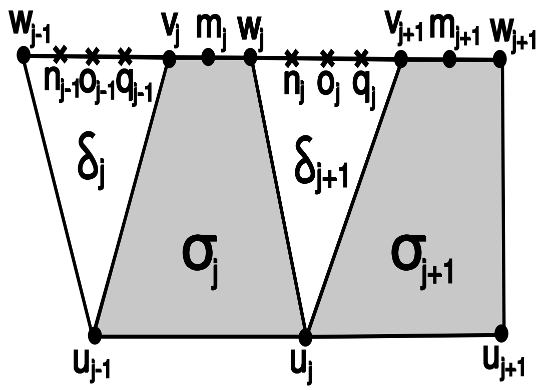

Figure 7

We note that the segment $[w_j, v_{j+1}]$ with respect to the metric d_j hits the segment $[u_{j-1}, u_{j+1}]$. Similarly, the segment $[w_j, u_{j-1}]$ with respect to the metric d_j is the concatenation of the segments $[w_j, u_j]$ and $[u_j, u_{j-1}]$ while the segment $[v_{j+1}, u_{j+1}]$ with respect to the metric d_j is the union of the segments $[v_{j+1}, u_j]$ and $[u_j, u_{j+1}]$. So the segment $[w_j, v_{j+1}]$ with respect to the metric d_j is the union of the segments $[w_j, u_j]$ and $[u_j, v_{j+1}]$, $j < k$.

Because the segment $[v_j, w_j]$ with respect to the metric d_j is the concatenation of the segments $[v_j, u_{j-1}], [u_{j-1}, u_j]$ and $[u_j, w_j]$, for any $p \in \overline{X_j \setminus \sigma_j}$ and for any $x \in [w_j, u_j]$, we have $d_j(p, x) = d_j(p, u_j) + d_j(u_j, x)$.

We show further that $T_j \cup \{[w_j, v_{j+1}]\}$ is a concave collection for d_j which covers $|X_j^{(1)}|$. Let α be a segment in T_j different from $[u_j, w_j], [w_j, m_j]$, or $[m_j, v_j]$. Let x_1 and x_2 be two distinct points on $[w_j, v_{j+1}]$ such that $d_j(x_1, x_2) = d_j(x_1, u_j) + d_j(u_j, x_2)$ and let y_1 and y_2 be two distinct points on α. Let x_m, y_m and x'_m be the midpoints of the segments $[x_1, x_2]$, $[y_1, y_2]$ and $[u_j, x_2]$. We note that $d_j(x_m, x'_m) = \frac{1}{2} d_j(x_1, u_j)$. Besides we note that $d_j(x_1, y_1) = d_j(x_1, u_j) + d_j(u_j, y_1)$. Because the segments $[y_1, y_2]$ and $[u_j, x_2]$ belong to T_j, we have:

$$d_j(y_m, x'_m) \leq \frac{1}{2}[d_j(y_1, u_j) + d_j(y_2, x_2)].$$

Altogether we have:

$$d_j(x_m, y_m) \leq d_j(x_m, x'_m) + d_j(x'_m, y_m) \leq$$

$$\leq \frac{1}{2}[d_j(x_1, u_j) + d_j(y_1, u_j) + d_j(y_2, x_2)] =$$

$$= \frac{1}{2}[d_j(x_1, y_1) + d_j(y_2, x_2)].$$

The above relation implies that $T_j \cup \{[w_j, v_{j+1}]\}$ is a concave collection for d_j.

One can similarly show that, if α is $[u_j, w_j], [w_j, m_j]$, or $[m_j, v_j]$, $T_j \cup \{[w_j, v_{j+1}]\}$ is a concave collection for d_j which covers $|X_j^{(1)}|$.

We note that the segment $[w_j, v_{j+1}]$ with respect to the metric d_j is the union of the segments $[w_j, u_j]$ and $[u_j, v_{j+1}]$, and that $T_j \cup \{[w_j, v_{j+1}]\}$ is a concave collection for d_j which covers $|X_j^{(1)}|$. Lemma 3.1 therefore implies that there exists a strongly convex metric d_{j+1} on X_{j+1} such that $d_{j+1}(x, y) = d_1(x, y)$ for all $x, y \in X_{j+1}$ and such that T_{j+1} is a concave collection for d_{j+1} which covers $|X_{j+1}^{(1)}|$.

It follows by induction that there exists a strongly convex metric $d' = d_k$ on $|X'| = |X_k|$ such that $d'(x, y) = d_k(x, y)$ for all $x, y \in |X'|$ and such that $T' = T_k$ is a concave collection for d' which covers $|X'^{(1)}|$.

□

The above theorem implies the following corollary.

Corollary 3.3. *Any collapsible hexagonal* 2*-complex has a strongly convex metric.*

Since hexagonal 2-complexes have the 12-property, they are, if their fundamental group vanishes, collapsible (see [10]). The above corollary therefore implies the main result of the paper.

Corollary 3.4. *Any finite simply connected hexagonal* 2*-complex has a strongly convex metric.*

Because strongly convex metric spaces are contractible and locally contractible, the following holds.

Corollary 3.5. *Any finite simply connected hexagonal* 2*-complex is contractible and locally contractible.*

We note that, due to their collapsibility, it was already clear that finite, simply connected hexagonal 2-complexes are contractible.

Acknowledgements. The authors are grateful to Professor Tudor Zamfirescu and to Professor Louis Funar for valuable advices and useful remarks. The second author thanks Professor Mircea Craioveanu for significant support. The paper was partially written during the second author's stay at the University of Augsburg. She thanks Professor Katrin Wendland for hospitality.

References

[1] M. Agoston, *Algebraic topology*, Marcel Dekker Inc., New York and Basel, 1976.

[2] A.-D. Aleksandrov, *Die innere Geometrie der konvexen Flaechen*, Akademie Verlag, Berlin, 1955.

[3] D. Andrica, I.-C. Lazăr, *Cubical* 2*-complexes with the* 8*-property admit a strongly convex metric*, Acta Universitatis Apulensis, **21**, 47-45, 2010.

[4] V. N. Berestovskii, *Spaces with bounded curvature and distance geometry*, Siberian Math. J., **27**, 8-19, 1995.

[5] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer, New York, 1999.

[6] D. Burago, Y. Burago, S. Ivanov, *A Course in Metric Geometry*, American Mathematical Society, Providence, Rhode Island, 2001.

[7] J.-M. Corson, B. Trace, *The* 6*-property for simplicial complexes and a combinatorial Cartan-Hadamard theorem for manifolds*, Proc. AMS, **126**, 917-924, 1998.

[8] T. Januszkiewicz, J. Swiatkowski, *Simplicial nonpositive curvature*, Publ. Math. IHES, 1 - 85, 2006.

[9] I.-C. Lazăr, *CAT(0) simplicial complexes of dimension* 2 *are collapsible*, Proceedings of the International Conference on Theory and Applications of Mathematics and Informatics (Eds.: D. Breaz, N. Breaz, D. Wainberg),

Alba Iulia, September, $9-11$, 2009, Acta Universitatis Apulensis, Special Issue, pages $507-530$, 2009.

[10] I.-C. Lazăr, *The collapsibility of hexagon* $2-$*complexes with the* $12-$*property*, to appear in An. Stiint. Univ. Ovidius Constanta, **19**, fascicola 1, 2011, 191-200.

[11] I.-C. Lazăr, *The collapsibility of square* $2-$*complexes with the* $8-$*property*, Ph.D. thesis, Proceedings of the 12th Symposium of Mathematics and its Applications (Eds: P. Găvruţă, O. Lipovan, W. Müller, D. Păunescu), Timişoara, November, $5-7, 2009$, pages $407-412$, 2009.

[12] I.-C. Lazăr, *The study of simplicial complexes of nonpositive curvature*, Ph.D. Thesis, Cluj University Press, 2010 (http://www.ioana-lazar.ro/phd.html).

[13] K. Menger, *Untersuchungen ueber allgemeine Metrik*, Math. Ann., **100**, 75-163, 1928.

[14] D. Rolfsen, *Characterizing the* 3*-cell by its metric*, Fund Math, **68**, $215-223$, 1970.

[15] D. Rolfsen, *Strongly convex metrics in cells*, Bull. Amer. Math. Soc., **74**, 171-175, 1968.

[16] W. White, *A* 2*-complex is collapsible if and only if it admits a strongly convex metric*, Fund. Math., **68**, 23-29, 1970.

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