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NOTES ON ANNIHILATOR CONDITIONS IN MODULES OVER COMMUTATIVE RINGS

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Abstract

Let M be a module over the commutative ring R. In this paper we introduce two new notions, namely *strongly coprimal* and *super coprimal modules*. Denote by $Z_R(M)$ the set of all zero-divisors of R on M. M is said to be *strongly coprimal* (resp. *super coprimal*) if for arbitrary $a, b \in Z_R(M)$ (resp. every finite subset F of $Z_R(M)$) the annihilator of $\{a, b\}$ (resp. F) in M is non-zero. In this paper we give some results on these classes of modules. Also we provide a relationship between the families of coprimal, strongly coprimal and super coprimal modules. We prove that if M is a coprimal module of finite Goldie dimension over a commutative ring, then M is super coprimal. Finally we show that every proper submodule of a module over a Prüfer domain of finite character can be expressed as a finite intersection of strongly primal submodules.

1 Introduction

Throughout this paper all rings are commutative with nonzero identity, and all modules are considered to be unitary. We wish to study properties of submodules of a module over a certain Prüfer domain, in particular, their decomposition into intersections of strongly primal submodules. So far, the literature on this subject is sparse and mostly restricted to the question of when or which submodules admit decompositions as intersections of finitely many primary

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submodules. We know that every submodule of a Noetherian module can be expressed as a finite intersection of irreducible submodules. Furthermore, in a Noetherian module, every irreducible submodule is primary. Hence if N is a proper submodule of the Noetherian module M, then N has a decomposition as an intersection of a finite number of primary submodules. This happens rarely in non-noetherian modules, because in general modules irreducible submodules fail to be primary. Therefore we look for another decomposition for submodules. We investigate decompositions of submodules of a module over a Prüfer domain into intersections of strongly primal submodules. As we intend to restrict our considerations to finite intersections, we assume to start with that our domain are of finite character; i.e. every non-zero element is contained but in a finite number of maximal ideals.

For a given ring R, an R-module M and a submodule N of M, we will denote by $(N :_R M)$ the residual of N by M i.e., the set of all r in R such that $rM \subseteq N$. The annihilator of M, denoted by $ann_R(M)$, is $(0 :_R M)$. For every subset S of R, we denote by $Ann_M(S)$ the set of elements $m \in M$ such that $ma = 0$ for each $a \in S$. An element $r \in R$ is called a zero-divisor on M provided that there exists $0 \neq m \in M$ such that $rm = 0$, that is $Ann_M(r) \neq 0$. We denote by $Z_R(M)$ the set of all zero-divisors of R on M. If we consider R as an R-module, then we write $Z(R)$ instead of $Z_R(R)$.

An annihilator condition on a commutative ring R is property (A). R is said to have property (A) if every finitely generated ideal I contained in $Z(R)$ has a nonzero annihilator ([8]). Y. Quentel introduced property (A) in [15], calling it condition (C). Faith in [5] studied rings with property (A) and called such rings McCoy. An example of a McCoy ring is a Noetherian ring. However, the property (A) fails for some non-Noetherian rings [11, p. 63]. To avoid the ambiguity we call such rings F-McCoy. Let M be an R-module. We define the R-module M to be an F-McCoy module provided that for every finitely generated ideal I of R with $I \subseteq Z_R(M)$, $Ann_M(I) \neq 0$. This is a natural extension of the concept of an F-McCoy ring. An example of an F-McCoy module is a finitely generated module over a commutative Noetherian ring (see [11, Theorem 82]).

Recently the concept of rings with property (A) has been generalized to noncommutative rings [9]. Let R be an associative ring with identity. We write $Z_l(R)$ and $Z_r(R)$ for the set of all left zero-divisors of R and the set of all right zero-divisors of R, respectively. Then the ring R has right (left) Property (A) if for every finitely generated two-sided ideal $I \subseteq Z_l(R)$ $(Z_r(R))$, there exists nonzero $a \in R$ $(b \in R)$ such that $Ia = 0$ $(bI = 0)$. A ring R is said to have Property (A) if R has right and left Property (A).

Nielsen in [14] defined another class of rings and called it McCoy. This paper is on the basis of some recent papers devoted to this new class of rings.

Let R be an associative ring with 1 (not necessarily commutative). R is said to be right McCoy when the equation $f(x)g(x) = 0$ over $R[x]$, where $f(x), g(x) \neq 0$, implies there exists a nonzero $r \in R$ with $f(x)r = 0$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy then R is called a McCoy ring. This class of McCoy rings includes properly the class of Armendariz rings introduced in [16], which is extensively studied in the last years.

Let R be a commutative ring with identity. Then concepts "F-McCoy ring" and "McCoy ring" are different concepts. In fact neither implies the other. For example, if R is a reduced ring, then it is McCoy by [14, Theorem 2]. But we know that there are reduced rings which are not F-McCoy. Also if we let $\mathbb{Z}_4$ to be the ring of integers modulo 4, then, by [9, Theorem 2.1], $M_2(\mathbb{Z}_4)$, the set of all 2×2 matrices over $\mathbb{Z}_4$, has Property (A) but it is not right McCoy by [17].

For the sake of completeness we give some definitions and results that we will use throughout. The concept of primality has been first considered by Fuchs in [6] for ideals, and then by Dauns [3] for modules. Let R be a commutative ring, M an R-module and N a submodule of M. An element $r \in R$ is called prime to N if $rm \in N$ ($m \in M$) implies that $m \in N$, that is $(N :_M r) = \{m \in M : rm \in N\} = N$. Denote by $S(N)$ the set of all elements of R that are not prime to N. Then N is said to be *primal* if $S(N)$ forms an ideal; this ideal is always a prime ideal, called the adjoint ideal P of N. In this case we also say that N is a P-primal submodule of M. If the zero submodule of M is primal, then M will be called a coprimal module [1]. It is clear that $S(0) = Z_R(M)$. Hence M is a coprimal R-module if $Z_R(M)$ forms an ideal of R. The ring R is coprimal if it is coprimal as an R-module (see [3]). It is easy to check that a submodule N of an R-module M is *primal* if and only if the factor module M/N is coprimal as an $R/(N :_R M)$-module. It has been proved in [4] that N is a P-primal submodule of M if and only if $(N :_R M) \subseteq P$ and $Z_{R/(N:_RM)}(M/N) = P/(N :_R M)$. Let M be a module over the commutative ring R. A submodule N of M is called irreducible if N can not be expressed as a finite intersection of proper submodules of M that contain properly N. It is proved in [3, Proposition B] that every irreducible submodule is primal.

Let R be a commutative ring. We say that an R-module M is *strongly coprimal* if for arbitrary $a, b \in Z_R(M)$ the annihilator of $\{a, b\}$ in M is nonzero. M is called *super coprimal* if for every finite subset F of $Z_R(M)$, the annihilator of the set F in M is nonzero.

Here we provide a brief summary of the paper. The relationship between the classes of coprimal, strongly coprimal and super coprimal submodules are given in Section 2. For example it is shown in Proposition 2.2 that every super coprimal module is strongly coprimal and every strongly coprimal module is coprimal. Also, every super coprimal module is F-McCoy, and M is super coprimal if and only if it simultaneously is coprimal and F-McCoy. Example 2.3 shows that the converse of these implications do not hold. Consider the module of polynomials $M[x]$. Then, $M[x]$ is a coprimal $R[x]$-module if and only if M is a coprimal and a F-McCoy R-module (see Corollary 2.10). Let N be an irreducible submodule of M. It is shown in Proposition 2.12 that M/N is a F-McCoy R-module. According to Example 2.3, a coprimal module need not be super coprimal. In Theorem 2.13 we show that, if M is a coprimal R-module of finite Goldie dimension, then M is super coprimal. In section 3 we first give some results concerning the relationship between the strongly primal submodules of M and strongly primal submodules of $T^{-1}M$, the module of fractions of M with respect to a multiplicatively closed subset T of R. Then we show that if M is a module over the Prüfer domain R of finite character, then every proper submodule of M can be written as an intersection of finitely many strongly primal submodules.

2 Results

We start with the following definitions:

Definition 2.1. Let R be a commutative ring and let M be an R-module.

(1) M is called *strongly coprimal* if for arbitrary $a, b \in Z_R(M)$ the annihilator of $\{a, b\}$ in M is nonzero.

(2) M is called *super coprimal* if for every finite subset F of $Z_R(M)$, the annihilator of the set F in M is nonzero.

The submodule N of the R-module M is called *strongly primal* (respectively *super primal*) if M/N is a strongly coprimal (respectively super coprimal) $R/(N :_R M)$-module.

The next result follows directly from definitions.

Proposition 2.2. *Let R be a commutative ring and let M be an R-module. Then the following statements hold:*

(1) Every super coprimal R-module is strongly coprimal, and every strongly coprimal R-module is coprimal.

(2) Every super coprimal R-module is F-McCoy.

(3) An R-module is super coprimal if and only if it is coprimal and F-McCoy.

It is shown in Examples 2.3 that the converse implications in Proposition 2.2 do not hold.

Examples 2.3. (1) An F-McCoy R-module need not be strongly coprimal. For example, let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $Z(R) = \{(0,0),(1,0),(0,1)\}$. Now assume that I is an ideal of R with $I \subseteq Z(R)$. Then $I = \{(0,0)\}$, $I = \{(0,0),(1,0)\}$ or $I = \{(0,0),(0,1)\}$; and in any case it is easy to see that $Ann_R(I) \neq 0$. Hence R is an F-McCoy ring. Now assume that $a_1 = (1,0)$ and $a_2 = (0,1)$. Then the ideal generated by a_1 and a_2 is equal to R, that is $< a_1, a_2 >= R$. Hence $Ann(< a_1, a_2 >) = 0$. Thus R is not strongly coprimal.

(2) Here we provide an example of an F-McCoy module which is not coprimal. Let $R = \mathbb{Z}$ and consider the R-module $M = \mathbb{Z}_2 \times \mathbb{Z}_3$. It is easy to check that $Z_R(M) = 2\mathbb{Z} \cup 3\mathbb{Z}$. Hence M is not a coprimal R-module. On the other hand, since R is a principal ideal domain, M is F-McCoy.

(3) In this example we use the concept of so-called $A+B$-rings introduced in [12]. Let K be a field, w, y and z algebraically independent indeterminates, $M = (w,y,z)K[w,y,z]$ and let $D = K[w,y,z]_M$. Clearly D is a local ring. Let Q be the maximal ideal of D and let $\mathcal{P}$ denote the set of height two primes of D. For each $P_\alpha \in \mathcal{P}$, let $Q_\alpha = Q/P_\alpha$. Let $\mathcal{I} = \mathcal{A} \times \mathbb{N}$ where $\mathcal{A}$ is an index set for $\mathcal{P}$ and let $B = \sum Q_i$ where $Q_i = Q_\alpha$, for each $i = (\alpha, n) \in \mathcal{I}$. Set $R = D + B$ the ring constructed from $D \times B$ by setting $(r,a)+(s,b) = (r+s, a+b)$ and $(r,a)(s,b) = (rs, rb+sa+ab)$. It is proved in [13, Example 5.1] that

(1) $Z(R) = Q + B$ is a prime ideal of R. So R is a coprimal ring.

(2) Every subset $\{a,b\}$ of $Z(R)$ has a nonzero annihilator in R.

(3) The subset $\{(w,0),(y,0),(z,0)\}$ of $Z(R)$ does not have a nonzero annihilator in R.

(4) R is not F-McCoy.

This example shows that:

(i) A coprimal module need not to be F-McCoy.

(ii) A coprimal module need not to be super coprimal.

(iii) A strongly coprimal module need not be super coprimal.

(iv) A strongly coprimal module need not to be F-McCoy.

Lemma 2.4. *Let I and P be a finitely generated ideals of a commutative ring R with P prime and $I \subseteq P$. If $I_P \neq 0$, then $(IP)_{(P)}$ is a super primal ideal of R.*

Proof. Set $J = IP$. It follows from $I_P \neq 0$ and Nakayama's Lemma that $J_P \neq I_P$ and so $J_{(P)} \neq I_{(P)}$. Therefore

$$J_{(P)} \subset I_{(P)} \subseteq (J_{(P)} :_R P).$$

So $xP \subseteq J_{(P)}$ for some $x \notin J_{(P)}$.

Now assume that $A = \{a_1, ..., a_n\}$ is a finite subset of $Z(R/J_{(P)})$. By [7, Lemma 1.7], $J_{(P)}$ is a P-primal ideal of R. Hence it follows from $Z_R(R/J_{(P)}) = P$ that $A \subseteq P$. Therefore the nonzero element $x + J_{(P)}$ of $R/J_{(P)}$ annihilates A. Thus $J_{(P)}$ is super primal. □

Theorem 2.5. *Let R be a commutative ring. The following statements are equivalent.*

(i) R is an arithmetical ring.

(ii) Every primal ideal of R is irreducible.

(iii) Every strongly primal ideal of R is irreducible.

Proof. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ follow from [7, Theorem 1.8] and the fact that every strongly primal ideal is primal. To prove $(iii) \Rightarrow (i)$, assume that M is a maximal ideal of R and $a, b \in M$. Assume that I is the ideal of R generated by the set $\{a, b\}$. We may assume that $I_M \neq 0$. Then $(IM)_{(M)}$ is an strongly primal ideal of R by Lemma 2.4. Hence $(IM)_{(M)}$ is an irreducible ideal of R by (iii). It follows that $(IM)_M$ is an irreducible ideal of R_M. Hence by the observation (†) given in the proof of [7, Theorem 1.8], either $R_M a \subseteq R_M b$ or $R_M b \subseteq R_M a$. Consequently R_M is a valuation ring, and so R is an arithmetical ring. □

Theorem 2.6. *Let R be a PID and M an R-module and N a submodule of M. The following statements are equivalent:*

(1) N is a primal submodule of M.

(2) N is an strongly primal submodule of M.

(3) N is a super primal submodule of M.

Proof. Since every super primal submodule is strongly primal and every strongly primal submodule is primal, what we need is to prove $(1) \Rightarrow (3)$. So assume that N is a P-primal submodule of M. Set $I = (N :_R M)$, $\bar{R} = R/I$, $\bar{P} = P/I$ and $\bar{M} = M/N$. Then $Z_{\bar{R}}(\bar{M}) = \bar{P}$ by [4, Theorem 2.3]. There exists $\bar{x} = x + I \in \bar{P}$ such that $Z_{\bar{R}}(\bar{M}) = \bar{R}\bar{x}$. Now assume that $\{\bar{a_1}, ..., \bar{a_n}\}$ is a finite subset of $Z_{\bar{R}}(\bar{M})$. Then, for every $1 \leq i \leq n$, $\bar{a_i} = \bar{r_i}\bar{x}$ for some $r_i \in R$. Since $\bar{x} \in Z_{\bar{R}}(\bar{M})$, there exists a nonzero $\bar{m}$ in $\bar{M}$ such that $\bar{x}\bar{m} = 0$. In this case $\bar{a_i}\bar{m} = 0$ for every $1 \leq i \leq n$. Thus the annihilator of $\{\bar{a_1}, ..., \bar{a_n}\}$ in $\bar{M}$ is nonzero. Consequently, N is a super primal submodule of M. □

Proposition 2.7. *Let R be a commutative ring, M an R-module and N a submodule of M. Then $f(x) \in R[x]$ is not prime to $N[x]$ if and only if $mf(x) \in N[x]$ for some $m \in M \backslash N$.*

Proof. Assume that $f \in R[x]$ is not prime to $N[x]$ and $mf(x) \notin N[x]$ for all $m \in M\backslash N$. As $f(x)$ is not prime to $N[x]$, there exists a polynomial $g(x)$ of smallest degree with the properties $g(x) \in M[x]\backslash N[x]$ and $f(x)g(x) \in N[x]$. Assume that $f(x) = \Sigma_{i=0}^{s} a_i x^i$ and $g(x) = \Sigma_{i=0}^{t} m_i x^i$. Then $m_t \notin N$ and $t \geq 1$. Since $m_t f(x) \notin N[x]$, there is $1 \leq i \leq s$ with $m_t a_i \notin N$, and hence $a_i g(x) \notin N[x]$. Let q be the largest integer for which $a_q g(x) \notin N[x]$. Then we have

$$(a_0 + a_1 x + ... + a_q x^q)(m_0 + m_1 x + ... + m_t x^t) \in N[x].$$

This shows that $a_q m_t \in N$. In this case $(a_q(g(x) - m_t x^t))f(x) \in N[x]$, which contradicts the minimality of $g(x)$. □

We will show in Proposition 2.11 that if we can find conditions under which $M[x]$ is a coprimal $R[x]$-module, then one can have conditions under which the module M is super coprimal and hence primal. In the following results we give sufficient conditions for $M[x]$ to be a coprimal $R[x]$-module.

Proposition 2.8. *Let R be a commutative ring, M an R-module and N an irreducible submodule of M. Then:*

(1) $N[x]$ is a primal submodule of $M[x]$.

(2) If P is the adjoint ideal of N, then $P[x]$ is the adjoint ideal of $N[x]$.

Proof. (1) Assume that $f(x) = \Sigma_{i=0}^{s} a_i x^i$ and $g(x) = \Sigma_{i=0}^{t} b_i x^i$ are not prime to $N[x]$. By Proposition 2.7, there exist $m, m' \in M\backslash N$ with $mf(x) \in N[x]$ and $m'g(x) \in N[x]$. As N is irreducible and $Rm + N$ and $Rm' + N$ are proper divisors of N, we have $N \subset (Rm + N) \cap (Rm' + N)$. So there exists $z \in (Rm + N) \cap (Rm' + N)$ with $z \notin N$. So $z = r_1 m + n_1 = r_2 m' + n_2$ in which $r_1, r_2 \in R$ and $n_1, n_2 \in N$. Clearly $r_1 m, r_2 m' \notin N$. Furthermore, $z(f(x) - g(x)) = (r_1 m + n_1)f(x) - (r_2 m' + n_2)g(x) \in N[x]$. So $f(x) - g(x)$ is not prime to $N[x]$, that is $N[x]$ is primal.

(2) As N is irreducible, it is primal by [3, Proposition B]. Assume that P is the adjoint ideal of N, and suppose that $f(x) = \Sigma_{i=0}^{s} a_i x^i$ is not prime to $N[x]$. By Proposition 2.7, $mf(x) \in N[x]$ for some $m \in M\backslash N$. Thus, for every $1 \leq i \leq s$, $ma_i \in N$, which implies that a_i is not prime to N and so $a_i \in P$. Therefore $f(x) \in P[x]$.

Now pick an element $g(x) = \Sigma_{i=0}^{t} b_i x^i$ in $P[x]$. Set $I = (N :_R M)$. If $g(x) \in I[x]$, then $g(x) \in S(N[x])$ since $S(I[x]) \subseteq S(N[x])$. Now assume that $g(x) \notin I[x]$. Let $S = \{s_1, ..., s_k\} \subseteq \{1, ..., s\}$ be such that $a_{s_i} \notin I$ for all $1 \leq i \leq k$. As $a_{s_i} \in P$, there exists $m_{s_i} \in M\backslash N$ with $a_{s_i} m_{s_i} \in N$. Since N is irreducible we have $N \subset \cap_{i=1}^{k}(Rm_{s_i} + N)$. There is $y \in \cap_{i=1}^{k}(Rm_{s_i} + N)$ with $y \notin N$. In this case $y = r_{s_1} m_{s_1} + n_1 = r_{s_2} m_{s_2} + n_2 = ... = r_{s_k} m_{s_k} + n_k$

in which $r_{s_i} \in R$ and $n_i \in N$ for $1 \leq i \leq k$. It is easy to see that $yg(x) \in N[x]$ with $y \notin N[x]$. So $g(x)$ is not prime to $N[x]$. We have already shown that $P[x]$ consists exactly of those elements of $R[x]$ that are not prime to $N[x]$. So $N[x]$ is $P[x]$-primal. □

Theorem 2.9. *Let R be a commutative ring, P a prime ideal of R, M an R-module and N a submodule of M. Then, $N[x]$ is a primal submodule of $M[x]$ with the adjoint prime ideal $P[x]$ if and only if N is a primal submodule of M with the adjoint prime ideal P and M/N is an F-McCoy $R/(N :_R M)$-module.*

Proof. Suppose first that $N[x]$ is a $P[x]$-primal submodule of $M[x]$. Set $I = (N :_R M)$ and let $A = (a_1 + I, a_2 + I, ..., a_k + I)$ be an ideal of R/I consisting entirely of zero-divisors of M/N. In this case $a_1, ..., a_k$ belong to $S(N) \subseteq S(N[x]) = P[x]$. Consequently $f(x) = \Sigma_{i=0}^{k} a_i x^i \in P[x]$. Therefore, by Proposition 2.7, $mf(x) \in N[x]$ for some $m \in M \backslash N$. This implies that $m + N$ is a nonzero element of M/N which annihilates A. Thus M/N is a McCoy R/I-module. This also shows that N is a P-primal submodule of M.

Conversely, assume that N is a P-primal submodule of M and M/N is a McCoy R/I-module. Suppose that $f(x) = \Sigma_{i=0}^{s} a_i x^i$ and $g(x) = \Sigma_{i=0}^{t} b_i x^i$ are not prime to $N[x]$. There are $m, m' \in M \backslash N$ with $mf(x) \in N[x]$ and $m'g(x) \in N[x]$ by Proposition 2.7. As N is assumed to be P-primal, this shows that $a_0, a_1, ..., a_s, b_0, b_1, ..., b_t$ all belong to P. It follows from [4, Theorem 2.3] that the ideal $T = (a_0 + I, ..., a_s + I, b_0 + I, ..., b_t + I)$ is contained in $P/I = Z_{R/I}(M/N)$. Since M/N is McCoy, there exists $0 \neq m + N \in M/N$ with $(m + N)T = 0$. Therefore $m(f(x) - g(x)) \in N[x]$ with $m \in M \backslash N$, that is $f(x) - g(x)$ is not prime to $N[x]$. So $N[x]$ is a primal submodule of $M[x]$.

Finally we show that the adjoint ideal of $N[x]$ is just $P[x]$. Assume that $g(x) = \Sigma_{i=0}^{k} c_i x^i \in P[x]$. In this case the ideal $U = (c_0 + I, c_0 + I, ..., c_k + I)$ is contained in $P/I = Z_{R/I}(M/N)$ by [4, Theorem 2.3]. Since M/N is McCoy, there is an element $0 \neq m + N \in M/N$ with $(m + N)U = 0$. This implies that $mc_i \in N$ for each $1 \leq i \leq k$ with $m \in M \backslash N$. This shows that $mg(x) \in N[x]$ with $m \in M \backslash N$. Thus $g(x)$ is not prime to $N[x]$, and hence $P[x] \subseteq S(N[x])$. On the other hand we may prove in the same way as in the proof of Proposition 2.8 that $S(N[x]) \subseteq P[x]$, and hence $S(N[x]) = P[x]$. □

Corollary 2.10. *Let R be a commutative ring and let M be an R-module. Then $M[x]$ is a primal $R[x]$-module if and only if M is a primal and a McCoy R-module.*

One can easily check that for every module M over a commutative ring R, $Z_R(M) \subseteq Z_{R[x]}(M[x])$ and it is easy to see that if $M[x]$ is a primal $R[x]$-module then $Z_{R[x]}(M[x]) = Z_R(M)[x]$. Combining this result with Proposition 2.7, we get:

Proposition 2.11. *Let R be a commutative ring and let M be an R-module. Then the following conditions are equivalent:*

(i) $M[x]$ *is a primal* $R[x]$*-module;*

(ii) M *is a super primal* R*-module;*

(iii) $M[x]$ *is a super primal* $R[x]$*-module;*

(iv) $M[x]$ *is a strongly primal* $R[x]$*-module.*

Proposition 2.12. *Let R be a commutative ring and let M be an R-module. If N is an irreducible submodule of M, then M/N is a McCoy R-module.*

Proof. Assume that N is an irreducible submodule of M. Then $N[x]$ is a primal submodule of $M[x]$ by Proposition 2.8. Now the result follows from Theorem 2.9. $\square$

A submodule N of M is called essential if for every nonzero submodule K of M, $N \cap K \neq 0$. The module M is uniform if every nonzero submodule of M is essential in M. By Proposition 2.8, every uniform R-module is coprimal.

Our next result provides a sufficient condition, via Goldie Dimension, under which an R module is super coprimal.

Theorem 2.13. *Let R be a commutative ring. If M is a coprimal R-module of finite Goldie dimension, then M is super coprimal. In particular, M is McCoy.*

Proof. Since M is of finite Goldie dimension, it contains an essential submodule N, $N = U_1 \oplus U_2 \oplus ... \oplus U_k$, where $U_1, U_2, ..., U_k$ are uniform submodules of M. For every element $a \in Z_R(M)$, $Ann_M(a) \neq 0$. Thus essentiality of N implies that $N \cap Ann_M(a) \neq 0$. Thus there are $m_i \in U_i$, not all equal to 0, such that $a(m_1 + m_2 + ... + m_k) = 0$. However, $am_i \in U_i$ implies $am_i = 0$ for all $1 \leq i \leq k$. This shows that if $m_i \neq 0$, then $a \in Z_R(U_i)$. It follows that $Z_R(M) = \cup_{i=1}^{k} Z_R(U_i)$. But, by Proposition 2.8 each U_i ($1 \leq i \leq k$) is a primal R-module; so $Z_R(U_i)$ is a prime ideal of R for every $1 \leq i \leq k$. Now Prime Avoidance Theorem implies that $Z_R(M) = Z_R(U_i)$ for some $1 \leq i \leq k$. Assume that $F = \{a_1, a_2, ..., a_t\}$ is a subset of R with $F \subseteq Z_R(M)$. Consequently $Ann_M(a_j) \cap U_i \neq 0$ for every $1 \leq j \leq t$. As U_i is uniform, $Ann_M(a_1) \cap Ann_M(a_2) \cap ... \cap Ann_M(a_t) \cap U_i \neq 0$. This implies that $Ann_M(F) \neq 0$, as required. $\square$

Corollary 2.14. *Let M be a module over a commutative ring R. Then under each of the following conditions every primal submodule of M is strongly primal (super primal).*

(1) Every quotient of M has finite Goldie dimension.

(2) Every submodule N of M contains a finitely generated submodule T, such that N/T has no maximal submodules.

Proof. It follows from Theorem 2.13 and [2]. □

3 Strongly primal decomposition

In this section we will show that if M is a module over a prüfer domain of finite character R, then every proper submodule of M can be expressed as an intersection of a finite number of strongly primal submodules.

Let R be a commutative ring, T a multiplicatively closed subset of R and M an R-module. Consider the $T^{-1}R$-module $T^{-1}M$, the module of fractions of M with respect to T. In Propositions 3.2 and 3.3 we discuss on the relationship between the strongly primal submodules of M and strongly primal submodules of $T^{-1}M$.

Lemma 3.1. *Let T be a multiplicatively closed subset of a ring R, M be an R-module and N be a strongly primal submodule of M with $S(N) \cap T = \emptyset$. If $m/s \in T^{-1}N$, then $m \in N$.*

Proof. It follows from [4, Lemma 2.5] since every strongly primal submodule is primal. □

Proposition 3.2. *Let T be a multiplicatively closed subset of R, M be an R-module and N be a strongly primal submodule of M with $S(N) \cap T = \emptyset$. Then $T^{-1}N$ is a strongly primal submodule of $T^{-1}M$.*

Proof. Let N be a strongly primal submodule of M with $S(N) \cap T = \emptyset$ and let $\{a/s, b/t\}$ be a subset of $Z_{T^{-1}R}(T^{-1}M/T^{-1}N)$. Then $\{a, b\}$ is a subset of $Z_R(M/N)$ by Lemma 3.1. So there exists an element $m \in M \backslash N$ such that $m + N$ annihilates $\{a, b\}$ in M/N. In this case $m/1 \notin T^{-1}N$ and $m/1 + T^{-1}N$ annihilates the set $\{a/s, b/t\}$ in $T^{-1}M/T^{-1}N$. Consequently $T^{-1}N$ is strongly primal. □

Let R be a commutative ring, M be an R-module and T be a multiplicatively closed set in R. If K is a submodule of $T^{-1}M$, set $K \cap M = f^{-1}(K)$ where $f : M \to T^{-1}M$ is the natural homomorphism given by $m \mapsto m/1$. Clearly, $K \cap M$ is a submodule of M.

Proposition 3.3. *Let T be a multiplicatively closed subset of a ring R, M be an R-module and K be a strongly primal submodule of the $T^{-1}R$-module $T^{-1}M$. Then $K \cap M$ is a strongly primal submodule of M.*

Proof. Let K be a strongly primal submodule of $T^{-1}M$. For every $a \in Z_R(M/(K \cap M)$, there exists $m \notin K \cap M$ with $am \in K \cap M$. In this case $m/1 \notin K$ and $(a/1)(m/1 + K) = 0$ imply that $a/1 \in Z_{T^{-1}R}(T^{-1}M/K)$. Now assume that $\{a, b\}$ is a subset of $Z_R(K \cap M)$. Then the set $\{a/1, b/1\}$ is contained in $Z_{T^{-1}R}(T^{-1}M/K)$. Since K is strongly primal, there exists an element $m/s \notin K$ with $am/s \in K$ and $bm/s \in K$. Then $m \notin K \cap M$ with $am, bm \in K \cap M$. This implies that $m + K \cap M \in M/(K \cap M)$ annihilates the set $\{a, b\}$ in $M/(K \cap M)$. So $K \cap M$ is strongly primal. □

Let R be a commutative ring with identity. R is called a *valuation ring* if the set of ideals of R is linearly ordered by inclusion. R is called an *arithmetical rings* if for every maximal ideal M of R, R_M is a valuation ring. We are interested in the question of when a proper submodule of a module over a commutative ring with identity is the intersection of a finite number of super primal submodules. We shall examine this question for the class of arithmetical rings. Of particular interest is the class of Prüfer domains, namely the arithmetical integral domains. The Prüfer domain R is said to be of finite character if every non-zero element of R is contained but in a finite number of maximal ideals.

Proposition 3.4. *Let R be a valuation ring, and let M be an R-module. Then every proper submodule of M is strongly primal (super primal).*

Proof. Let N be a proper submodule of M. Assume that $a, b \in R$ are two zero-divisors on M/N. We can assume that $b = ra$ for some $r \in R$. Then there exists $m \in M \backslash N$ such that $a(m+N) \in N$ and $b(m+N) = o$, that is the set $\{a, b\}$ has a nonzero annihilator in M/N. Hence N is a strongly primal submodule of M. □

Corollary 3.5. *Let R be an arithmetical ring, P be a maximal ideal of R and M be an R-module. Then every proper submodule of the R_P-modules M_P is strongly primal.*

Let R be a commutative ring, M be an R-module and T be a multiplicatively closed subset of R. For every submodule N of M, let

$$N_T = \{m \in M : sm \in N \text{ for some } s \in T\}.$$

It is clear that N_T is a submodule of M containing N. Also if $(N :_R M) \cap T \neq \emptyset$, then $N_T = M$. N_T is called the T-component of N. Let P be a prime ideal of a commutative ring R and set $T_P = R \backslash P$. Then $m \in N_{T_P}$ if and only if $(N :_R m) \nsubseteq P$. Furthermore $N_{T_P} = N_P \cap M$ where N_P is the localization of N at P. We denote N_{T_P} by $N_{(P)}$.

Theorem 3.6. *Let R be an arithmetical ring, and let M be an R-module. Then, for every non-zero submodule N of M and every maximal ideal P containing $(N :_R M)$, $N_{(P)}$ is a strongly primal submodule of M.*

Proof. Clearly, N_P is a proper submodule of M_P. As R_P is a valuation ring, N_P is a strongly primal submodule of M_P by Proposition 3.5. Now $N_{(P)} = N_P \cap M$ is a strongly primal submodule of M by Proposition 3.3. $\square$

Let R be a commutative ring, M an R-module and N an R-submodule of M. Denote by $Max(R)$ the set of all maximal ideals of R. Then $N = \bigcap_{P \in Max(R)} N_{(P)}$.

Theorem 3.7. *Let R be a Prüfer domain of finite character, M be an R-module and N be a proper submodule of M. Then N is the intersection of a finite number of strongly primal submodules.*

Proof. Since R is a domain of finite character, there are only a finite number of maximal ideals, say $P_1, P_2, ..., P_k$, containing $(N :_R M)$. Also if P is a maximal ideal of R not containing $(N :_R M)$, $N_{(P)} = M$, and if P contains $(N :_R M)$, then $N_{(P)}$ is a strongly primal submodule of M by Theorem 3.6. Therefore $N = N_{(P_1)} \cap N_{(P_2)} \cap ... \cap N_{(P_k)}$ is a decomposition of N as the intersection of strongly primal submodules. $\square$

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