An. Şt. Univ. Ovidius Constanţa Vol. 18(2), 2010, 241–252

SIGNED TOTAL K-DOMINATION NUMBERS OF DIRECTED GRAPHS

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Abstract

Let $k \geq 1$ be an integer, and let $D = (V, A)$ be a finite and simple digraph in which $d_D^-(v) \geq k$ for all $v \in V$. A function $f : V \longrightarrow \{-1, 1\}$ is called a signed total k-dominating function (STkDF) if $f(N^-(v)) \geq k$ for each vertex $v \in V$. The weight $w(f)$ of f is defined by $w(f) = \sum_{v \in V} f(v)$. The signed total k-domination number for a digraph D is $\gamma_{kS}^t(D) = \min\{w(f) \mid f \text{ is a STkDF of } D\}$. In this paper, we initiate the study of signed total k-domination in digraphs and we present some sharp lower bounds for $\gamma_{kS}^t(D)$ in terms of the order, the maximum and minimum outdegree and indegree and the chromatic number.

1 Introduction

Let D be a finite simple digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. A digraph without directed cycles of length 2 is an *oriented graph*. The order $n = n(D)$ of a digraph D is the number of its vertices. We write $d_D^+(v)$ for the outdegree of a vertex v and $d_D^-(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For every vertex $v \in V$, let $N_D^-(v)$ be the set consisting of all vertices of D from which arcs go into v. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$

Key Words: signed total k-dominating function, signed total k-domination number, directed graph.
2010 Mathematics Subject Classification: 05C20, 05C69, 05C45.
Received: January, 2010
Accepted: September, 2010

and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v. Consult [7] for the notation and terminology which are not defined here. For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$ the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. In this note, we consider only finite simple digraphs D.

Although domination and other related concepts have been extensively studied for undirected graphs, the respective analogue on digraphs have not received much attention. A survey of results on domination in directed graphs by Ghoshal, Lasker and Pillone is found in chapter 15 of Haynes et al., [1], but most of the results in this survey chapter deal with the concepts of kernels and solutions in digraphs and on domination in tournaments.

The concept of the signed total k-domination number $\gamma^t_{kS}(G)$, of an undirected graph is introduced by Wang in [5]. In the special case when $k = 1$, $\gamma^t_{kS}(G)$ is the *signed total domination number* introduced in [8] and investigated in [2]. Here we transfer this concept to digraphs, and then we present some sharp lower bounds on signed total k-domination number of digraphs.

Let $k \geq 1$ be an integer and let $D = (V, A)$ be a finite simple digraph such that $\delta^-(D) \geq k$. A signed total k-dominating function (abbreviated STkDF) of D is a function $f : V \longrightarrow \{-1, 1\}$ such that $f(N^-(v)) \geq k$ for every $v \in V$. The signed total k-domination number for a digraph D is

$$\gamma^t_{kS}(D) = \min\{w(f) \mid f \text{ is a STkDF of } D\}.$$

A $\gamma^t_{kS}(D)$-function is a STkDF of D of weight $\gamma^t_{kS}(D)$. As the assumption $\delta^-(D) \geq k$ is clearly necessary, we will always assume that when we discuss $\gamma^t_{kS}(D)$ all digraphs involved satisfy $\delta^-(D) \geq k$ and thus $n(D) \geq k + 1$. In the special case when $k = 1$, $\gamma^t_{kS}(D)$ is the *signed total domination number* investigated in [4]. For any STkDF f of D we define $P = \{v \in V \mid f(v) = 1\}$ and $M = \{v \in V \mid f(v) = -1\}$.

We make use of the following results and observations in this paper.

Theorem A. (Szekeres-Wilf [3]) For any graph G,

$$\chi(G) \leq 1 + \max\{\delta(H) \mid H \text{ is a subgraph of } G\}.$$

Theorem B. (Sheikholeslami [4]) Let D be a digraph of order n with $\delta^-(D) \geq 1$, and let $r \geq 1$ be an integer such that $\delta^+(D) \geq r$. Then

$$\gamma^t_{1S}(D) \geq 2(\chi(G) + r - \Delta(G)) - n,$$

where G is the underlying graph of D.

Observation 1. For any digraph D, $\gamma^t_{kS}(D) \equiv n \pmod 2$.

Proof. Let f be a $\gamma^t_{kS}(D)$-function . Obviously $n = |P| + |M|$ and $\gamma^t_{kS}(D) = |P| - |M|$. Therefore, $n - \gamma^t_{kS}(D) = 2|M|$ and the result follows. □

Observation 2. Let u be a vertex of indegree at most $k+1$ in D. If f is a STkDF on D, then f assigns 1 to each vertex of $N^-_D(u)$.

Proof. Since $f(N^-_D(u)) \geq k$ and $|N^-_D(u)| \leq k+1$, the results follows. □

Observation 3. Let D be a digraph of order n. Then $\gamma^t_{kS}(D) = n$ if and only if $k \leq \delta^-(D) \leq k+1$ and for each $v \in V(D)$ there exists a vertex $u \in N^+(v)$ with indegree at most $k+1$.

Proof. If $k \leq \delta^-(D) \leq k+1$ and for each $v \in V(D)$ there exists a vertex $u \in N^+(v)$ with indegree at most $k+1$, then trivially $\gamma^t_{ks}(D) = n$.

Conversely, assume that $\gamma^t_{kS}(D) = n$. By assumption $k \leq \delta^-(D)$. Suppose to the contrary that $\delta^-(D) > k+1$ or that there exists a vertex $v \in V(D)$ such that $d^-(u) \geq k+2$ for each $u \in N^+(v)$. If $\delta^-(D) > k+1$, define $f : V(D) \to \{-1, 1\}$ by $f(v) = -1$ for some fixed v and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Obviously, f is a signed total k-dominating function of D with weight less than n, a contradiction. Thus $k \leq \delta^-(D) \leq k+1$. Now let $v \in V(D)$ and $d^-(u) \geq k+2$ for each $u \in N^+(v)$. Define $f : V(D) \to \{-1, 1\}$ by $f(v) = -1$ and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Again, f is a signed total k-dominating function of D, a contradiction. This completes the proof. □

Corollary 4. If D is a digraph with $\Delta^-(D) \leq k+1$, then $\gamma^t_{kS}(D) = n(D)$.

Next we determine the exact value of the signed total k-domination number for particular types of tournaments. Let n be an odd positive integer. We have $n = 2r+1$, where r is a positive integer. We define the circulant tournament $\mathrm{CT}(n)$ with n vertices as follows. The vertex set of $\mathrm{CT}(n)$ is $V(\mathrm{CT}(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$. For each i, the arcs are going from u_i to $u_{i+1}, u_{i+2}, \ldots, u_{i+r}$, where the indices are taken modulo n.

Proposition 5. Let $n = 2r+1$ where r is a positive integer and let $1 \leq k \leq r$ be an integer. Then

$$\gamma^t_{kS}(\mathrm{CT}(n)) = \begin{cases} 2k+1 & \text{if } r \equiv k \pmod 2 \\ 2k+3 & \text{if } r \equiv k+1 \pmod 2. \end{cases}$$

Proof. Let f be a $\gamma^t_{kS}(CT(n))$-function. Without loss of generality, we may assume $f(u_0) = 1$. Consider the sets $N^-(u_0)$ and $N^-(u_{r+1})$. Since f is a STkDF on $\mathrm{CT}(n)$, we have $f(N^-(u_0)) \geq k$, $f(N^-(u_{r+1}) \geq k$ if $r \equiv k \pmod 2$

and $f(N^-(u_0)) \geq k+1$, $f(N^-(u_{r+1}) \geq k+1$ when $r \equiv k+1 \pmod 2$. Therefore

$$\begin{array}{rcl} \omega(f) & = & f(V(\mathrm{CT}(n))) \\ & = & f(u_0) + f(N^-(u_0)) + f(N^-(u_{r+1}) \\ & \geq & \left\{ \begin{array}{ll} 2k+1 & \text{if } r \equiv k \pmod 2 \\ 2k+3 & \text{if } r \equiv k+1 \pmod 2. \end{array} \right. \end{array}$$

This implies that

$$\gamma^t_{kS}(\mathrm{CT}(n)) \geq \left\{ \begin{array}{ll} 2k+1 & \text{if } r \equiv k \pmod 2 \\ 2k+3 & \text{if } r \equiv k+1 \pmod 2. \end{array} \right.$$

If $n = 3, 5$, then obviously $\gamma^t_{kS}(\mathrm{CT}(n)) = n$. If $k = r$ or $k = r-1$, then obviously $\gamma^t_{kS}(\mathrm{CT}(n)) = n$. Thus we assume that $n \geq 7$ and $k \leq r-2$. Suppose now that $s = \lfloor \frac{r-k}{2} \rfloor$, $V^- = \{u_1, \ldots, u_s, u_{r+1}, \ldots, u_{r+s}\}$ and $V^+ = V(\mathrm{CT}(n)) - V^-$. Define $f : V(\mathrm{CT}(n)) \to \{-1, 1\}$ by $f(u_0) = 1$, $f(v) = 1$ if $v \in V^+$ and $f(v) = -1$ when $v \in V^-$. For any vertex $v \in V(\mathrm{CT}(n))$ we have $|N^-(v)| = r$ and $|N^-(v) \cap V^-| \leq s$. Therefore $f(N^-(v)) = r - 2s \geq k$ and so f is a STkDF on $\mathrm{CT}(n)$. Now we have

$$\gamma^t_{kS}(\mathrm{CT}(n)) \leq \omega(f) = \left\{ \begin{array}{ll} 2k+1 & \text{if } r \equiv k \pmod 2 \\ 2k+3 & \text{if } r \equiv k+1 \pmod 2. \end{array} \right.$$

This completes the proof. □

2 Main results

In this section we present some sharp lower bounds for $\gamma^t_{kS}(D)$ in terms of the order, the maximum degree and the chromatic number of D. Recall that the complement of a graph G is denoted $\overline{G}$.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. We denote the associated digraph $D(K_n)$ of the complete graph K_n of order n by K^*_n.

Theorem 6. Let $k \geq 1$ be an integer and let D be a digraph of order n with $\delta^-(D) \geq k$. Then

$$\gamma^t_{kS}(D) \geq 2(k+1) - n.$$

Furthermore, the bound is sharp for the digraph $D = K^*_{k+1} \vee \overline{K_{n-(k+1)}}$ in which the edges are oriented from $V(K^*_n)$ to $V(\overline{K_{n-(k+1)}})$.

Proof. Let f be a STkDF on D and let $v \in V$. Then f assigns 1 to at least k vertices in $N_D^-(v)$, say $u_1, u_2, \ldots, u_k$, and also f assigns 1 to at least a vertex in $N_D^-(u_1) - \{u_2, u_3, \ldots, u_k\}$. Therefore $|M| \leq n - (k+1)$ which implies that

$$\gamma_{kS}^t(D) = |P| - |M| \geq (k+1) - (n - (k+1)) = 2(k+1) - n,$$

as desired.

Let $D = K_{k+1}^* \vee \overline{K_{n-(k+1)}}$ in which the edges are oriented from $V(K_n^*)$ to $V(\overline{K_{n-(k+1)}})$. Define $f : V(D) \longrightarrow \{-1, 1\}$ by $f(v) = 1$ if $v \in V(K_{k+1}^*)$, and $f(v) = -1$ if $v \in V(\overline{K_{n-(k+1)}})$. Obviously, f is a STkDF of D and $w(f) = 2(k+1) - n$. Hence, $\gamma_{kS}^t(D) = 2(k+1) - n$. This completes the proof. □

For oriented graphs we will present a better lower bound on the signed total k-domination number.

Theorem 7. Let $k \geq 1$ be an integer, and let D be an oriented graph of order n with $\delta^-(D) \geq k$. Then

$$\gamma_{kS}^t(D) \geq 2(2k+1) - n,$$

and this bound is sharp.

Proof. Let f be a STkDF on D and let $v \in V$. Each vertex $v \in P$ has at least k in-neighbors in P. This implies that

$$\frac{|P|(|P|-1)}{2} \geq |A(D[P])| \geq k|P|$$

and thus $|P| \geq 2k+1$. Therefore $|M| \leq n-(2k+1)$, and we obtain the desired lower bound as follows

$$\gamma_{kS}^t(D) = |P| - |M| \geq (2k+1) - (n - (2k+1)) = 2(2k+1) - n.$$

Let $\{u_1, u_2, \ldots, u_{2k+1}\}$ be the vertex set of an arbitrary k-regular tournament T_{2k+1}. Now let D consists of T_{2k+1} and $n - (2k+1)$ further vertices $v_1, v_2, \ldots, v_{n-(2k+1)}$ such that $u_i \to v_j$ for each $i \in \{1, 2, \ldots, 2k+1\}$ and each $j \in \{1, 2, \ldots, n-(2k+1)\}$. Define $f : V(D) \longrightarrow \{-1, 1\}$ by $f(x) = 1$ if $x \in \{u_1, u_2, \ldots, u_{2k+1}\}$ and $f(x) = -1$ if $x \in \{v_1, v_2, \ldots, v_{n-(2k+1)}\}$. Obviously, f is a STkDF of D and $w(f) = 2(2k+1)-n$. Hence $\gamma_{kS}^t(D) = 2(2k+1)-n$. □

Theorem 8. Let $k \geq 1$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq k$. Then

$$\gamma_{kS}^t(D) \geq n \left(\frac{2\left\lceil \frac{\delta^-(D)+k}{2} \right\rceil - \Delta^+(D)}{\Delta^+(D)} \right).$$

Proof. Let f be a $\gamma_{kS}^t(D)$-function, and let s be the number of arcs from the set P to the set M. If x is an arbitrary vertex of D, then $f(N^-(x)) \geq k$ implies that $|E(P,x)| \geq |E(M,x)| + k$. Thus

$$\delta^-(D) \leq d^-(x) = |E(P,x)| + |E(M,x)| \leq 2|E(P,x)| - k,$$

and we obtain $|E(P,x)| \geq \left\lceil \frac{\delta^-(D)+k}{2} \right\rceil$ for each $x \in V(D)$. Hence we deduce that

$$s = \sum_{x \in M} |E(P,x)| \geq \sum_{x \in M} \left\lceil \frac{\delta^-(D)+k}{2} \right\rceil = |M| \left\lceil \frac{\delta^-(D)+k}{2} \right\rceil. \qquad (1)$$

Since $|E(D[P])| = \sum_{y \in P} |E(P,y)| \geq |P| \left\lceil \frac{\delta^-(D)+k}{2} \right\rceil$, it follows that

$$\begin{aligned} s &= \sum_{y \in P} d^+(y) - |E(D[P])| \\ &\leq \sum_{y \in P} d^+(y) - |P| \left\lceil \frac{\delta^-(D)+k}{2} \right\rceil \\ &\leq |P|\Delta^+(D) - |P| \left\lceil \frac{\delta^-(D)+k}{2} \right\rceil. \qquad (2) \end{aligned}$$

Inequalities (1) and (2) imply that

$$|M| \leq \frac{|P|\Delta^+(D) - |P| \left\lceil \frac{\delta^-(D)+k}{2} \right\rceil}{\left\lceil \frac{\delta^-(D)+k}{2} \right\rceil}.$$

Since $\gamma_{kS}^t(D) = |P| - |M|$ and $n = |P| + |M|$, the last inequality leads to

$$\begin{aligned} \gamma_{kS}^t(D) &\geq |P| - \frac{|P|\Delta^+(D) - |P| \left\lceil \frac{\delta^-(D)+k}{2} \right\rceil}{\left\lceil \frac{\delta^-(D)+k}{2} \right\rceil} \\ &= \left(\frac{n + \gamma_{kS}^t(D)}{2} \right) \frac{2\left\lceil \frac{\delta^-(D)+k}{2} \right\rceil - \Delta^+(D)}{\left\lceil \frac{\delta^-(D)+k}{2} \right\rceil}, \end{aligned}$$

and this yields to the desired result immediately. □

To see the sharpness of the last result, let $D = K_n^*$. If $k = n-1$ or $k = n-2$, then Theorem 8 leads to $\gamma_{kS}^t(D) \geq n$ and thus $\gamma_{kS}^t(D) = n$.

If $D(G)$ is the associate digraph of a graph G, then $N_{D(G)}^-(v) = N_G(v)$ for each $v \in V(G) = V(D(G))$. Thus the following useful remark is valid.

Remark 9. *If $D(G)$ is the associate digraph of a graph G, then $\gamma^t_{kS}(D(G)) = \gamma^t_{kS}(G)$.*

There are a lot of interesting applications of Remark 9, as for example the following two results.

Corollary 10. Let $k \geq 1$ be an integer, and let G be a graph of order n with $\delta(G) \geq k$. Then

$$\gamma^t_{kS}(G) \geq n\left(\frac{2\left\lceil\frac{\delta(G)+k}{2}\right\rceil - \Delta(G)}{\Delta(G)}\right).$$

Proof. Since $\delta(G) = \delta^-(D(G))$, $\Delta(G) = \Delta^+(D(G))$ and $n = n(D(G))$, it follows from Theorem 8 and Observation 9 that

$$\begin{aligned}\gamma^t_{kS}(G) = \gamma^t_{kS}(D(G)) &\geq \frac{2\left\lceil\frac{\delta^-(D(G))+k}{2}\right\rceil - \Delta^+(D(G))}{\Delta^+(D(G))}\cdot n\\ &= \left(\frac{2\left\lceil\frac{\delta(G)+k}{2}\right\rceil - \Delta(G)}{\Delta(G)}\right)n.\end{aligned}$$

□

Corollary 11. (Wang [6] 2010) Let $k \geq 1$ be an integer, and let G be an r-regular graph of order n with $r \geq k$. Then $\gamma^t_{kS}(G) \geq kn/r$ if $k + r$ is even and $\gamma^t_{kS}(G) \geq (k+1)n/r$ if $k + r$ is odd.

The special case $k = 1$ in Corollary 11 was given by Zelinka [8] in 2001. Counting the arcs from M to P, we next prove an analogue to Theorem 8

Theorem 12. Let $k \geq 1$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq k$ and $\delta^+(D) \geq 1$. Then

$$\gamma^t_{kS}(D) \geq n\left(\frac{\delta^+(D) - 2\left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor}{\delta^+(D)}\right).$$

Proof. Let f be a $\gamma^t_{kS}(D)$-function, and let s be the number of arcs from M to P. If x is an arbitrary vertex of D, then

$$\Delta^-(D) \geq d^-(x) = |E(P,x)| + |E(M,x)| \geq 2|E(M,x)| + k$$

and thus $|E(M,x)| \leq \left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor$ for each $x \in V(D)$. Hence we deduce that

$$s = \sum_{x\in P}|E(M,x)| \leq |P|\left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor. \quad (3)$$

and

$$s=\sum_{y\in M}d^+(y)-|E(D[M])|\geq |M|\delta^+(D)-|M|\left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor. \qquad (4)$$

Inequalities (3) and (4) imply that

$$|P|\geq \frac{|M|\delta^+(D)-|M|\left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor}{\left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor}.$$

Since $\gamma^t_{kS}(D)=|P|-|M|$ and $n=|P|+|M|$, the last inequality leads to

$$\begin{aligned}\gamma^t_{kS}(D) &\geq \frac{|M|\delta^+(D)-|M|\left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor}{\left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor}-|M|\\ &= \left(\frac{n-\gamma^t_{kS}(D)}{2}\right)\frac{\delta^+(D)-2\left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor)}{\left\lfloor\frac{\Delta^-(D)-k}{2}\right\rfloor},\end{aligned}$$

and this yields to the desired result immediately. □

Using Observation 9 and Theorem 12, we obtain an analogue to Corollary 10, and this also leads to Corollary 11.

Theorem 13. Let $k\geq 1$ be an integer, and let D be a digraph of order n with $\delta^-(D)\geq k$. Then

$$\gamma^t_{kS}(D)\geq \frac{\delta^+(D)+2k-\Delta^+(D)}{\delta^+(D)+\Delta^+(D)}\cdot n.$$

Proof. If f is a $\gamma^t_{kS}(D)$-function, then

$$\begin{aligned}nk &= \sum_{x\in V}k\leq \sum_{x\in V}f(N^-(x))=\sum_{x\in V}d^+(x)f(x)\\ &= \sum_{x\in P}d^+(x)-\sum_{x\in M}d^+(x)\\ &\leq |P|\Delta^+(D)-|M|\delta^+(D)\\ &= |P|(\Delta^+(D)+\delta^+(D))-n\delta^+(D).\end{aligned}$$

This implies that

$$|P|\geq \frac{n(\delta^+(D)+k)}{\delta^+(D)+\Delta^+(D)},$$

and hence we obtain the desired bound as follows

$$\begin{aligned}\gamma^t_{kS}(D) &= |P| - |M| = 2|P| - n \\ &\geq \frac{2n(\delta^+(D) + k)}{\delta^+(D) + \Delta^+(D)} - n \\ &= \left(\frac{\delta^+(D) + 2k - \Delta^+(D)}{\delta^+(D) + \Delta^+(D)}\right) n.\end{aligned}$$

□

Using Remark 9, we obtain the following analogue for graphs. The special case $k = 1$ is close to a result by Henning [2] (cf. Theorem 4 in [2]).

Corollary 14. If $k \geq 1$ is an integer, and G is a graph of order n with $\delta(G) \geq k$, then

$$\gamma^t_{kS}(G) \geq \left(\frac{\delta(G) + 2k - \Delta(G)}{\delta(G) + \Delta(G)}\right) n.$$

Theorem 15. Let $k \geq 1$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq k$. Then

$$\gamma^t_{kS}(D) \geq \frac{kn + |A(D)| - n\Delta^+(D)}{\Delta^+(D)}.$$

Proof. If f is a $\gamma^t_{kS}(D)$-function, then

$$\begin{aligned}nk &\leq \sum_{x \in V} f(N^-(x)) = \sum_{x \in V} d^+(x) f(x) \\ &= \sum_{x \in P} d^+(x) - \sum_{x \in M} d^+(x) \\ &= 2\sum_{x \in P} d^+(x) - \sum_{x \in V} d^+(x) \\ &\leq 2|P|\Delta^+(D) - |A(D)|.\end{aligned}$$

This implies that

$$|P| \geq \frac{kn + |A(D)|}{2\Delta^+(D)},$$

and hence we obtain the desired bound as follows

$$\begin{aligned}\gamma^t_{kS}(D) &= |P| - |M| = 2|P| - n \\ &\geq \frac{kn + |A(D)|}{\Delta^+(D)} - n \\ &= \frac{kn + |A(D)| - n\Delta^+(D)}{\Delta^+(D)}.\end{aligned}$$

□

Theorem 16. Let $r \geq k \geq 1$ be integers, and let D be a digraph of order n such that $\delta^-(D) \geq k$ and $\delta^+(D) \geq r$. Then

$$\gamma^t_{kS}(D) \geq 2(\chi(G) + r + k - 1 - \Delta(G)) - n,$$

where G is the underlying graph of D.

Proof. By Theorem B we may assume $k \geq 2$. Since $d^-(x) \geq k$ for each $x \in V(D)$, we have $\Delta(G) \geq r + k \geq 2k$. Let f be a $\gamma^t_{kS}(D)$-function. First let $\Delta(G) = 2k$. Then $d^-(x) = d^+(x) = k$. It follows from Observation 3 that $\gamma^t_{kS}(D) = n$ and the result follows.

Now, let $\Delta(G) \geq 2k + 1$. Suppose $\alpha = \frac{\Delta(G)-r-k}{2}$. We claim that $r \leq \Delta(G)-k-1$. Let, to the contrary, $r \geq \Delta(G)-k$. Since $d^+(x)+d^-(x) \leq \Delta(G)$, by the assumption we have $d^-(x) \leq k$ for each $x \in V(D)$. Thus

$$n(\Delta(G) - k) \leq \sum_{x \in V(D)} d^+(x) = \sum_{x \in V(D)} d^-(x) \leq nk,$$

which implies $\Delta(G) \leq 2k$, a contradiction. Therefore $\alpha > 0$. For each $x \in M$, $|E(P, x)| \geq |E(M, x)| + k$ and so

$$\Delta(G) \geq \deg(x) = |E(P, x)| + |E(M, x)| + d^+(x) \geq r + 2|E(M, x)| + k,$$

which implies $|E(M, x)| \leq \alpha$. Let $H = D[M]$ be the subdigraph induced by M and let $H' = G[M]$ be the underlying graph of H.

Suppose H_1 is an induced subgraph of H. Then $d^-_{H_1}(x) \leq |E(M, x)| \leq \alpha$ for each $x \in V(H_1)$, and hence

$$\sum_{x \in V(H_1)} d^+_{H_1}(x) = \sum_{x \in V(H_1)} d^-_{H_1}(x) \leq \alpha|V(H_1)|.$$

Therefore there exists a vertex $x \in V(H_1)$ such that $d^+_{H_1}(x) \leq \alpha$. This implies that $\delta(H'_1) \leq 2\alpha$, where H'_1 is the underlying graph of H_1. By Theorem A,

$$\begin{aligned}
\chi(H') &\leq 1 + \max\{\delta(H'') \mid H'' \text{ is a subgraph of } H'\} \\
&= 1 + \max\{\delta(H'_1) \mid H'_1 \text{ is an induced subgraph of } H'\} \\
&\leq 1 + 2\alpha.
\end{aligned}$$

Since $2|P| - n = \gamma^t_{kS}(D)$, it follows that

$$\chi(G) \leq \chi(G[P]) + \chi(G[M]) \leq |P| + 1 + 2\alpha = 1 + 2\alpha + \frac{n + \gamma^t_{kS}(D)}{2}.$$

Thus

$$\gamma_{kS}^t(D) \geq 2(\chi(G) + k + r - 1 - \Delta(G)) - n,$$

as required. □

Acknowledgement

This research was in part supported by a grant from IPM (No. 88050041).

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