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WAVELET PACKETS ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract

The objective of this paper is to construct wavelet packets associated with multiresolution analysis on locally compact Abelian groups. Moreover, from the collection of dilations and translations of the wavelet packets, we characterize the subcollections which form an orthonormal basis for $L^2(G)$.

1 Introduction

The classic multiresolution analysis (MRA) as introduced by Mallat in [24], is simply an increasing sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ such that $\cap_{j\in\mathbb{Z}}V_j = \{0\}$, $\cup_{j\in\mathbb{Z}}V_j$ is dense in $L^2(\mathbb{R})$, and which satisfies $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of a function $\varphi, \{\varphi(x-k) : k \in \mathbb{Z}\}$ represents a complete orthonormal system for V_0. The function φ is called the *scaling function* or the *father wavelet*. In recent years, the notion of MRA and wavelets have been generalized in many different settings [5, 6, 12, 19, 24]. In his papers, Lang [13–15] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathcal{C}$ by following procedures of Mallat [18] and Daubechies [6] via scaling fillers and these wavelets turn out to be certain lacunary Walsh series on the real line. Subsequently, Farkov [7] extended the results of Lang in the wavelet analysis on the Cantor

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dyadic group $\mathcal{C}$ to the locally compact Abelian group G which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. Recently, Protasov and Farkov [20] have provided the construction of the dyadic compactly supported wavelets in $L^2(\mathbb{R}^+)$ and studied compactly supported non-trivial solutions of the refinement equations generating multiresolution analysis in $L^2(\mathbb{R}^+)$. More results in this direction can be found in [1, 8, 9] and the references therein.

For a given multiresolution analysis and the corresponding orthonormal wavelet basis of $L^2(\mathbb{R})$, wavelet packets were constructed by Coifman, Meyer and Wickerhauser [4]. This construction is an important generalization of wavelet analysis. Wavelet packet functions consist of a rich family of building block functions and are localized in time, but offer more flexibility than wavelets in representing different kinds of signals. The power of wavelet packets lies in the fact that we have much more freedom in selecting which basis functions are to be used to represent the given function. The above cited work is a good source of the basis selection criteria and applications to image processing. The concept of wavelet packet was subsequently generalized to $\mathbb{R}^d$ by taking tensor products, whereas Shen [23] formulated non-tensor products wavelets in $L^2(\mathbb{R}^s)$. Other notable generalizations are the non-orthogonal version of wavelet packets (see Chui and Li [3]) and the wavelet frame packets (see Chen [2]) on $\mathbb{R}$ for dilation 2. Long and Chen [16, 17] investigated the orthogonal, biorthogonal and frame packets on $\mathbb{R}^d$ for dyadic dilation.

The main tool in obtaining wavelet packets is the so-called splitting trick, which is a well known technique in constructing wavelet bases. So, the main purpose of this paper is to give a construction of wavelet packets associated with p-multiresolution analysis on locally compact Abelian groups using the splitting trick for wavelets.

This paper is organized as follows: In Section 2, we state some basic preliminaries, notations and definitions including the Walsh-Fourier transform, Walsh functions and polynomials. In Section 3, we establish necessary and sufficient conditions for shifts of a function $\varphi \in L^2(G)$ to be an orthonormal system for $L^2(G)$ (*splitting lemma*). In Theorem 3.4, we prove that the integer translates of the basic wavelet packets form an orthonormal basis for $L^2(G)$. Moreover, from the collection of dilations and translations of the wavelet packets, we characterize the sub-collections which form orthonormal basis for $L^2(G)$ in Theorem 3.7.

2 Preliminaries

All the definitions and properties in this section can be found in [7, 10, 11, 22]. Let p be a fixed natural number greater than 1. Let G be a locally compact Abelian group of the form

$$x = (x_j) = (..., 0, 0, x_k, x_{k+1}, x_{k+2}, ...),$$

where $x_j \in \{0, 1, ..., p-1\}$ for $j \in \mathbb{Z}$ and $x_j = 0$ for $j < k = k(x)$. The group operation on G is denoted by $\oplus$ and is defined as coordinatewise addition modulo p. The topology on G is defined by complete system of neighbourhoods of zero as

$$U_\ell = \big\{(x_j) \in G : x_j = 0 \text{ for } j \leq \ell\big\}, \quad \ell \in \mathbb{Z}.$$

Clearly each neighbourhood U_ℓ is a subgroup of G, $U_{\ell+1} \subset U_\ell$ for $\ell \in \mathbb{Z}$ and $\bigcap U_\ell = \{0\}$. Set $U = U_0$ and denote by $\ominus$ the operation inverse to $\oplus$.

For $1 \leq q \leq \infty$, we denote $L^q(G)$, as the Lebesgue spaces of Borel's subsets of G defined by the Haar measure μ with $\mu(U_0) = \mu(U) = 1$. Let G^* be the dual group of G consisting of all sequences of the form

$$\xi = (\xi_j) = (..., 0, 0, \xi_k, \xi_{k+1}, \xi_{k+2}, ...),$$

where $\xi_j \in \{0, 1, ..., p-1\}$ for $j \in \mathbb{Z}$ and $\xi_j = 0$ for $j < k = k(x)$. The operations addition modulo p, neighbourhoods U^*_ℓ and the Haar measure μ^* for G^* are introduced as above for G. Suppose that H is a discrete subgroup of G of the form

$$H = \big\{(x_j) \in G : x_j = 0 \text{ for } j > 0\big\}.$$

Then it is easy to verify that the quotient group $H/A(H)$ contains p elements and the annihilator $H^\perp$ of H consists of all sequences $\{\xi_j\}$ of G^* which satisfy $\xi_j = 0$ for $j > 0$, where A is an automorphism of G (see [10]). Therefore, for $x \in G$ and $\xi \in G^*$, we have

$$\chi(x, \xi) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} x_{-j}\xi_{j-1}\right). \tag{2.1}$$

Let $\mathbb{R}^+ = [0, \infty)$. We define a map $\lambda : G \to \mathbb{R}^+$ by

$$\lambda(x) = \sum_{j\in\mathbb{Z}} x_j p^{-j}, \quad x \in G.$$

Note that λ takes the subgroup U onto the interval $[0, 1]$ and defines an isomorphism of the measure spaces (G, μ) and $(\mathbb{R}^+, \nu)$, where ν is the Lebesgue measure on $\mathbb{R}^+$. Also, the image of H under λ is the set of non-negative integers: $\lambda(H) = \mathbb{Z}^+$. Thus, for every $\alpha \in \mathbb{Z}^+$, let $h_{[\alpha]}$ denote the element of H such that $\lambda(h_{[\alpha]}) = \alpha$. In a similar fashion, we can define a map $\lambda^* : G^* \to \mathbb{R}^+$, the automorphism $B \in \text{Aut}G^*$, the subgroup U^* and the element $\xi_{[\alpha]}$ of $H^\perp$ as we did for G. We note that $\chi(Ax, \xi) = \chi(x, B\xi)$ for all $x \in G$, $\xi \in G^*$.

We now, define the generalized Walsh functions $\{W_\alpha\}$ for the group G as

$$W_\alpha(x) = \chi(x, \xi_{[\alpha]}), \qquad \alpha \in \mathbb{Z}^+, \quad x \in G.$$

and for G^* by

$$W^*_\alpha(\xi) = \chi(h_{[\alpha]}, \xi), \qquad \alpha \in \mathbb{Z}^+, \quad \xi \in G^*.$$

It is shown in [11] that the systems $\{W_\alpha\}_{\alpha\in\mathbb{Z}^+}$ and $\{W^*_\alpha\}_{\alpha\in\mathbb{Z}^+}$ are orthonormal basis of $L^2(U)$ and $L^2(U^*)$ respectively.

The Walsh-Fourier transform of a function $f \in L^1(G)$ is defined by

$$\hat{f}(\xi) = \int_G f(x)\overline{\chi(x,\xi)}\, d\mu(x),$$

where $\chi(x, \xi)$ is given by (2.1). The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [11, 22]). In particular, if $f \in L^2(G)$, then $\hat{f} \in L^2(G)$ and

$$\|\hat{f}\|_{L^2(G)} = \|f\|_{L^2(G)}.$$

As in [7, 8] we note, that for any function $\varphi \in L^2(G)$, we have

$$\begin{aligned}
\int_G \varphi(x)\overline{\varphi(x \ominus h)}\, d\mu(x) &= \int_{G^*} |\hat{\varphi}(\xi)|^2\, \overline{\chi(h,\xi)}\, d\mu^*(\xi) \\
&= \sum_{h^*\in H^\perp} \int_{U^*\oplus h^*} |\hat{\varphi}(\xi \ominus h^*)|^2\, \overline{\chi(h, \xi \ominus h^*)}\, d\mu^*(\xi) \\
&= \int_{U^*} \overline{\chi(h,\xi)} \sum_{h^*\in H^\perp} |\hat{\varphi}(\xi \ominus h^*)|^2\, d\mu^*(\xi).
\end{aligned}$$

Therefore, the necessary and sufficient condition for the system $\{\varphi(. \ominus h) : h \in H\}$ to be orthonormal in $L^2(G)$ is that

$$\sum_{h^* \in H^\perp} \left| \hat{\varphi}(\xi \ominus h^*) \right|^2 = 1 \qquad a.e.\ \xi \in G^*. \tag{2.2}$$

Now, we recall the definition of multiresolution analysis in $L^2(G)$ and some of its properties.

Definition 2.1. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(G)$ is called a multiresolution analysis of $L^2(G)$ if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(G)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(iii) $f \in V_j$ if and only if $f(A.) \in V_{j+1}$ for all $j \in \mathbb{Z}$,

(iv) there exists a function φ in $L^2(G)$, called the scaling function, such that the system of functions $\{\varphi(. \ominus h) : h \in H\}$ forms an orthonormal basis for V_0.

Given a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$, we define another sequence $\{W_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(G)$ by $W_j = V_{j+1} \ominus V_j,\ j \in \mathbb{Z}$. These subspaces inherit the scaling property of $\{V_j\}$, namely

$$f \in W_j \quad \text{if and only if} \quad f(A.) \in W_{j+1}. \tag{2.3}$$

Moreover, the subspaces $\{W_j\}$ are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(G) = \bigoplus_{j\in\mathbb{Z}} W_j \tag{2.4}$$

$$= V_0 \oplus \Big(\bigoplus_{j\geq 0} W_j \Big). \tag{2.5}$$

A set of functions $\{\psi_1, \psi_2, ..., \psi_{p-1}\}$ in $L^2(G)$ is said to be a set of basic wavelets associated with the multiresolution analysis if the collection $\{\psi_\ell(. \ominus h) : 1 \leq \ell \leq p-1,\ h \in H\}$ forms an orthonormal basis for W_0.

Now, in view of (2.3) and (2.4), it is clear that if $\{\psi_1, \psi_2, ..., \psi_{p-1}\}$ is a basic set of wavelets, then

$$\Big\{ p^{j/2} \psi_\ell(A^j. \ominus h) : j \in \mathbb{Z},\ h \in H, 1 \leq \ell \leq p-1 \Big\}$$

forms an orthonormal basis for $L^2(G)$ (see [7, 8, 24]). If we take $p = 2$, then only one wavelet ψ can be obtained and the system $\{2^{j/2}\psi(A^j. \ominus h) : j \in \mathbb{Z},\ h \in H\}$ forms an orthonormal basis for $L^2(G)$ (see [15]).

We denote $\psi_0 = \varphi$, the scaling function, and consider $p-1$ functions $\psi_\ell,\ 1 \leq \ell \leq p-1$ in W_0 as possible candidates for wavelets. Since $p^{-1}\psi_\ell(A^{-1}.) \in V_{-1} \subset V_0$, it follows from property (iv) of MRA that for each $\ell,\ 0 \leq \ell \leq p-1$, there exists a sequence $\{a^\ell_\alpha : \alpha \in \mathbb{Z}^+\}$ with $\sum_{\alpha\in\mathbb{Z}^+} |a^\ell_\alpha|^2 < \infty$ such that

$$p^{-1}\psi_\ell\left(A^{-1}x\right) = \sum_{\alpha\in\mathbb{Z}^+} a^\ell_\alpha \varphi(x \ominus h_{[\alpha]}). \tag{2.6}$$

Taking Walsh- Fourier transform, we get

$$\hat{\psi}_\ell\left(A\xi\right) = m_\ell(\xi)\, \hat{\varphi}(\xi), \tag{2.7}$$

where

$$m_\ell(\xi) = \sum_{\alpha\in\mathbb{Z}^+} a^\ell_\alpha\, \overline{W^*_\alpha(\xi)}. \tag{2.8}$$

The functions $m_\ell,\ 1 \leq \ell \leq p-1$, are in $L^2(G)$ such that

$$M(\xi) = \left(m_\ell(\xi + B^{-1}\xi_{[k]})\right)_{\ell,k=0}^{p-1}$$

is an unitary matrix (see [21, 24]).

3 Main Results

The following lemma give a necessary and sufficient condition for shifts of a function $\varphi \in L^2(G)$ to be an orthonormal system in $L^2(G)$.

Lemma 3.1.(*The splitting lemma*). Let $\varphi \in L^2(G)$ such that the system $\{p^{1/2}\varphi(Ax \ominus h)\}_{h\in H}$ is orthonormal. Let V be its closed linear span. Also, let ψ_ℓ and m_ℓ be the functions defined by (2.6) and (2.8), respectively. Then

$$\{\psi_\ell(x \ominus h) : 0 \leq \ell \leq p-1,\ h \in H,\ x \in G\}$$

is an orthonormal system if and only if

$$\sum_{h^*\in H^\perp} m_\ell(\xi \oplus B^{-1}h^*)\,\overline{m_r(\xi \oplus B^{-1}h^*)} = \delta_{\ell r},\ , \qquad 0 \leq \ell,\, r \leq p-1. \tag{3.1}$$

Moreover, $\{\psi_\ell(x \ominus h) : 0 \le \ell \le p-1,\, h \in H,\, x \in G\}$ is an orthonormal basis of V whenever it is orthonormal.

Proof. For $0 \le \ell \le p-1$ and $h \in H$, we have

$$\begin{aligned}
\big\langle \psi_\ell(x), \psi_r(x \ominus h)\big\rangle &= \Big\langle \big(\psi_\ell(x)\big)^\wedge, \big(\psi_r(x \ominus h)\big)^\wedge \Big\rangle \\
&= \int_{G^*} \hat{\psi}_\ell(\xi)\, \overline{\hat{\psi}_r(\xi) W_\alpha^*(\xi)}\, d\mu^*(\xi) \\
&= \int_{G^*} m_\ell(B^{-1}\xi)\hat{\varphi}(B^{-1}\xi)\, \overline{m_r(B^{-1}\xi)\hat{\varphi}(B^{-1}\xi)}\, W_\alpha^*(\xi) d\mu^*(\xi) \\
&= \int_{U^* \oplus h^*} \sum_{h^* \in H^\perp} m_\ell(B^{-1}(\xi \oplus h^*))\, \overline{m_r(B^{-1}(\xi \oplus h^*))} \\
&\qquad \times \hat{\varphi}(B^{-1}(\xi \oplus h^*))\, \overline{\hat{\varphi}(B^{-1}(\xi \oplus h^*))}\, W_\alpha^*(\xi)\, d\mu^*(\xi) \\
&= \sum_{h^* \in H^\perp} m_\ell(B^{-1}\xi \oplus B^{-1}h^*)\, \overline{m_r(B^{-1}\xi \oplus B^{-1}h^*)} \\
&\times \int_{U^* \oplus h^*} \sum_{h^* \in H^\perp} \hat{\varphi}\big(B^{-1}(\xi \oplus h^*)\big)\, \overline{\hat{\varphi}\big(B^{-1}(\xi \oplus h^*)\big)}\, W_\alpha^*(\xi) d\mu^*(\xi) \\
&= \sum_{h^* \in H^\perp} m_\ell(B^{-1}\xi \oplus B^{-1}h^*)\, \overline{m_r(B^{-1}\xi \oplus B^{-1}h^*)} \\
&\qquad \times \int_{U^*} \sum_{t \in H^\perp} \big|\hat{\varphi}\big(B^{-1}(\xi \oplus k) \oplus t\big)\big|^2\, W_\alpha^*(\xi) d\mu^*(\xi) \\
&= \int_{U^*} \left(\sum_{h^* \in H^\perp} m_\ell(B^{-1}\xi \oplus B^{-1}h^*)\, \overline{m_r(B^{-1}\xi \oplus B^{-1}h^*)} \right) \\
&\qquad \times W_\alpha^*(\xi) d\mu^*(\xi). \qquad \big(\text{by } (2.2)\big)
\end{aligned}$$

Therefore,

$$\big\langle \psi_\ell(x), \psi_r(x \ominus h)\big\rangle = \delta_{\ell r}\delta_{0h}$$

$$\Leftrightarrow \quad \sum_{h^* \in H^\perp} m_\ell(B^{-1}\xi \oplus B^{-1}h^*)\, \overline{m_r(B^{-1}\xi \oplus B^{-1}h^*)} = \delta_{\ell r}, \quad a.e.\ \xi \in G^*$$

$$\Leftrightarrow \quad \sum_{h^*\in H^\perp} m_\ell(\xi \oplus B^{-1}h^*)\,\overline{m_r(\xi \oplus B^{-1}h^*)} = \delta_{\ell r}, \quad a.e.\ \xi \in G^*.$$

We have proved the first part of the lemma.

We, now show the orthonormality of the system

$$\mathcal{F} = \{\psi_\ell(x \ominus h) : 0 \le \ell \le p-1,\ h \in H,\ x \in G\}.$$

Let $\mathcal{F}$ is an orthonormal system, then we want to show that this system is an orthonormal basis for V. Let $f \in V$, so there exists $\{a^\ell_\alpha\}_{\ell=0,\alpha\in\mathbb{Z}^+}^{p-1} \in \ell^2(\mathbb{Z}^+)$ such that

$$f(x) = \sum_{\alpha\in\mathbb{Z}^+} a^\ell_\alpha\, p^{1/2}\varphi(Ax \ominus h_{[\alpha]}).$$

Assume that $f \perp \psi_\ell(x \ominus h)$, for all $h \in H$, $x \in G$, $0 \le \ell \le p-1$, then we claim that $f = 0$. For all ℓ, α such that $0 \le \ell \le p-1, \alpha \in \mathbb{Z}^+$, we have

$$\begin{aligned}
0 &= \big\langle \psi_\ell(x \ominus h), f(x)\big\rangle \\
&= \Big\langle \psi_\ell(x \ominus h), \sum_{\alpha\in\mathbb{Z}^+} a^\ell_\alpha\, p^{1/2}\varphi(Ax \ominus h_{[\alpha]})\Big\rangle \\
&= \Big\langle (\psi_\ell(x \ominus h))^\wedge, \Big(\sum_{\alpha\in\mathbb{Z}^+} a^\ell_\alpha\, p^{1/2}\varphi(Ax \ominus h_{[\alpha]})\Big)^\wedge\Big\rangle \\
&= \int_{G^*} \hat\psi_\ell(\xi)\overline{W^*_\alpha(\xi)} \sum_{\alpha\in\mathbb{Z}^+} \overline{a^\ell_\alpha}\, p^{-1/2}\overline{\hat\varphi(B^{-1}\xi)}W^*_\alpha(B^{-1}\xi)d\mu^*(\xi) \\
&= p^{-1/2}\int_{G^*} m_\ell(B^{-1}\xi)\hat\varphi(B^{-1}\xi)\,\overline{W^*_\alpha(\xi)} \\
&\qquad \times \sum_{\alpha\in\mathbb{Z}^+} \overline{a^\ell_\alpha}\,\overline{\hat\varphi(B^{-1}\xi)}W^*_\alpha(B^{-1}\xi)d\mu^*(\xi) \\
&= p^{1/2}\int_{G^*} m_\ell(\xi)\hat\varphi(\xi)\overline{W^*_\alpha(\xi)} \sum_{\alpha\in\mathbb{Z}^+} \overline{a^\ell_\alpha}\,\overline{\hat\varphi(\xi)}\,\overline{W^*_\alpha(B\xi)}d\mu^*(\xi) \\
&= p^{1/2}\sum_{\alpha\in\mathbb{Z}^+} \overline{a^\ell_\alpha}\, m_\ell(\xi)\overline{W^*_\alpha(\xi)} \sum_{h^*\in H^\perp}\int_{U^*\oplus h^*} |\hat\varphi(\xi)|^2\,\overline{W^*_\alpha(B\xi)}d\mu^*(\xi)
\end{aligned}$$

$$= p^{1/2} \sum_{\alpha\in\mathbb{Z}^+} \overline{a_\alpha^\ell}\, m_\ell(\xi)\overline{W_\alpha^*(\xi)} \int_{U^*} \sum_{h^*\in H^\perp} |\hat{\varphi}(\xi \ominus h^*)|^2\, \overline{W_\alpha^*(B\xi)} d\mu^*(\xi)$$

$$= p^{1/2} \int_{U^*} \left(\sum_{\alpha\in\mathbb{Z}^+} \overline{a_\alpha^\ell}\, m_\ell(\xi)\overline{W_\alpha^*(\xi)} \right) \overline{W_\alpha^*(B\xi)} d\mu^*(\xi). \qquad \big(\text{by } (2.2)\big)$$

Since $\{p^{1/2}W_\alpha^*(B\xi) : \alpha \in \mathbb{Z}^+\}$ is an orthonormal basis for $L^2(U^*)$, the above equation give

$$\sum_{\alpha\in\mathbb{Z}^+} \overline{a_\alpha^\ell}\, m_\ell(\xi)\overline{W_\alpha^*(\xi)} = 0, \qquad a.e. \ \text{for } \ell = 1, ..., p-1.$$

Now for $\ell = 1, ..., p-1$, we have

$$A^\ell(\xi) = \sum_{\alpha\in\mathbb{Z}^+} \overline{a_\alpha^\ell}\, \overline{W_\alpha^*(\xi)}. \tag{3.2}$$

So we have

$$\overline{A^\ell(\xi)}\, m_\ell(\xi) = 0, \qquad \ell = 1, ..., p-1. \tag{3.3}$$

Equation (3.1) is equivalent to saying that for $\ell = 1, ..., p-1$ and for $a.e.\ \xi \in G^*$, the functions $\{m_\ell\}$ are mutually orthogonal and each has norm 1. Equation (3.3) says that the vector

$$\{A^\ell(\xi) : \ell = 1, ..., p-1,\ \xi \in G^*\} \tag{3.4}$$

is orthogonal to each member of the above orthonormal basis of $\mathbb{C}^{+p}$. Hence the vector in the expression (3.4) is zero. Inparticular, $A^\ell(\xi) = 0$ for $\ell = 1, ..., p-1$. That is, $a_\alpha^\ell = 0,\ \ell = 1, ..., p-1,\ \alpha \in \mathbb{Z}^+$. Therefore, $f = 0$. □

Using this splitting lemma, one can split an arbitrary Hilbert space into mutually orthogonal subspaces.

Corollary 3.2. Let $\{E_\alpha : \alpha \in \mathbb{Z}^+\}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$, and $m_\ell,\ 0 \le \ell \le p-1$, be as in Lemma 3.1 satisfying (3.1). Define

$$F_\alpha^\ell = \sum_{\alpha\in\mathbb{Z}^+} p^{1/2} a_{\alpha-p\ell} E_\alpha, \qquad \alpha \in \mathbb{Z}^+,\ 0 \le \ell \le p-1$$

then $\left\{F_\alpha^\ell : \alpha \in \mathbb{Z}^+,\, 0 \le \ell \le p-1\right\}$ is an orthonormal basis for its closed linear span $\mathcal{H}_\ell$ and $\mathcal{H} = \bigoplus_{\ell=0}^{p-1} \mathcal{H}_\ell$.

Proof. Let $\varphi \in L^2(G)$ be such that $\{\varphi(x \ominus h) : h \in \mathbb{H},\, x \in G\}$ is an orthonormal system. Let $V = \overline{span}\left\{p^{1/2}\varphi(Ax \ominus h) : h \in H,\, x \in G\right\}$. Define a linear operator T from the Hilbert space $\mathcal{H}$ into V by $T(p^{1/2}\varphi(Ax \ominus h)) = E_\alpha$. Let ψ_ℓ be as in (2.6). Then, $T(p^{1/2}\varphi(Ax \ominus h)) = F_\alpha^\ell$. The corollary now follows from the splitting lemma. □

Now, if φ is the scaling function associated with given MRA. Then there exists the function m_0 such that

$$\hat{\varphi}(\xi) = m_0(B^{-1}\xi)\hat{\varphi}(B^{-1}\xi)$$

where $\quad m_0(\xi) = \sum_{\alpha \in \mathbb{Z}^+} a_\alpha \overline{W_\alpha^*(\xi)}, \qquad \sum_{\alpha \in \mathbb{Z}^+} |\alpha_k|^2 < +\infty.$

Applying the splitting lemma to the space V_1, we get the functions ω_ℓ, $0 \le \ell \le p-1$, where

$$\hat{\omega}_\ell(\xi) = m_\ell(B^{-1}\xi)\hat{\varphi}(B^{-1}\xi) \tag{3.5}$$

such that $\{\omega_\ell(x \ominus h) : 0 \le \ell \le p-1, h \in H,\, x \in G\}$ forms an orthonormal basis for V_1. Observe that $\omega_0 = \varphi$, the scaling function and ω_ℓ, $1 \le \ell \le p-1$, are the basic p-wavelets.

We now define ω_n for each integer $n \ge 0$. Suppose that $s \ge 0$, ω_s already defined. Then define ω_{s+pr}, $0 \le s \le p-1$, by

$$\omega_{s+pr}(x) = p \sum_{\alpha \in \mathbb{Z}^+} a_\alpha^s\, \omega_r(Ax \ominus h_{[\alpha]}). \tag{3.6}$$

Note that (3.6) defines ω_n for all $n \ge 0$. Taking Walsh-Fourier transform in both sides of (3.6), we get

$$\big(\omega_{s+pr}\big)^\wedge(\xi) = m_s(B^{-1}\xi)\, \hat{\omega}_r(B^{-1}\xi), \qquad 0 \le s \le p-1. \tag{3.7}$$

The functions $\{\omega_n : n \ge 0\}$ will be called the basic p-wavelet packets associated with multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$.

We now obtain the expression for the Fourier transform of the p-wavelet packets in terms of the functions m_ℓ as:

Proposition 3.3. Let $\{\omega_n : n \geq 0\}$ be the basic p-wavelet packets constructed above and

$$n = \sum_{j=0}^{k} \mu_j p^j \ , \qquad \mu_j \in \{0, 1, 2, ..., p-1\} \, , \ \mu_k \neq 0, \ k = k(n) \in \mathbb{Z}^+ \quad (3.8)$$

be the unique expansion of the integer n in the base p. Then

$$\hat{\omega}_n(\xi) = m_{\mu_0}(\xi) m_{\mu_1}(B^{-1}\xi) m_{\mu_2}(B^{-2}\xi)...m_{\mu_k}(B^{-k}\xi)\,\hat{\varphi}(B^{-k}\xi). \qquad (3.9)$$

Proof. We say that an integer n has length k if it has an expansion as in (3.8). We use induction on the length of n to prove the proposition. Since ω_0 is the scaling function and ω_ℓ, $1 \leq \ell \leq p-1$, are the wavelets, it follows from (3.5) that the claim is true for all n of length 1. Assume that it holds for all integers of length k. Then an integer t of length $k+1$ is of the form $t = \mu + pn$ where $0 \leq \mu \leq p-1$, and n has length k. Suppose that n has the expansion (3.8), then from (3.7) and (3.9), we have

$$\begin{aligned}
\hat{\omega}_t(\xi) &= \hat{\omega}_{\mu+pn}(\xi) \\
&= m_\mu(B^{-1}\xi)\,\hat{\omega}_n(B^{-1}\xi) \\
&= m_\mu(B^{-1}\xi) m_{\mu_1}(B^{-1}\xi) m_{\mu_2}(B^{-2}\xi)...m_{\mu_k}(B^{-(k+1)}\xi)\,\hat{\varphi}(B^{-(k+1)}\xi).
\end{aligned}$$

Since $t = \mu + pn$, $\omega_t(\xi)$ has the desired form, and the induction is complete . □

The purpose of the construction of p-wavelet packets is to show that their translates form an orthonormal basis for $L^2(G)$. This is proved in the following theorem.

Theorem 3.4. Let $\{\omega_n : n \geq 0\}$ be the basic p-wavelet packets associated with the multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$. Then

(i) $\left\{\omega_n(. \ominus h) : p^j \leq n \leq p^{j+1} - 1, h \in H\right\}$ is an orthonormal basis of $W_j, j \geq 0$.

(ii) $\left\{\omega_n(. \ominus h) : 0 \leq n \leq p^j - 1, h \in H\right\}$ is an orthonormal basis of $V_j, j \geq 0$.

(iii) $\{\omega_n(. \ominus h) : n \geq 0, h \in H\}$ is an orthonormal basis of $L^2(G)$.

Proof. We prove the theorem by induction on j. Since $\{\omega_n : 1 \le n \le p-1\}$ are the basic p-wavelets, so (i) is true for $j = 0$. Let us assume that it holds for j. By (2.3) and the assumption, we have

$$\left\{p^{1/2}\omega_n(A.\ominus h) : p^j \le n \le p^{j+1}-1,\, h \in H\right\}$$

is an orthonormal basis of W_{j+1}. Set $E_n = \overline{\text{span}}\left\{p^{1/2}\omega_n(A.\ominus h) : h \in H\right\}$ so that

$$W_{j+1} = \bigoplus_{n=p^j}^{p^{j+1}-1} E_n. \tag{3.10}$$

By applying the splitting lemma to E_n, we get the functions $g_\ell^n, 0 \le \ell \le p-1$, defined by

$$\left(g_\ell^n\right)^\wedge(\xi) = m_\ell(B^{-1}\xi)\,\hat{\omega}_n(B^{-1}\xi), \quad 0 \le \ell \le p-1 \tag{3.11}$$

such that $\{g_\ell^n(.\ominus h) : 0 \le \ell \le p-1,\, h \in H\}$ is an orthonormal basis of E_n.

Now, if n has the expansion as in (3.8). Then, using (3.9), we get

$$\left(g_\ell^n\right)^\wedge(\xi) = m_\ell(B^{-1}\xi)m_{\mu_1}(B^{-1}\xi)m_{\mu_2}(B^{-2}\xi)...m_{\mu_k}(B^{-(k+1)}\xi)\,\hat{\varphi}(B^{-(k+1)}\xi).$$

But the expression on the right-hand side is precisely $\hat{\omega}_m(\xi)$, where

$$m = \ell + p\mu_1 + p^2\mu_2 + ... + p^j\mu_j = \ell + pn.$$

Hence, we get $g_\ell^n = \omega_{\ell+pn}$. Since

$$\left\{\ell + pn : 0 \le \ell \le p-1,\; p^j \le n \le p^{j+1}-1\right\}$$
$$= \left\{n : 0 \le \ell \le p-1,\; p^{j+1} \le n \le p^{j+2}-1\right\}.$$

Thus we have proved (i) for $j+1$ and the induction is complete. Part (ii) follows from the fact that $V_j = V_0 \oplus W_0 \oplus ... \oplus W_{j-1}$ and (iii) from the decomposition (2.4). □

We define now the general p-wavelet packets of $L^2(G)$ as:

Let $\{\omega_n : n \ge 0\}$ be the basic p-wavelet packets associated with the multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(G)$. The collection of functions

$$\mathcal{F} = \left\{p^{j/2}\omega_n(A^j.\ominus h) : n \ge 0,\, h \in H,\; j \in \mathbb{Z}\right\}$$

will be called the general p-wavelet packets associated with $\{V_j\}_{j\in\mathbb{Z}}$.

Obviously, the system of functions in $\mathcal{F}$ is overcomplete in $L^2(G)$. For example the subcollection with $j = 0$, $n \geq 0$, $h \in H$, is the basic p-wavelet packet basis constructed in the previous section. Secondly, the subcollection with $n = 1, 2, ..., p-1$, $j \in \mathbb{Z}, h \in H$, is the p-wavelet basis. Now, we prove several decompositions of the wavelet subspaces W_j.

For $n \geq 0$ and $j \in \mathbb{Z}$, define the subspaces

$$S_j^n = \overline{span}\left\{p^{j/2}\omega_n(A^j. \ominus h) : h \in H\right\}.$$

Since ω_0 is the scaling function and $\omega_n,\ 1 \leq n \leq p-1$, are the basic p-wavelets, we observe that

$$S_j^0 = V_j, \qquad S_j^1 = W_j = \bigoplus_{r=1}^{p-1} S_j^r, \qquad j \in \mathbb{Z}$$

so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$, can be written as

$$S_{j+1}^0 = \bigoplus_{r=0}^{p-1} S_j^r.$$

This fact can be generalized to decompose S_{j+1}^n into $p-1$ orthogonal subspaces as:

Proposition 3.5. If $n \geq 0$ and $j \in \mathbb{Z}$, we have

$$S_{j+1}^n = \bigoplus_{\ell=0}^{p-1} S_j^{\ell+pn}. \tag{3.12}$$

Proof. By definition

$$S_{j+1}^n = \overline{span}\left\{p^{(j+1)/2}\omega_n(A^{j+1}. \ominus h) : h \in H\right\}.$$

Let $g_\alpha(x) = p^{(j+1)/2}\omega_n(A^{j+1}. \ominus h_{[\alpha]}),\ \alpha \in \mathbb{Z}^+$. Then $\{g_\alpha : \alpha \in \mathbb{Z}^+\}$ is an orthonormal basis for the Hilbert space U_{j+1}^n. For $0 \leq \ell \leq p-1$, define

$$F_t^\ell(x) = \sum_{\alpha\in\mathbb{Z}^+} p^{1/2} a_{\alpha-pt}^\ell\, g_t(x), \qquad t \in \mathbb{Z}^+,$$

and $\mathcal{H}_\ell = \overline{span}\left\{F_t^\ell : t \in \mathbb{Z}^+\right\}$. Then, by Corollary 3.2, we have

$$S_{j+1}^n = \bigoplus_{\ell=0}^{p-1} \mathcal{H}_\ell.$$

Now

$$\begin{aligned} F_t^\ell(x) &= \sum_{\alpha\in\mathbb{Z}^+} p^{1/2} a_{\alpha-pt}^\ell \, g_t(x) \\ &= \sum_{\alpha\in\mathbb{Z}^+} p^{1/2} a_\alpha^\ell \, g_{t+pt}(x) \\ &= \sum_{\alpha\in\mathbb{Z}^+} a_\alpha^\ell p^{(j+2)/2} \omega_n\big(A^{j+1}x \ominus h_{[\alpha]} \ominus At\big) \\ &= p^{j/2} \sum_{\alpha\in\mathbb{Z}^+} p\, a_\alpha^\ell \, \omega_n\big(A(A^j x \ominus t) \ominus h_{[\alpha]}\big) \\ &= p^{j/2} \omega_{\ell+pn}\big(A^j x \ominus t\big) \qquad \big(\text{by } (3.6)\big). \end{aligned}$$

Hence,

$$\mathcal{H}_\ell = S_j^{\ell+pn} \quad \text{and} \quad S_{j+1}^n = \bigoplus_{\ell=0}^{p-1} S_j^{\ell+pn}. \qquad \square$$

The above decomposition can be used to obtain various decompositions of the wavelet subspaces $W_j, \ j \geq 0$.

Theorem 3.6. If $j \geq 0$, then

$$\begin{aligned} W_j = \bigoplus_{r=1}^{p-1} S_j^r = \bigoplus_{r=p}^{p^2-1} S_{j-1}^r = \ldots\ldots\ldots = \bigoplus_{r=p^m}^{p^{m+1}-1} S_{j-m}^r, \quad m \leq j \\ = \bigoplus_{r=p^j}^{p^{j+1}-1} S_0^r \ . \end{aligned} \tag{3.13}$$

Proof. The proof is obtained by repeated application of the previous proposition. $\square$

By using Theorem 3.6 we can construct many orthonormal bases of $L^2(G)$. We have the following decomposition:

$$L^2(G) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus ...$$

Therefore, for each $j \geq 0$, we can choose any of the decomposition of W_j obtained above. For example, if we do not want to decompose any W_j, then we have the usual wavelet decomposition. On the other hand, if we prefer the last decomposition in (3.13) for each W_j, then we get the p-wavelet packet decomposition. There are other decompositions as well. Observe that in (3.13), the lower index of S_j^n's are decreased by 1 in each successive step. If we keep some of these spaces fixed and choose to decompose others by using (3.12), then we get decompositions of W_j which do not appear in (3.13). So there is certain interplay between the indices $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$.

Let $\mathbb{T} \subset \mathbb{N}_0 \times \mathbb{Z}$. Our aim is to characterize the sets $\mathbb{T}$ such that the collection

$$\mathcal{F}_{\mathbb{T}} = \left\{ p^{j/2} \omega_n (A^j . \ominus h) : h \in H,\ (n,j) \in S \right\}$$

will form an orthonormal basis of $L^2(G)$. In other words, we are looking for those subsets $\mathbb{T}$ of $\mathbb{N}_0 \times \mathbb{Z}$ for which

$$\bigoplus_{(n,j)\in\mathbb{T}} S_j^n = L^2(G)\ . \tag{3.14}$$

Theorem 3.7. Let $\{\omega_n : n \geq 0\}$ be the basic p-wavelet packets associated with the multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ and $\mathbb{T} \subset \mathbb{N}_0 \times \mathbb{Z}$. Then, $\mathcal{F}_{\mathbb{T}}$ is an orthonormal basis of $L^2(G)$ if and only if $\{I_{n,j} : (n,j) \in \mathbb{T}\}$ is a partition of $\mathbb{N}_0$, where $I_{n,j} = \left\{ \ell \in \mathbb{N}_0 : p^j n \leq \ell \leq p^j(n+1) - 1 \right\}$.

Proof. By using Proposition 3.5 repeatedly, we have

$$\begin{aligned} S_j^n &= \bigoplus_{\ell=0}^{p-1} S_{j-1}^{\ell+pn} = \bigoplus_{\ell=pn}^{p(n+1)-1} S_{j-1}^{\ell} = \bigoplus_{\ell=pn}^{p(n+1)-1} \left[\bigoplus_{m=0}^{p-1} S_{j-2}^{p\ell+m} \right] \\ &= \bigoplus_{\ell=p^2 n}^{p^2(n+1)-1} S_{j-2}^{\ell} = ... = \bigoplus_{\ell=p^j n}^{p^j(n+1)-1} S_0^{\ell} = \bigoplus_{\ell\in I_{n,j}} S_0^{\ell}. \end{aligned}$$

Therefore,

$$\bigoplus_{(n,j)\in\mathbb{T}} S_j^n = \bigoplus_{(n,j)\in\mathbb{T}} \bigoplus_{\ell\in I_{n,j}} S_0^\ell.$$

But we have already proved in Theorem 3.4 (iii) that $L^2(G) = \bigoplus_{\ell\in\mathbb{N}_0} S_0^\ell$. Hence, the necessary and sufficient condition for the equation (3.14) to be true is that $\{I_{n,j} : (n,j) \in \mathbb{T}\}$ is a partition of $\mathbb{N}_0$. □

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