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MODULE STRUCTURES ON ITERATED DUALS OF BANACH ALGEBRAS

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Abstract

Let $\mathcal{A}$ be a Banach algebra and $(\mathcal{A}'', \Box)$ be its second dual with first Arens product. We consider three $(\mathcal{A}'', \Box)$-bimodule structures on forth dual and four $(\mathcal{A}'', \Box)$-bimodule structures on fifth dual of a Banach algebra. This paper determines the conditions that make these structures equal. Among other results we show that if $\mathcal{A}''$ is weakly amenable with some conditions, then $\mathcal{A}$ is 3 -weakly amenable.

1 Introduction

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-module, that is X is a Banach space and an $\mathcal{A}$-module such that the module operations $(a, x) \longmapsto a \cdot x$ and $(a, x) \longmapsto x \cdot a$ from $\mathcal{A} \times X$ into X are jointly continuous. The dual space X' of X is also a Banach $\mathcal{A}$-module by the following module actions:

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle, \qquad \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle, \qquad (a \in \mathcal{A},\ x \in X,\ f \in X').$$

We set $X'' = (X')'$, and so on, and we regard X as a subspace of X'' in the standard way. Also $X''' = (X'')'$,...

Let X be a Banach $\mathcal{A}$-module. Then a continuous linear map $D : \mathcal{A} \longrightarrow X$ is called a *derivation* if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in \mathcal{A}).$$

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For $x \in X$ we define $\delta_x : \mathcal{A} \longrightarrow X$ as follows:

$$\delta_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A}),$$

it is easy to show that δ_x is a derivation. Such derivations are called *inner derivations*. $\mathcal{A}$ is called *amenable*, if every derivation $D : \mathcal{A} \longrightarrow X'$ is inner, for each Banach $\mathcal{A}$-module X. If every derivation from $\mathcal{A}$ into $\mathcal{A}'$ is inner, $\mathcal{A}$ is called *weakly amenable*. Let $n \in \mathbb{N}$. A Banach algebra $\mathcal{A}$ is called *n-weakly amenable* if every derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is inner [4], where $\mathcal{A}^{(n)}$ is the n-th dual of $\mathcal{A}$ that is a Banach $\mathcal{A}$-module. We regard $\mathcal{A}$ as a subspace of $\mathcal{A}''$ by canonical embedding $\hat{}: \mathcal{A} \to \mathcal{A}''; a \mapsto \hat{a}$. We write $\widehat{\mathcal{A}}$ as the image of $\mathcal{A}$ under this mapping.

Let X, Y and Z be normed spaces and let $f : X \times Y \longrightarrow Z$ be a continuous bilinear map. Then the adjiont of f is defined by

$$f' : Z' \times X \longrightarrow Y', \qquad \langle f'(z', x), y\rangle = \langle z', f(x, y)\rangle \qquad (z' \in Z', x \in X, y \in Y).$$

Since f' is a continuous bilinear map, this process may be repeated to define $f'' = (f')' : Y'' \times Z' \longrightarrow X'$, and then $f''' = (f'')' : X'' \times Y'' \longrightarrow Z''$. The map f''' is the unique extension of f such that $X'' \longrightarrow Z''; x'' \mapsto f'''(x'', y'')$ is $weak^* - weak^*$ continuous for all $y'' \in Y''$ and $Y'' \longrightarrow Z''; y'' \mapsto f'''(x, y'')$ is $weak^* - weak^*$ continuous for all $x \in X$. Let now $f^t : Y \times X \longrightarrow Z$ be the transpose of f defined by $f^t(y, x) = f(x, y)$ for all $x \in X$ and $y \in Y$. Then f^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $(f^t)''' : Y'' \times X'' \longrightarrow Z''$. The bilinear map f is called *Arens regular* if $f''' = ((f^t)''')^t$ (see [1, 2, 7, 8] and [13]). Let $x'' \in X''$ and $y'' \in Y''$. Then there exist nets $(x_\alpha) \subset X$ and $(y_\beta) \subset Y$ with $\widehat{x}_\alpha \xrightarrow{w^*} x''$ and $\widehat{y}_\beta \xrightarrow{w^*} y''$. We have

$$f'''(x'', y'') = \lim_\alpha \lim_\beta f\widehat{(x_\alpha, y_\beta)},$$

$$((f^t)''')^t(x'', y'') = \lim_\beta \lim_\alpha f\widehat{(x_\alpha, y_\beta)}.$$

Let $\mathcal{A}$ be a Banach algebra, and let $\pi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ denote the product of $\mathcal{A}$, so that $\pi(a, b) = ab$ $(a, b \in \mathcal{A})$. for F and G in $\mathcal{A}''$, we denote $\pi'''(F, G)$ and $((\pi^t)''')^t(F, G)$ by symbols $F\Box G$ and $F\Diamond G$, respectively. These are called *first and second Arens products* on $\mathcal{A}''$. These products are defined in stages as follows. For every $F, G \in \mathcal{A}''$, $f \in \mathcal{A}'$ and $a, b \in \mathcal{A}$, we define $f \cdot a, a \cdot f, G \cdot f$ and $f \cdot F$ in $\mathcal{A}'$; $F\Box G$ and $F\Diamond G$ in $\mathcal{A}''$ by

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle, \qquad \langle a \cdot f, b\rangle = \langle f, ba\rangle,$$

$$\langle G \cdot f, a\rangle = \langle G, f \cdot a\rangle, \qquad \langle f \cdot F, a\rangle = \langle F, a \cdot f\rangle,$$

$$\langle F\Box G, f\rangle = \langle F, G \cdot f\rangle, \qquad \langle F\Diamond G, f\rangle = \langle G, f \cdot F\rangle.$$

$\mathcal{A}''$ is a Banach algebra with (above) Arens products. In fact

$$F\Box G = w^* - \lim_{\alpha} w^* - \lim_{\beta} \widehat{a_\alpha b_\beta}$$

$$F\Diamond G = w^* - \lim_{\beta} w^* - \lim_{\alpha} \widehat{a_\alpha b_\beta},$$

where $F = w^* - \lim_\alpha \widehat{a}_\alpha$ and $G = w^* - \lim_\beta \widehat{b}_\beta$. The algebra $\mathcal{A}$ is Arens regular whenever the map π is Arens regular that is, whenever the first and second Arens products of $\mathcal{A}''$ coincide. Recall that a Banach algebra $\mathcal{A}$ is said to be *dual* if there is a closed submodule $\mathcal{A}_0$ of $\mathcal{A}'$ such that $\mathcal{A} = {\mathcal{A}_0}'$.

Definition 1.1. The Banach algebra $\mathcal{A}$ has strongly double limit property (SDLP) if for each bounded net (a_α) in $\mathcal{A}$ and each bounded net (f_β) in $\mathcal{A}'$, $\lim_\alpha \lim_\beta \langle f_\beta, a_\alpha\rangle = \lim_\beta \lim_\alpha \langle f_\beta, a_\alpha\rangle$, whenever both iterated limits exist.

This definition has been introduced in [14]. Medghalchi and Yazdanpanah in [14] showed that every reflexive Banach algebra has (SDLP). We know that reflexivity is equivalent with double limit property [3, Theorem A.3.31], so the (SDLP) is equivalent with reflexivity. Now suppose that the Banach algebra $\mathcal{A}$ has (SDLP), then for each $f \in \mathcal{A}'$ and bounded nets $(a_\alpha), (b_\beta)$ in $\mathcal{A}$, we have

$$\lim_{\beta}\lim_{\alpha}\langle b_\beta \cdot f, a_\alpha\rangle = \lim_{\alpha}\lim_{\beta}\langle b_\beta \cdot f, a_\alpha\rangle,$$

which means that for each $f \in \mathcal{A}'$, the map $a \mapsto a.f$, $\mathcal{A} \longrightarrow \mathcal{A}'$ is weakly compact by [3, Theorem 2.6.17], i.e., $\mathcal{A}$ is Arens regular. Hence (SDLP) is stronger than Arens regularity. On the other hand this two are not equivalent in general. We know $C([0,1])$ is an Arens regular Banach algebra. If we consider the sequence (f_m) in $C([0,1])$ defined by $f_m(x) = \frac{m}{m+\frac{1}{x}}$ for $0 < x \leq 1$ and $f_m(0) = 0$ for all $m \in \mathbb{N}$, and assume that sequence (μ_n) is in $M([0,1]) = C([0,1])^*$ (the set of all regular Borel measures on $[0,1]$), where μ_n is the point mass at $\frac{1}{n}$, for all $n \in \mathbb{N}$. Then, we easily see that

$$\lim_{m}\lim_{n}\langle \mu_n, f_m\rangle = 0 \neq 1 = \lim_{n}\lim_{m}\langle \mu_n, f_m\rangle.$$

Therefore $C([0,1])$ has not (SDLP). Also there are Arens regular Banach algebras which are not reflexive as Banach spaces. For example, the disc algebra $A(\mathbb{D})$ is Arens regular [16] but not reflexive [15].

One may consider the question of how $\mathcal{A}$ inherits the amenability or weak amenability of $\mathcal{A}''$. For amenability the answer is positive (see [12]). So for weak amenability, this problem was considered by few authors and a positive answer has been given in each of the following cases:

- $\mathcal{A}$ is a left ideal in $\mathcal{A}''$ [12].
- $\mathcal{A}$ is a dual Banach algebra [11].
- $\mathcal{A}$ is Arens regular and every derivation from $\mathcal{A}$ into $\mathcal{A}'$ is weakly compact [5].
- $\mathcal{A}$ has (SDLP) [14].
- $\mathcal{A}$ is a right ideal in $\mathcal{A}''$ and $\mathcal{A}''\mathcal{A} = \mathcal{A}''$ [9].

In section two of this paper, we put many module structures on forth dual $\mathcal{A}^{(4)}$ and show that these module structures are not always equal, and we show when these module structures are equal. By using part two, we make four module structures on $\mathcal{A}^{(5)}$. This is done in section three, where these module structures on $\mathcal{A}^{(5)}$ are not always equal. In section four we show that with some module structures on $\mathcal{A}^{(5)}$, weak amenability $\mathcal{A}''$ implies weak amenability $\mathcal{A}$. This is a question that if $\mathcal{A}''$ is 3 -weakly amenable, is $\mathcal{A}$ 3-weakly amenable? We show that the 3 -weak amenability of $\mathcal{A}''$ implies the 3 -weak amenability of $\mathcal{A}$ if $D''(\mathcal{A}'') \cdot \mathcal{A}^{(4)} \subseteq \widehat{\mathcal{A}'}$, for each derivation $D : \mathcal{A} \longrightarrow \mathcal{A}'''$. It is known that every $(n + 2)$-weakly amenable Banach algebra is n-weakly amenable for $n \geq 1$ [4]. In particular the 3-weak amenability of $\mathcal{A}$ implies the weak amenability of $\mathcal{A}$. Does weak amenability imply 3 -weak amenability? The answer is negative. Yong Zhang [19] gave an example of a weakly amenable Banach algebra that it is not 3-weakly amenable, but he had showed in [20] that if $\mathcal{A}$ is weakly amenable with a left (right) bounded approximate identity such that it is a left (right) ideal in $\mathcal{A}''$, then $\mathcal{A}$ is (2n+1)-weakly amenable for $n \geq 1$. A different proof are provided by Dales, Ghahramani and Grønbæk in [4] in which $\mathcal{A}$ is an ideal in $\mathcal{A}''$. Finally we put some conditions on $\mathcal{A}$ and $\mathcal{A}''$ such that if $\mathcal{A}$ is weakly amenable, then $\mathcal{A}$ is 3-weakly amenable. For the remainder of this paper, $\mathcal{A}''$ is regarded as a Banach algebra with respect to the first Arens product $\Box$.

2 $\mathcal{A}''$- bimodule structures on forth dual of a Banach algebra

$\mathcal{A}'''$ has two $\mathcal{A}''$-bimodule structures. First we regard $\mathcal{A}'''$, as the dual space of $\mathcal{A}''$, ($\mathcal{A}''' = (\mathcal{A}'')'$) and so $\mathcal{A}'''$ can be made into an $\mathcal{A}''$-bimodule by the following actions

$$\langle \lambda \cdot F, G\rangle = \langle \lambda, F\Box G\rangle, \qquad \langle F \cdot \lambda, G\rangle = \langle \lambda, G\Box F\rangle, \qquad (\lambda \in \mathcal{A}'''; F, G \in \mathcal{A}'').$$

In the second way, $\mathcal{A}'''$, as the second dual of $\mathcal{A}'$, ($\mathcal{A}''' = (\mathcal{A}')''$), can be an $\mathcal{A}''$-bimodule by the following formula. For $\lambda \in \mathcal{A}'''$ and $F \in \mathcal{A}''$, we have

$$\lambda \circ F = w^* - \lim_i w^* - \lim_\alpha \widehat{f_i \cdot a_\alpha}, \qquad F \circ \lambda = w^* - \lim_\alpha w^* - \lim_i \widehat{a_\alpha \cdot f_i},$$

where $F = w^* - \lim_\alpha \widehat{a}_\alpha$ in $\mathcal{A}''$and $\lambda = w^* - \lim_i \widehat{f}_i$ in $\mathcal{A}'''$, such that (a_α) and (f_i) are nets in $\mathcal{A}$ and $\mathcal{A}'$ respectively. In fact $\lambda \circ F$ and $F \circ \lambda$ are extensions of module actions $(f, a) \longrightarrow f \cdot a$ $(\mathcal{A}' \times \mathcal{A} \longrightarrow \mathcal{A}')$ and $(a, f) \longrightarrow a \cdot f$ $(\mathcal{A} \times \mathcal{A}' \longrightarrow \mathcal{A}')$.

These two $\mathcal{A}''$-bimodule structures on $\mathcal{A}'''$ are considered in [10] and have been shown that two right $\mathcal{A}''$-bimodule actions on $\mathcal{A}'''$ always coincide but left $\mathcal{A}''$-bimodule structures on $\mathcal{A}'''$ are not always equal. Now the Banach algebra $\mathcal{A}^{(4)}$ has three $\mathcal{A}''$-bimodule structures.

(a) We consider $\mathcal{A}^{(4)} = (\mathcal{A}''')'$ in which $\mathcal{A}''' = (\mathcal{A}')''$, so $\mathcal{A}^{(4)}$ can be an $\mathcal{A}''$-bimodule by following actions

$$\langle F \circ \Lambda, \lambda\rangle = \langle \Lambda, \lambda \circ F\rangle, \qquad \langle \Lambda \circ F, \lambda\rangle = \langle \Lambda, F \circ \lambda\rangle$$

where $F \in \mathcal{A}'', \lambda \in \mathcal{A}'''$ and $\Lambda \in \mathcal{A}^{(4)}$.

(b) We consider $\mathcal{A}^{(4)} = (\mathcal{A}''')'$ in which $\mathcal{A}''' = (\mathcal{A}'')'$, so $\mathcal{A}^{(4)}$ can be an $\mathcal{A}''$-bimodule by following right and left module actions

$$\langle F \cdot \Lambda, \lambda\rangle = \langle \Lambda, \lambda \cdot F\rangle, \qquad \langle \Lambda \cdot F, \lambda\rangle = \langle \Lambda, F \cdot \lambda\rangle$$

where $F \in \mathcal{A}'', \lambda \in \mathcal{A}'''$ and $\Lambda \in \mathcal{A}^{(4)}$.

(c) Let $\mathcal{A}^{(4)} = (\mathcal{A}'')''$ be as the second dual of $\mathcal{A}''$. Take $\Lambda \in \mathcal{A}^{(4)}$, $F \in \mathcal{A}''$ and bounded nets $(F_\alpha) \subset \mathcal{A}'', (a_\beta) \subset \mathcal{A}$ with $\widehat{F}_\alpha \xrightarrow{w^*} \Lambda$ and $\widehat{a}_\beta \xrightarrow{w^*} F$. Two module actions are defined by

$$F \bullet \Lambda = w^* - \lim_\beta \lim_\alpha \widehat{a_\beta \cdot F_\alpha} \qquad \Lambda \bullet F = w^* - \lim_\alpha \lim_\beta \widehat{F_\alpha \cdot a_\beta}.$$

Hence $F \bullet \Lambda$ and $\Lambda \bullet F$ are extensions of module actions $(a, F) \longrightarrow a \cdot F$ $(\mathcal{A} \times \mathcal{A}'' \longrightarrow \mathcal{A}'')$ and $(F, a) \longrightarrow F \cdot a$ $(\mathcal{A}'' \times \mathcal{A} \longrightarrow \mathcal{A}'')$.

We show that these three $\mathcal{A}''$-bimodule structures on $\mathcal{A}^{(4)}$ are not always equal. Suppose that $\Lambda \in \mathcal{A}^{(4)}$, $\lambda \in \mathcal{A}'''$, $F \in \mathcal{A}''$ and bounded nets $(G_\alpha) \subset \mathcal{A}'', (f_\gamma) \subset \mathcal{A}', (a_\beta) \subset \mathcal{A}$ by $\widehat{G}_\alpha \xrightarrow{w^*} \Lambda, \widehat{f}_\gamma \xrightarrow{w^*} \lambda$ and $\widehat{a}_\beta \xrightarrow{w^*} F$, then

$$\begin{aligned}\langle \Lambda \circ F, \lambda\rangle &= \langle \Lambda, F \circ \lambda\rangle \\ &= \lim_\alpha \langle \widehat{G}_\alpha, F \circ \lambda\rangle \\ &= \lim_\alpha \langle F \cdot \lambda, G_\alpha\rangle \\ &= \lim_\alpha \lim_\beta \lim_\gamma \langle G_\alpha, a_\beta \cdot f_\gamma\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle \Lambda \cdot F, \lambda \rangle &= \langle \Lambda, F \cdot \lambda \rangle \\ &= \lim_{\alpha} \langle \widehat{G}_\alpha, F \cdot \lambda \rangle \\ &= \lim_{\alpha} \langle \lambda, G_\alpha \Box F \rangle \\ &= \lim_{\alpha} \lim_{\gamma} \langle \widehat{f}_\gamma, G_\alpha \Box F \rangle \\ &= \lim_{\alpha} \lim_{\gamma} \langle G_\alpha, F \cdot f_\gamma \rangle.\end{aligned}$$

For structure (c), we have

$$\begin{aligned}\langle \Lambda \bullet F, \lambda \rangle &= \lim_{\alpha} \lim_{\beta} \langle \widehat{G_\alpha \cdot a_\beta}, \lambda \rangle \\ &= \lim_{\alpha} \lim_{\beta} \langle \lambda, G_\alpha \cdot a_\beta \rangle \\ &= \lim_{\alpha} \lim_{\beta} \lim_{\gamma} \langle \widehat{f}_\gamma, G_\alpha \cdot a_\beta \rangle \\ &= \lim_{\alpha} \lim_{\beta} \lim_{\gamma} \langle G_\alpha, a_\beta \cdot f_\gamma \rangle.\end{aligned}$$

We see two right actions in parts (a) and (c) are equal and different from the action of (b). For left actions, suppose that $\Lambda \in \mathcal{A}^{(4)}$, $\lambda \in \mathcal{A}'''$, $F \in \mathcal{A}''$ and bounded nets $(G_\alpha) \subset \mathcal{A}'', (f_\gamma) \subset \mathcal{A}', (a_\beta) \subset \mathcal{A}$ with $\widehat{G}_\alpha \xrightarrow{w^*} \Lambda, \widehat{f}_\gamma \xrightarrow{w^*} \lambda$ and $\widehat{a}_\beta \xrightarrow{w^*} F$, then

$$\begin{aligned}\langle F \circ \Lambda, \lambda \rangle &= \langle \Lambda, \lambda \circ F \rangle \\ &= \lim_{\alpha} \langle \widehat{G}_\alpha, \lambda \circ F \rangle \\ &= \lim_{\alpha} \langle \lambda \circ F, G_\alpha \rangle \\ &= \lim_{\alpha} \lim_{\gamma} \lim_{\beta} \langle G_\alpha, f_\gamma \cdot a_\beta \rangle,\end{aligned}$$

and

$$\begin{aligned}\langle F\cdot\Lambda,\lambda\rangle &= \langle\Lambda,\lambda\cdot F\rangle\\ &= \lim_\alpha\langle\widehat{G}_\alpha,\lambda\cdot F\rangle\\ &= \lim_\alpha\langle\lambda,F\Box G_\alpha\rangle\\ &= \lim_\alpha\lim_\gamma\langle\widehat{f}_\gamma,F\Box G_\alpha\rangle\\ &= \lim_\alpha\lim_\gamma\langle F,G_\alpha\cdot f_\gamma\rangle\\ &= \lim_\alpha\lim_\gamma\lim_\beta\langle\widehat{a}_\beta,G_\alpha\cdot f_\gamma\rangle\\ &= \lim_\alpha\lim_\gamma\lim_\beta\langle G_\alpha,f_\gamma\cdot a_\beta\rangle.\end{aligned}$$

For the structure (c), we have

$$\begin{aligned}\langle F\bullet\Lambda,\lambda\rangle &= \lim_\beta\lim_\alpha\langle\widehat{a_\beta\cdot G_\alpha},b'''\rangle\\ &= \lim_\beta\lim_\alpha\langle\lambda,a_\beta\cdot G_\alpha\rangle\\ &= \lim_\beta\lim_\alpha\lim_\gamma\langle\widehat{f}_\gamma,a_\beta\cdot G_\alpha\rangle\\ &= \lim_\beta\lim_\alpha\lim_\gamma\langle G_\alpha,f_\gamma\cdot a_\beta\rangle.\end{aligned}$$

We see that left actions in parts (a) and (b) are equal and different from the action of (c). We put some conditions on $\mathcal{A}$ and show that with this conditions all $\mathcal{A}''$-bimodule structures on $\mathcal{A}^{(4)}$ are equal. First we bring some simple, but useful lemmas.

Lemma 2.1. *If $\mathcal{A}$ is Arens regular, then, for the bounded nets (F_α) and (G_β) in $\mathcal{A}''$,*
$(w^*-\lim_\alpha F_\alpha)\Box(w^*-\lim_\beta G_\beta) = w^*-\lim_\alpha w^*-\lim_\beta(F_\alpha\Box G_\beta) = w^*-\lim_\beta w^*-\lim_\alpha(F_\alpha\Box G_\beta)$. ∎

Lemma 2.2. *Let the Banach algebra $\mathcal{A}$ with one of the following conditions*

(i) The map $\varphi:\mathcal{A}'\times\mathcal{A}\longrightarrow\mathcal{A}';((f,a)\longrightarrow f\cdot a)$ is Arens regular,

(ii) The map $\psi:\mathcal{A}''\longrightarrow\mathcal{A}'';(G\longrightarrow G\Box F)$ is weak-compact for every $F\in\mathcal{A}''$,

(iii) The map $\phi:\mathcal{A}''\longrightarrow\mathcal{A}'';(G\longrightarrow G\Box F)$ is w^*-w-continuous for every $F\in\mathcal{A}''$.

Then for each bounded net (a_α) in $\mathcal{A}$ and $\lambda\in\mathcal{A}'''$,

$$\langle\lambda,(w^*-\lim_\alpha\widehat{a}_\alpha)\Box F\rangle = \lim_\alpha\langle\lambda,\widehat{a}_\alpha\Box F\rangle \qquad (1)$$

Proof. (i) Let $\lambda = w^* - \lim_\beta \widehat{f}_\beta$, where (f_β) is a bounded net in $\mathcal{A}'$, then we have

$$\begin{aligned}
\langle \lambda, (w^* - \lim_\alpha \widehat{a}_\alpha)\Box F\rangle &= \lim_\beta \langle (w^* - \lim_\alpha \widehat{a}_\alpha)\Box F, f_\beta\rangle \\
&= \lim_\beta \lim_\alpha \langle \widehat{a}_\alpha \Box F, f_\beta\rangle \\
&= \lim_\beta \lim_\alpha \langle F, f_\beta \cdot a_\alpha\rangle \\
&= \langle w^* - \lim_\beta w^* - \lim_\alpha \widehat{f_\beta \cdot a_\alpha}, F\rangle \\
&= \langle w^* - \lim_\alpha w^* - \lim_\beta \widehat{f_\beta \cdot a_\alpha}, F\rangle \\
&= \lim_\alpha \langle w^* - \lim_\beta \widehat{f}_\beta, a_\alpha \Box F\rangle \\
&= \lim_\alpha \langle \lambda, \widehat{a}_\alpha \Box F\rangle.
\end{aligned}$$

(ii) From the double limit property of weak compact operator ψ, we see
$$\lim_\beta \lim_\alpha \langle \widehat{a}_\alpha \Box F, f_\beta\rangle = \lim_\alpha \lim_\beta \langle \widehat{a}_\alpha \Box F, f_\beta\rangle,$$
hence

$$\begin{aligned}
\langle \lambda, (w^* - \lim_\alpha \widehat{a}_\alpha)\Box F\rangle &= \lim_\beta \langle (w^* - \lim_\alpha \widehat{a}_\alpha)\Box F, f_\beta\rangle \\
&= \lim_\beta \lim_\alpha \langle \widehat{a}_\alpha \Box F, f_\beta\rangle \\
&= \lim_\alpha \lim_\beta \langle \widehat{a}_\alpha \Box F, f_\beta\rangle \\
&= \lim_\alpha \langle \lambda, \widehat{a}_\alpha \Box F\rangle.
\end{aligned}$$

(iii) Equation (1) is a consequence of $w^* - w-$continuity of ϕ. ■

Lemma 2.3. *If for every $G \in \mathcal{A}''$ the map $\rho : \mathcal{A}'' \longrightarrow \mathcal{A}''; (F \longrightarrow G\Box F)$ is w^*-w-continuous, then for every bounded net (F_j) in $\mathcal{A}''$*
$$\langle \lambda, G\Box(w^* - \lim_j F_j)\rangle = \lim_j \langle \lambda, G\Box F_j\rangle, \qquad (\lambda \in \mathcal{A}''').$$

Proof. It is similar to part (iii) of Lemma 2.2. ■

Lemma 2.4. *Let $\mathcal{A}$ be an Arens regular Banach algebra. If the map $\varphi : \mathcal{A}'' \longrightarrow \mathcal{A}''; (F \longrightarrow G\Box F)$ is weak-compact or w^*-w-continuous for every $G \in \mathcal{A}''$, then*
$$\langle \lambda, w^* - \lim_\alpha w^* - \lim_j (\widehat{a}_\alpha \Box F_j)\rangle = \lim_\alpha \lim_j \langle \lambda, \widehat{a}_\alpha \Box F_j\rangle. \quad (2)$$
for all $\lambda \in \mathcal{A}'''$, bounded nets (a_α) and (F_j) in $\mathcal{A}$ and $\mathcal{A}''$, respectively.

Proof. Let (f_β) be a bounded net in $\mathcal{A}'$ such that $\widehat{f_\beta} \overset{w^*}{\longrightarrow} \lambda$. Then

$$\begin{aligned}\langle \lambda, w^* - \lim_\alpha w^* - \lim_j (\widehat{a}_\alpha \Box F_j)\rangle &= \lim_\beta \langle (w^* - \lim_\alpha w^* - \lim_j (\widehat{a}_\alpha \Box F_j), f_\beta\rangle \\ &= \lim_\beta \lim_\alpha \lim_j \langle \widehat{a}_\alpha \Box F_j, f_\beta\rangle \\ &= \lim_\alpha \lim_j \lim_\beta \langle \widehat{a}_\alpha \Box F_j, f_\beta\rangle \\ &= \lim_\alpha \lim_j \langle \lambda, \widehat{a}_\alpha \Box F_j\rangle.\end{aligned}$$

Since φ is $w^* - w-$continuous, the equation (2) is obtained immediately.■

Proposition 2.5. *Let $\mathcal{A}$ be a Banach algebra. If one of the following conditions holds, then the two $\mathcal{A}''$-module actions in (a), (c) coincide.*

(i) The Banach algebra $\mathcal{A}$ and the map $\varphi : \mathcal{A}' \times \mathcal{A} \longrightarrow \mathcal{A}'; ((f, a) \longrightarrow f \cdot a)$ are Arens regular and the map $\psi : \mathcal{A}'' \longrightarrow \mathcal{A}''; (F \longrightarrow F\Box G)$ is w^*-w-continuous for every $G \in \mathcal{A}''$.

(ii) The Banach algebra $\mathcal{A}$ is Arens regular and the map $\phi : \mathcal{A}'' \longrightarrow \mathcal{A}''; (F \longrightarrow F\Box G)$ is weak-compact for every $G \in \mathcal{A}''$.

(iii) For bounded nets (G_α), (f_γ) and (a_β) in $\mathcal{A}'', \mathcal{A}'$ and $\mathcal{A}$, respectively, we have

$$\lim_\alpha \lim_\gamma \lim_\beta \langle G_\alpha, f_\gamma \cdot a_\beta\rangle = \lim_\beta \lim_\alpha \lim_\gamma \langle G_\alpha, f_\gamma \cdot a_\beta\rangle.$$

Proof. We know that the two right $\mathcal{A}''$-module actions on $\mathcal{A}''''$ in (a) and (c) are equal to

$$\lim_\alpha \lim_\beta \lim_\gamma \langle G_\alpha, a_\beta \cdot f_\gamma\rangle,$$

in which (G_α), (f_γ) and (a_β) are bounded nets in $\mathcal{A}'', \mathcal{A}'$ and $\mathcal{A}$, respectively. For left $\mathcal{A}''$-module actions on $\mathcal{A}''''$ it is enough to show the following equality

$$\lim_\alpha \lim_\gamma \lim_\beta \langle G_\alpha, f_\gamma \cdot a_\beta\rangle = \lim_\beta \lim_\alpha \lim_\gamma \langle G_\alpha, f_\gamma \cdot a_\beta\rangle.$$

(i) By Arens regularity of the map φ we have

$$\lim_\gamma \lim_\beta \langle G_\alpha, f_\gamma . a_\beta\rangle = \lim_\beta \lim_\gamma \langle G_\alpha, f_\gamma . a_\beta\rangle.$$

Now suppose that $\lambda = w^* - \lim_\beta \widehat{f_\beta}$, by Lemma 2.1 and Lemma 2.4 we have

$$\begin{aligned}
\lim_\alpha \lim_\gamma \lim_\beta \langle G_\alpha, f_\gamma \cdot a_\beta\rangle &= \lim_\alpha \lim_\beta \lim_\gamma \langle G_\alpha, f_\gamma \cdot a_\beta\rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle \widehat{f}_\gamma, \widehat{a}_\beta \Box G_\alpha\rangle \\
&= \lim_\alpha \lim_\beta \langle \lambda, \widehat{a}_\beta \Box G_\alpha\rangle \\
&= \langle \lambda, w^* - \lim_\alpha w^* - \lim_\beta (\widehat{a}_\beta \Box G_\alpha)\rangle \\
&= \langle \lambda, w^* - \lim_\beta w^* - \lim_\alpha (\widehat{a}_\beta \Box G_\alpha)\rangle \\
&= \lim_\beta \lim_\alpha \lim_\gamma \langle G_\alpha, f_\gamma \cdot a_\beta\rangle.
\end{aligned}$$

(ii) By weak compactness of ϕ, we have

$$\lim_\gamma \lim_\beta \langle G_\alpha, f_\gamma \cdot a_\beta\rangle = \lim_\gamma \lim_\beta \langle \widehat{f}_\gamma, \widehat{a}_\beta \Box G_\alpha\rangle = \lim_\beta \lim_\gamma \langle \widehat{f}_\gamma, \widehat{a}_\beta \Box G_\alpha\rangle.$$

It is easy to check that the map ϕ is $w^* - w-$continuous, and so the rest of the proof is the same as the proof of part (i).

(iii) It is clear. ■

3 $\mathcal{A}''$- bimodule structures on fifth dual of a Banach algebra

Let $\mathcal{A}$ be a Banach algebra. We consider four $\mathcal{A}''$- bimodule structures on $\mathcal{A}^{(5)}$.

(I) We consider $\mathcal{A}^{(5)} = (\mathcal{A}^{(4)})'$ in which $\mathcal{A}^{(4)}$ has an $\mathcal{A}''$-bimodule structure as in part **(c)** in Section 2. Therefore $\mathcal{A}^{(5)}$ is the dual space of $\mathcal{A}^{(4)}$, by the following actions

$$\langle F \bullet \Psi, \Lambda\rangle = \langle \Psi, \Lambda \bullet F\rangle, \qquad \langle \Psi \bullet F, \Lambda\rangle = \langle \Psi, F \bullet \Lambda\rangle$$

where $F \in \mathcal{A}'', \Lambda \in \mathcal{A}^{(4)}$ and $\Psi \in \mathcal{A}^{(5)}$. In this case we have $\mathcal{A}^{(5)} = ((\mathcal{A}'')'')'$.

(II) We consider $\mathcal{A}^{(5)} = (\mathcal{A}^{(4)})'$ in which $\mathcal{A}^{(4)}$ has an $\mathcal{A}''$-bimodule structure as in part **(b)** in Section 2, so the left action $\mathcal{A}''$ on $\mathcal{A}^{(5)} = (((\mathcal{A}'')')')'$ is defined by

$$\begin{aligned}
\langle F \cdot \Psi, \Lambda\rangle &= \langle \Psi, \Lambda \cdot F\rangle, \\
\langle \Lambda \cdot F, \lambda\rangle &= \langle \Lambda, F \cdot \lambda\rangle, \\
\langle F \cdot \lambda, G\rangle &= \langle \lambda, G \Box F\rangle.
\end{aligned}$$

where $F, G \in \mathcal{A}'', \lambda \in \mathcal{A}''', \Lambda \in \mathcal{A}^{(4)}$ and $\Psi \in \mathcal{A}^{(5)}$. The right action is defined in a similar way.

(III) Let $\mathcal{A}^{(5)} = (\mathcal{A}''')''$ be as the second dual of $\mathcal{A}'''$ in which $\mathcal{A}''' = ((\mathcal{A}')')'$ is an $\mathcal{A}$-bimodule. Take $\Psi \in \mathcal{A}^{(5)}$, $F \in \mathcal{A}''$ and bounded nets $(\lambda_\alpha) \subset \mathcal{A}''', (a_\beta) \subset \mathcal{A}'$ with $\widehat{\lambda}_\alpha \xrightarrow{w^*} \Psi$ and $\widehat{a}_\beta \xrightarrow{w^*} F$. Two module actions is defined by

$$F \circ \Psi = w^* - \lim_\beta \lim_\alpha \widehat{a_\beta \cdot \lambda_\alpha} \qquad \Psi \circ F = w^* - \lim_\alpha \lim_\beta \widehat{\lambda_\alpha \cdot a_\beta}.$$

In fact $F \circ \Psi$ and $\Psi \circ F$ are extension of module actions $(a, \lambda) \longrightarrow a \cdot \lambda$ $(\mathcal{A} \times \mathcal{A}''' \longrightarrow \mathcal{A}''')$ and $(\lambda, a) \longrightarrow \lambda \cdot a$ $(\mathcal{A}''' \times \mathcal{A} \longrightarrow \mathcal{A}''')$.

(IV) We consider $\mathcal{A}^{(5)} = (\mathcal{A}^{(4)})'$ in which $\mathcal{A}^{(4)}$ has an $\mathcal{A}''$-bimodule structure as in part **(a)** in Section 2, hence the $\mathcal{A}''$-module actions on $\mathcal{A}^{(5)}$ are defined by

$$\langle F \star \Psi, \Lambda \rangle = \langle \Psi, \Lambda \circ F \rangle, \qquad \langle \Psi \star F, \Lambda \rangle = \langle \Psi, F \circ \Lambda \rangle,$$

where $F \in \mathcal{A}'', \Lambda \in \mathcal{A}^{(4)}$ and $\Psi \in \mathcal{A}^{(5)}$.

Suppose that $\Psi \in \mathcal{A}^{(5)}$, $\Lambda \in \mathcal{A}^{(4)}$, $F \in \mathcal{A}''$ and bounded nets $(\lambda_\alpha) \subset \mathcal{A}''', (G_\gamma) \subset \mathcal{A}'', (a_\beta) \subset \mathcal{A}'$ by $\widehat{\lambda}_\alpha \xrightarrow{w^*} \Psi, \widehat{G}_\gamma \xrightarrow{w^*} \Lambda$ and $\widehat{a}_\beta \xrightarrow{w^*} F$, then

$$\begin{aligned}
\langle F \bullet \Psi, \Lambda \rangle &= \langle \Psi, \Lambda \bullet F \rangle \\
&= \lim_\alpha \langle \widehat{\lambda}_\alpha, \Lambda \bullet F \rangle \\
&= \lim_\alpha \lim_\gamma \lim_\beta \langle \lambda_\alpha, G_\gamma \cdot a_\beta \rangle,
\end{aligned}$$

and

$$\begin{aligned}
\langle F \circ \Psi, \Lambda \rangle &= \lim_\beta \lim_\alpha \langle \widehat{a_\beta \cdot \lambda_\alpha}, \Lambda \rangle \\
&= \lim_\beta \lim_\alpha \lim_\gamma \langle \widehat{G}_\gamma, a_\beta \cdot \lambda_\alpha \rangle \\
&= \lim_\beta \lim_\alpha \lim_\gamma \langle \lambda_\alpha, G_\gamma \cdot a_\beta \rangle,
\end{aligned}$$

so $F \circ \Psi$ and $F \bullet \Psi$ are not always equal. But

$$\begin{aligned}
\langle \Psi \circ F, \Lambda \rangle &= \lim_\alpha \lim_\beta \langle \widehat{\lambda_\alpha \cdot a_\beta}, \Lambda \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle \widehat{G}_\gamma, \lambda_\alpha \cdot a_\beta \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle \lambda_\alpha, a_\beta \cdot G_\gamma \rangle \\
&= \lim_\alpha \langle F \bullet \Lambda, \lambda_\alpha \rangle \\
&= \langle \Psi, F \bullet \Lambda \rangle \\
&= \langle \Psi \bullet F, \Lambda \rangle.
\end{aligned}$$

Hence the two right $\mathcal{A}''$-bimodule structure parts (I) and (III) on $\mathcal{A}^{(5)}$ always coincide. Also we can show that left (right)$\mathcal{A}''$-module action on $\mathcal{A}^{(5)}$ in part (II) is

$$\langle F\cdot\Psi,\Lambda\rangle=\lim_{\alpha}\lim_{\gamma}\langle\lambda_\alpha,G_\gamma\Box F\rangle\quad(\langle\Psi\cdot F,\Lambda\rangle=\lim_{\alpha}\lim_{\gamma}\langle\lambda_\alpha,F\Box G_\gamma\rangle),$$

hence the $\mathcal{A}''$-bimodule structure part (II) is different from (I) and (III). For two $\mathcal{A}''$-module action on $\mathcal{A}^{(5)}$ in part (IV), we have

$$\langle F\star\Psi,\Lambda\rangle=\lim_{\alpha}\lim_{\gamma}\langle F\circ\lambda_\alpha,G_\gamma\rangle,\qquad\langle\Psi\star F,\Lambda\rangle=\lim_{\alpha}\lim_{\gamma}\langle\lambda_\alpha\circ F,G_\gamma\rangle.$$

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra. Suppose that the map $\varphi:\mathcal{A}\times\mathcal{A}''\longrightarrow\mathcal{A}'';((a,F)\longrightarrow a\cdot F)$ is Arens regular and the map $\psi:\mathcal{A}''\longrightarrow\mathcal{A}'';(G\longrightarrow G\Box F)$ is w*-w-continuous for every $F\in\mathcal{A}''$. Then the two right $\mathcal{A}''$-module actions on $\mathcal{A}^{(5)}$in (II) and (III) are equal.*

Proof. By w^*-w-continuity of ψ we must prove the following equality

$$\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\langle\lambda_\alpha,a_\beta\cdot G_\gamma\rangle=\lim_{\alpha}\lim_{\gamma}\lim_{\beta}\langle\lambda_\alpha,a_\beta\cdot G_\gamma\rangle,\tag{3}$$

for bounded nets $(\lambda_\alpha),(G_\gamma)$ and (a_β) in $\mathcal{A}''',\mathcal{A}''$ and $\mathcal{A}'$, respectively. By Arens regularity of φ, we have

$$\lim_{\beta}\lim_{\gamma}\langle\lambda_\alpha,a_\beta\cdot G_\gamma\rangle=\lim_{\beta}\lim_{\gamma}\langle\widehat{a_\beta\cdot G_\gamma},\lambda_\alpha\rangle=\lim_{\gamma}\lim_{\beta}\langle\widehat{a_\beta\cdot G_\gamma},\lambda_\alpha\rangle,$$

and so (3) is true. ■

Lemma 3.2. *Let $\mathcal{A}$ be a Banach algebra. Assume that the Banach algebra $\mathcal{A}$ and the map $\varphi:\mathcal{A}''\times\mathcal{A}'''\longrightarrow\mathcal{A}''';((F,\lambda)\longrightarrow F\cdot\lambda)$ is Arens regular and the map $\psi:\mathcal{A}''\longrightarrow\mathcal{A}'';(G\longrightarrow F\Box G)$ is w*-w-continuous for every $F\in\mathcal{A}''$. Then two left $\mathcal{A}''$-module actions on $\mathcal{A}^{(5)}$in (II) and (III) are equal.*

Proof. By w^*-w-continuity of ψ, it is enough to prove the following equality for bounded nets $(\lambda_\alpha),(G_\gamma)$ and (a_β) in $\mathcal{A}''',\mathcal{A}''$ and $\mathcal{A}'$, respectively.

$$\lim_{\alpha}\lim_{\gamma}\lim_{\beta}\langle\lambda_\alpha,G_\gamma\Box\widehat{a}_\beta\rangle=\lim_{\beta}\lim_{\alpha}\lim_{\gamma}\langle\lambda_\alpha,G_\gamma\Box\widehat{a}_\beta\rangle.\tag{4}$$

By using Lemma 2.4 we can write

$$\lim_\gamma \lim_\beta \langle \lambda_\alpha, G_\gamma \Box \widehat{a}_\beta \rangle = \lim_\beta \lim_\gamma \langle \lambda_\alpha, G_\gamma \Box \widehat{a}_\beta \rangle.$$

Then, by Arens regularity of φ, we see

$$\begin{aligned}
\lim_\alpha \lim_\gamma \lim_\beta \langle \lambda_\alpha, G_\gamma \Box \widehat{a}_\beta \rangle &= \lim_\alpha \lim_\beta \lim_\gamma \langle \lambda_\alpha, G_\gamma \Box \widehat{a}_\beta \rangle \\
&= \lim_\alpha \lim_\beta \langle \lambda_\alpha, (w^* - \lim_\alpha G_\gamma) \Box \widehat{a}_\beta \rangle \\
&= \lim_\alpha \lim_\beta \langle \widehat{\widehat{a}_\beta \cdot \lambda_\alpha}, (w^* - \lim_\alpha G_\gamma)\widehat{\ } \rangle \\
&= \lim_\beta \lim_\alpha \langle \widehat{\widehat{a}_\beta \cdot \lambda_\alpha}, (w^* - \lim_\alpha G_\gamma)\widehat{\ } \rangle \\
&= \lim_\beta \lim_\alpha \lim_\gamma \langle a_\beta \cdot \lambda_\alpha, G_\gamma \rangle \\
&= \lim_\beta \lim_\alpha \lim_\gamma \langle \lambda_\alpha, G_\gamma \Box \widehat{a}_\beta \rangle.
\end{aligned}$$

4 3-weak amenability of the second dual

Let $D : \mathcal{A} \longrightarrow \mathcal{A}'''$ be a derivation. Then $D'' : \mathcal{A}'' \longrightarrow \mathcal{A}^{(5)} = (\mathcal{A}''')''$ the second transpose of D is a derivation (see [3] and [11]), that means that for every $F, G \in \mathcal{A}''$

$$D''(F \Box G) = D(F) \circ G + F \circ D(G).$$

But $D'' : \mathcal{A}'' \longrightarrow \mathcal{A}^{(5)} = (\mathcal{A}'')'''$ is not always a derivation. In the following we put other conditions on D such that D'' is a derivation (also see Theorem 4.3).

Proposition 4.1. *Let $\mathcal{A}$ be a Banach algebra and let $D : \mathcal{A} \longrightarrow \mathcal{A}'''$ be a derivation. Then $D'' : \mathcal{A}'' \longrightarrow \mathcal{A}^{(5)} = (\mathcal{A}'')'''$ is a derivation if and only if $D''(\mathcal{A}'') \cdot \mathcal{A}^{(4)} \subseteq \widehat{\mathcal{A}'}$.*

Proof. Let $F, G \in \mathcal{A}''$. Then there are nets (a_α) and (b_β) in $\mathcal{A}$ which converge to F and G in the w^*- topology of $\mathcal{A}''$ respectively. Clearly D'' is w^*- continuous. Then

$$\begin{aligned}
D''(F \Box G) &= w^* - \lim_\alpha w^* - \lim_\beta D(a_\alpha b_\beta) \\
&= w^* - \lim_\alpha w^* - \lim_\beta D(a_\alpha) \cdot b_\beta + w^* - \lim_\alpha w^* - \lim_\beta a_\alpha \cdot D(b_\beta) \\
&= D''(F) \cdot G + \lim_\alpha a_\alpha \cdot D''(G).
\end{aligned}$$

By (5), it is easy to see that D'' is a derivation if and only if for every $F, G \in \mathcal{A}''$ that $F = w^* - \lim_\alpha a_\alpha$, the following equality holds

$$F \cdot D''(G) = w^* - \lim_\alpha a_\alpha \cdot D''(G) \qquad (6).$$

The relation (6) is true if and only if for every

$$\Lambda \in \mathcal{A}^{(4)}, \langle F \cdot D''(G), \Lambda \rangle = \lim_\alpha \langle a_\alpha \cdot D''(G), \Lambda \rangle (7).$$

Also (7) holds if and only if

$$\langle D''(G).\Lambda, F \rangle = \lim_\alpha \langle D''(G) \cdot \Lambda, a_\alpha \rangle \qquad (8).$$

So (8) holds if and only if $D''(G).\Lambda : \mathcal{A}'' \to \mathbb{C}$ is $w^* - w^*-$continuous. This means that $D''(G).\Lambda \in \widehat{\mathcal{A}'}$. ■

Corollary 4.2 *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}''$ is 3-weakly amenable. If $D''(\mathcal{A}'') \cdot \mathcal{A}^{(4)} \subseteq \widehat{\mathcal{A}'}$, for each derivation $D : \mathcal{A} \longrightarrow \mathcal{A}'''$. Then $\mathcal{A}$ is 3 -weakly amenable.* ■

Let $\mathcal{A}$ be a Banach algebra and let $\iota : \mathcal{A}'' \longrightarrow \mathcal{A}^{(4)}$ be an injective map ($\langle \iota(F), \lambda \rangle = \langle \lambda, F \rangle$) for $F \in \mathcal{A}''$ and $\lambda \in \mathcal{A}'''$. Then ι is an $\mathcal{A}$-bimodule homomorphism . Also ι is an $\mathcal{A}''$-bimodule homomorphism with the module structures (a) and (b) on $\mathcal{A}^{(4)}$, but it is not always an $\mathcal{A}''$-bimodule homomorphism with the module structures (c). Therefore the adjoint of ι (ι^*) is an $\mathcal{A}''$-bimodule homomorphism with the module structures (a) and (b). Let X be a Banach space. For $n \in \mathbb{Z}^+$, we denote $X^\perp$, the subspace of $X^{(2n+1)}$ annihilating $\widehat{X}$, where $X^{(2n+1)}$ is the (2n+1)-th dual of X, i.e. $X^\perp = \{\lambda \in X^{(2n+1)}; \langle \lambda, x \rangle = 0, \quad x \in X\}$. For the Banach algebra $\mathcal{A}$, $(\mathcal{A}'')^\perp$ is clearly w^*-closed $\mathcal{A}''$-submodule of $\mathcal{A}^{(5)}$. Now we get the main theorem of this paper.

Theorem 4.3. *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}''$ is weakly amenable. Suppose that one the following conditions holds*

(1) $D''(\mathcal{A}'') \cdot \mathcal{A}^{(4)} \subseteq \widehat{\mathcal{A}'}$, for each derivation $D : \mathcal{A} \longrightarrow \mathcal{A}'''$.

(2) Conditions (i) of Lemma 3.1 and Lemma 3.2 are true.

(3) $\lim_\alpha \lim_\gamma \lim_\beta \langle \lambda_\alpha, F_\gamma \cdot a_\beta \rangle = \lim_\beta \lim_\alpha \lim_\gamma \langle \lambda_\alpha, F_\gamma \cdot a_\beta \rangle$, for every bounded nets (λ_α) in $\mathcal{A}'''$, (F_γ) in $\mathcal{A}''$ and every net (a_β) in $\mathcal{A}$.

Then $\mathcal{A}$ is 3 -weakly amenable.

Proof. Suppose that one of conditions (1) or (2) holds, so the two $\mathcal{A}''$-bimodule structures in parts (II) and (III) on $\mathcal{A}^{(5)}$ are equal. We know $(\mathcal{A}'')''' = (\mathcal{A}'')' \oplus (\mathcal{A}'')^\perp$. In other words $\mathcal{A}^{(5)} = (\mathcal{A}'')'''$ is a direct summand of $\mathcal{A}''-$ submodules of $\mathcal{A}^{(5)}$. Let $P : (\mathcal{A}'')''' \longrightarrow (\mathcal{A}'')'$ be the projection defined by the above direct sum. Suppose $D : \mathcal{A} \longrightarrow \mathcal{A}'''$ is a derivation. Then we can show that P is an $\mathcal{A}''$-module homomorphism. Thus $P \circ D'' : \mathcal{A}'' \longrightarrow (\mathcal{A}''')'' = (\mathcal{A}'')''' \longrightarrow (\mathcal{A}'')'$ is a derivation. Since $\mathcal{A}''$ is weakly amenable, there exists $\theta_0 \in (\mathcal{A}'')'$ such that $P \circ D = \delta_{\theta_0}$. On the other hand D is the restriction of $P \circ D''$ to $\mathcal{A}$. Thus $D = \delta_{\theta_0}$.

Now assume that the condition (3) holds, then the two $\mathcal{A}''$-bimodule structures in parts (I) and (III) on $\mathcal{A}^{(5)}$ are equal. Suppose $D : \mathcal{A} \longrightarrow \mathcal{A}'''$ is a derivation. Then $\iota^* \circ D'' : \mathcal{A}'' \longrightarrow (\mathcal{A}''')'' = (\mathcal{A}^{(4)})' \longrightarrow (\mathcal{A}'')'$ is a derivation. Due to the weak amenability of $\mathcal{A}''$, there exists $\theta_0 \in (\mathcal{A}'')'$ such that $\iota^* o D'' = \delta_{\theta_0}$. For every $a \in \mathcal{A}$ and $F \in \mathcal{A}''$, we have

$$\langle \iota^* \circ D''(a), F\rangle = \langle D''(a), \iota(F)\rangle = \langle \iota(F), D(a)\rangle = \langle D(a), F\rangle.$$

So D is the restriction of $\iota^* \circ D''$ to $\mathcal{A}$. Thus $D = \delta_{\theta_0}$. Therefore $\mathcal{A}$ is 3 -weakly amenable. ■

By applying Theorem 4.3 we have the following results.

Corollary 4.4. *Let $\mathcal{A}$ be a Banach algebra such that one of the conditions (1) up to (4) of Theorem 4.3 holds. If $\mathcal{A}''$ is weakly amenable, then $\mathcal{A}$ is weakly amenable.*

Proof. It follows immediately from Theorem 4.3 and [4, Proposition 1.2]. ■

Corollary 4.5. *Let $\mathcal{A}$ be a Banach algebra such that one of the conditions (1) up to (3) of Theorem 4.3 holds. If $\mathcal{A}''$ is 3 -weakly amenable, then $\mathcal{A}$ is 3 -weakly amenable.*

Proof. By Theorem 4.3 and [4, Proposition 1.2], $\mathcal{A}$ is 3 -weakly amenable. ■

The following Theorem has been proved in [10].

Theorem 4.6. *Let $\mathcal{A}$ be an Arens regular Banach algebra. Suppose that for every continuous derivation $D : \mathcal{A}'' \longrightarrow \mathcal{A}'''$ and every F in $\mathcal{A}'', D(F)$ and D are w^*-continuous. If $\mathcal{A}$ is weakly amenable , then so is $\mathcal{A}''$.* ■

Let one of the conditions of Theorem 4.3 holds. Then we have

Corollary 4.7. *Under assumptions of Theorem 4.6, if $\mathcal{A}$ is weakly amenable, then $\mathcal{A}$ is 3-weakly amenable.*

Proof. It is an immediate consequence of Theorem 4.3 and Theorem 4.6. ■

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