An. Şt. Univ. Ovidius Constanţa Vol. 18(1), 2010, 219–228

STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR λ-STRICTLY PSEUDO-CONTRACTIVE MAPPINGS IN HILBERT SPACES

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Abstract

Let H be a real Hilbert space. Let $T : H \to H$ be a λ-strictly pseudo-contractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$. For given $x_0 \in H$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0.$$

Under some mild conditions on parameters $\{\alpha_n\}$ and $\{\beta_n\}$, we prove that the sequence $\{x_n\}$ converges strongly to a fixed point of T in Hilbert spaces.

1 Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. And $T : C \to C$ is said to be a strictly pseudo-contractive mapping if there exists a constant $0 \leq \lambda < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda\|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

Key Words: λ-strictly pseudo-contractive mapping; fixed point; iterative algorithm; strong convergence; Hilbert space.

Mathematics Subject Classification: 47H05; 47H10; 47H17

Received: May, 2009

Accepted: January, 2010

for all $x, y \in C$. For such a case, we also say that T is a λ-strictly pseudo-contractive mapping. It is clear that, in a real Hilbert space H, (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\lambda}{2}\|(I - T)x - (I - T)y\|^2, \qquad (1.2)$$

for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T.

It is clear that the class of strictly pseudo-contractive mappings strictly includes the class of non-expansive mappings. Iterative methods for non-expansive mappings have been extensively investigated in the literature; see [1]-[11],[13] and the references therein. Related work can be found in [12],[14]-[22].

However iterative methods for strictly pseudo-contractive mappings are far less developed than those for non-expansive mappings though Browder and Petryshyn initiated their work in 1967; the reason is probably that the second term appearing in the right-hand side of (1.1) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strictly pseudo-contractive mapping T. However, on the other hand, strictly pseudo-contractive mappings have more powerful applications than non-expansive mappings do in solving inverse problems; see Scherzer [12]. Therefore it is interesting to develop the iterative methods for strictly pseudo-contractive mappings. As a matter of fact, Browder and Petryshyn [2] show that if a λ-strictly pseudo-contractive mapping T has a fixed point in C, then starting with an initial $x_0 \in C$, the sequence $\{x_n\}$ generated by the recursive formula:

$$x_{n+1} = \alpha x_n + (1 - \alpha)Tx_n, \quad n \geq 0,$$

where α is a constant such that $\lambda < \alpha < 1$, converges weakly to a fixed point of T.

Recently, Marino and Xu [7] have extended Browder and Petryshyn's result by proving that the sequence $\{x_n\}$ generated by the following Mann's algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0$$

converges weakly to a fixed point of T, provided the control sequence $\{\alpha_n\}$ satisfies the conditions that $\lambda < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - \lambda)(1 - \alpha_n) = \infty$. However, this convergence is in general not strong. Very recently, Mainge [6] studied some new iterative methods for strictly pseudo-contractive mappings. He obtained some strong convergence theorems by using the new iterative methods.

It is our purpose in this paper that we introduce a new iterative algorithm for λ-strictly pseudo-contractive mappings as follows:

Let H be a real Hilbert space. Let $T : H \to H$ be a λ-strictly pseudo-contractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$. For given $x_0 \in H$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0. \tag{1.3}$$

Under some mild conditions, we prove that the proposed iterative algorithm (1.3) converges strongly to a fixed point of a λ-strictly pseudo-contractive mapping T in Hilbert spaces.

2 Preliminaries

In this section, we collect the following well-known lemmas.

Lemma 2.1. *Let H be a real Hilbert space. Then there holds the following well-known results:*

(i) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$*;*

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H$*.*

You can find the following lemma in [7],[22].

Lemma 2.2. *(Demi-closed principle) Let C be a nonempty closed convex of a real Hilbert space H. Let $T : C \to C$ be a λ-strictly pseudo-contractive mapping. Then $I - T$ is demi-closed at 0, i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.3. ([7]) *Let H be a real Hilbert space. If $\{x_n\}$ is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2, \forall y \in H.$$

Lemma 2.4. ([16]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$*;*

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$*.*

Then $\lim_{n\to\infty} a_n = 0$*.*

3 Main Results

Theorem 3.1. *Let H be a real Hilbert space. Let $T : H \to H$ be a λ-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$. Assume that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [\epsilon, (1-\lambda)(1-\alpha_n))$ *for some* $\epsilon > 0$.

Then the sequence $\{x_n\}$ generated by (1.3) strongly converges to a fixed point of T.

Proof. First, we prove that the sequence $\{x_n\}$ is bounded.

Take $p \in F(T)$. From 1.3), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n)(x_n-p) + \beta_n(Tx_n - p) - \alpha_n p\| \\ &\leq \|(1-\alpha_n-\beta_n)(x_n-p) + \beta_n(Tx_n - p)\| + \alpha_n\|p\|. \quad (3.1)\end{aligned}$$

Combining (1.1) and (1.2), we have

$$\begin{aligned} & \|(1-\alpha_n-\beta_n)(x_n-p) + \beta_n(Tx_n - p)\|^2 \\ = \; & (1-\alpha_n-\beta_n)^2\|x_n-p\|^2 + \beta_n^2\|Tx_n-p\|^2 \\ & +2(1-\alpha_n-\beta_n)\beta_n\langle Tx_n - p, x_n - p\rangle \\ \leq \; & (1-\alpha_n-\beta_n)^2\|x_n-p\|^2 + \beta_n^2[\|x_n-p\|^2 + \lambda\|x_n - Tx_n\|^2] \\ & +2(1-\alpha_n-\beta_n)\beta_n[\|x_n-p\|^2 - \frac{1-\lambda}{2}\|x_n - Tx_n\|^2] \\ = \; & (1-\alpha_n)^2\|x_n-p\|^2 + [\lambda\beta_n^2 - (1-\lambda)(1-\alpha_n-\beta_n)\beta_n]\|x_n - Tx_n\|^2 \\ = \; & (1-\alpha_n)^2\|x_n-p\|^2 + \beta_n[\beta_n - (1-\alpha_n)(1-\lambda)]\|x_n - Tx_n\|^2 \\ \leq \; & (1-\alpha_n)^2\|x_n-p\|^2, \end{aligned}$$

which implies that

$$\|(1-\alpha_n-\beta_n)(x_n-p) + \beta_n(Tx_n - p)\| \leq (1-\alpha_n)\|x_n - p\|. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1-\alpha_n)\|x_n-p\| + \alpha_n\|p\| \\ &\leq \max\{\|x_n - p\|, \|p\|\}.\end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|p\|\}.$$

Hence, $\{x_n\}$ is bounded.

Taking $y = p$ in (1.1), we have

$$\begin{aligned} & \|Tx-p\|^2 \le \|x-p\|^2 + \lambda\|x-Tx\|^2 \\ \Rightarrow\quad & \langle Tx-p, Tx-p\rangle \le \langle x-p, x-Tx\rangle + \langle x-p, Tx-p\rangle + \lambda\|x-Tx\|^2 \\ \Rightarrow\quad & \langle Tx-p, Tx-x\rangle \le \langle x-p, x-Tx\rangle + \lambda\|x-Tx\|^2 \\ \Rightarrow\quad & \langle Tx-x, Tx-x\rangle + \langle x-p, Tx-x\rangle \le \langle x-p, x-Tx\rangle + \lambda\|x-Tx\|^2 \\ \Rightarrow\quad & (1-\lambda)\|Tx-x\|^2 \le 2\langle x-p, x-Tx\rangle. \end{aligned} \tag{3.3}$$

From (1.3), (3.3) and Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|(1-\alpha_n-\beta_n)x_n + \beta_n Tx_n - p\|^2 \\ &= \|(x_n-p) - \beta_n(x_n-Tx_n) - \alpha_n x_n\|^2 \\ &\le \|(x_n-p) - \beta_n(x_n-Tx_n)\|^2 - 2\alpha_n\langle x_n, x_{n+1}-p\rangle \\ &= \|x_n-p\|^2 - 2\beta_n\langle x_n-Tx_n, x_n-p\rangle + \beta_n^2\|x_n-Tx_n\|^2 \\ &\quad -2\alpha_n\langle x_n, x_{n+1}-p\rangle \\ &\le \|x_n-p\|^2 - \beta_n(1-\lambda)\|x_n-Tx_n\|^2 + \beta_n^2\|x_n-Tx_n\|^2 \\ &\quad -2\alpha_n\langle x_n, x_{n+1}-p\rangle \\ &= \|x_n-p\|^2 - \beta_n[(1-\lambda)-\beta_n]\|x_n-Tx_n\|^2 \\ &\quad -2\alpha_n\langle x_n, x_{n+1}-p\rangle. \end{aligned} \tag{3.4}$$

Since $\{x_n\}$ is bounded, so there exists a constant $M \ge 0$ such that

$$-2\langle x_n, x_{n+1}-p\rangle \le M \quad \text{for all} \quad n \ge 0.$$

Consequently, from (3.4), we get

$$\|x_{n+1}-p\|^2 - \|x_n-p\|^2 + \beta_n[(1-\lambda)-\beta_n]\|x_n-Tx_n\|^2 \le M\alpha_n. \tag{3.5}$$

Now we divide two cases to prove that $\{x_n\}$ converges strongly to p.

Case 1. Assume that the sequence $\{\|x_n-p\|\}$ is a monotonically decreasing sequence. Then $\{\|x_n-p\|\}$ is convergent. Clearly, we have

$$\|x_{n+1}-p\|^2 - \|x_n-p\|^2 \to 0,$$

this together with (C1) and (3.5) imply that

$$\|x_n - Tx_n\| \to 0. \tag{3.6}$$

By Lemma 2.2 and (3.6), it is easy to see that $\omega_w(x_n) \subset F(T)$, where $\omega_w(x_n) = \{x : \exists x_{n_i} \rightharpoonup x\}$ is the weak ω-limit set of $\{x_n\}$. This implies that $\{x_n\}$

converges weakly to a fixed point x^* of T. Indeed, if we take $x^*, \tilde{x} \in \omega_w(x_n)$ and let $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be sequences of $\{x_n\}$ such that

$$x_{n_i} \rightharpoonup x^* \ \text{ and } \ x_{m_j} \rightharpoonup \tilde{x}, \ \text{ respectively.}$$

Since $\lim_{n\to\infty} \|x_n - z\|$ exists for $z \in F(T)$. Therefore, by Lemma 2.3, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\|^2 &= \lim_{j\to\infty} \|x_{m_j} - x^*\|^2 \\
&= \lim_{j\to\infty} \|x_{m_j} - \tilde{x}\|^2 + \|\tilde{x} - x^*\|^2 \\
&= \lim_{i\to\infty} \|x_{n_i} - \tilde{x}\|^2 + \|\tilde{x} - x^*\|^2 \\
&= \lim_{i\to\infty} \|x_{n_i} - x^*\|^2 + 2\|\tilde{x} - x^*\|^2 \\
&= \lim_{n\to\infty} \|x_n - x^*\|^2 + 2\|\tilde{x} - x^*\|^2.
\end{aligned}$$

Hence, $\tilde{x} = x^*$.

Next, we prove that $\{x_n\}$ strongly converges to x^*.

Setting $y_n = (1-\beta_n)x_n + \beta_n Tx_n, n \geq 0$. Then, we can rewrite (1.3) as

$$x_{n+1} = y_n - \alpha_n x_n, n \geq 0.$$

It follows that

$$\begin{aligned}
x_{n+1} &= (1-\alpha_n)y_n - \alpha_n(x_n - y_n) \\
&= (1-\alpha_n)y_n - \alpha_n\beta_n(x_n - Tx_n).
\end{aligned} \tag{3.7}$$

At the same time, we note that

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|x_n - x^* - 2\beta_n(x_n - Tx_n)\|^2 \\
&= \|x_n - x^*\|^2 - 2\beta_n\langle x_n - Tx_n, x_n - x^*\rangle + \beta_n^2\|x_n - Tx_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \beta_n[(1-\lambda) - \beta_n]\|x_n - Tx_n\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Applying Lemma 2.1 to (3.7), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(y_n - x^*) - \alpha_n\beta_n(x_n - Tx_n) - \alpha_n x^*\|^2 \\
&\leq (1-\alpha_n)^2\|y_n - x^*\|^2 - 2\alpha_n\beta_n\langle x_n - Tx_n, x_{n+1} - x^*\rangle \\
&\quad -2\alpha_n\langle x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 - 2\alpha_n\beta_n\langle x_n - Tx_n, x_{n+1} - x^*\rangle \\
&\quad -2\alpha_n\langle x^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.8}$$

It is clear that $\lim_{n\to\infty}\langle x_n - Tx_n, x_{n+1} - x^*\rangle = 0$ and $\lim_{n\to\infty}\langle x^*, x_{n+1} - x^*\rangle = 0$. Hence, applying Lemma 2.4 to (3.8), we immediately deduce that $x_n \to x^*$.

Case 2. Assume that $\{\|x_n - p\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n - p\|^2$ and let $\tau : N \to N$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in N : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (3.5), it is easy to see that

$$\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \leq \frac{M\alpha_{\tau(n)}}{\beta_{\tau(n)}[(1-\lambda) - \beta_{\tau(n)}]} \to 0,$$

thus

$$\|x_{\tau(n)} - Tx_{\tau(n)}\| \to 0.$$

By the similar argument as above in Case 1, we conclude immediately that $x_{\tau(n)}$ weakly converges to x^* as $\tau(n) \to \infty$. At the same time, we note that, for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \\ &\leq \alpha_{\tau(n)}[2\beta_{\tau(n)}\langle x_{\tau(n)} - Tx_{\tau(n)}, x^* - x_{\tau(n)+1}\rangle + 2\langle x^*, x^* - x_{\tau(n)+1}\rangle \\ &\quad -\|x_{\tau(n)} - x^*\|^2], \end{aligned}$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \leq 2\beta_{\tau(n)}\langle x_{\tau(n)} - Tx_{\tau(n)}, x^* - x_{\tau(n)+1}\rangle + 2\langle x^*, x^* - x_{\tau(n)+1}\rangle.$$

Hence, we deduce that

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\| = 0.$$

Therefore,

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim_{n\to\infty}\Gamma_n = 0$, this is, $\{x_n\}$ converges strongly to x^*. This completes the proof. $\square$

From Theorem 3.1, we can obtain the following corollary.

Corollary 3.2. *Let H be a real Hilbert space. Let $T : H \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$. Assume that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [\epsilon, (1-\lambda)(1-\alpha_n))$ *for some* $\epsilon > 0$.

Then the sequence $\{x_n\}$ generated by (1.3) strongly converges to a fixed point of T.

Remark 3.3. It is well-known that the normal Mann iteration has only weak convergence. However, our algorithm which is similar to the normal Mann iteration has strong convergence.

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