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SIGNED (b, k)-MATCHINGS IN GRAPHS

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Abstract

Let G be a simple graph without isolated vertices with vertex set $V(G)$ and edge set $E(G)$, b be a positive integer and k an integer with $1 \leq k \leq |V(G)|$. A function $f : E(G) \to \{-1, 1\}$ is said to be a signed (b, k)-matching of G if $\sum_{e \in E(v)} f(e) \leq b$ for at least k vertices v of G, where $E(v)$ is the set of all edges at v. The value $\max \sum_{e \in E(G)} f(e)$, taking over all signed (b, k)-matching f of G, is called the signed (b, k)-matching number of G and is denoted by $\beta'_{(b,k)}(G)$. In this paper we initiate the study of the signed (b, k)-matching number in graphs and present some bounds for this parameter.

1 Introduction

Structural and algorithmic aspects of covering vertices by edges have been extensively studied in graph theory. A matching (edge cover) of a graph G is a set C of edges of G such that each vertex of G is incident to at most (at least) one edge of C: Let b be a fixed positive integer. A simple b-matching (simple b-edge cover) of a graph G is a set C of edges of G such that each vertex of G is incident to at most (at least) b edges of C: The maximum (minimum) size of a simple b-matching (simple b-edge cover) of G is called b-matching number (b-edge cover number), denoted by $\beta_b(G)$ ($\rho_b(G)$). The (simple) b-matching problems have been widely studied in, for instance, [1, 3, 4, 6, 7]. For an

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excellent survey of results on matchings, edge covers, b-matchings and b-edge covers, see Schrijver [8].

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [12] for terminology and notation which are not defined here and consider simple graphs without isolated vertices only. For every nonempty subset E' of $E(G)$, the subgraph of G whose vertex set is the set of vertices of the edges in E' and whose edge set is E', is called the subgraph of G induced by E' and denoted by $G[E']$.

For a function $f : E(G) \longrightarrow \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e\in S} f(e)$. The *edge-neighborhood* $E_G(v)$ of a vertex $v \in V(G)$ is the set of all edges incident to v. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e\in E_G(v)} f(e)$. Let b be a positive integer and k an integer with $1 \le k \le n$. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed* (b, k)*-matching* (SbkM) of G, if $f(v) \le b$ for at least k vertices v of G. The *signed* (b, k)*-matching number* of a graph G is $\beta'_{(b,k)}(G) = \max\{\sum_{e\in E(G)} f(e) \mid f$ is a SbkM on $G\}$. The signed (b, k)-matching f of G with $f(E(G)) = \beta'_{(b,k)}(G)$ is called $\beta'_{(b,k)}(G)$*-matching.* For any signed (b, k)-matching f of G we define $P = \{e \in E \mid f(e) = 1\}$, $M = \{e \in E \mid f(e) = -1\}$, $B_f = \{v \in V \mid f(v) \le b\}$.

If $b = 1$ and $k = n$, then the signed (b, k)-matching number is called the *signed matching number.* The signed matching number was introduced by Wang in [9] and denoted by $\beta'_1(G)$.

If $b = 1$ and $1 \le k \le n$, then the signed (b, k)-matching number is called the *signed* k*-submatching number.* The signed k-submatching number was introduced by Ghameshlou et al. in [2] and Wang in [11] and denoted by $\beta'_{(1,k)}(G)$.

When b is an arbitrary positive integer and $k = n$, then the signed (b, k)-matching number is called the *signed* b*-matching number.* The signed b-matching number was introduced by Wang in [10] and denoted by $\beta'_b(G)$.

The purpose of this paper is to initialize the study of the signed (b, k)-matching number $\beta'_{b,k}(G)$ and established some bounds for this parameter. Here are some results on $\beta'_1(G)$, $\beta'_b(G)$ and $\beta'_{(1,k)}(G)$.

The proof of the following Theorems can be found in [2], [9], [10] and [11].

Theorem A. *For any graph G of order $n \ge 2$ without isolated vertices, $\beta'_1(G) \ge -1$.*

Theorem B. *For any graph G of order $n \ge 2$ with no isolated vertex and each positive integer $b < \Delta(G)$, $\beta'_b(G) \le \lfloor \frac{bn}{2} \rfloor$.*

Theorem C. *For any graph G of order $n \ge 2$ and without isolated vertices,*

$$\beta'_{(1,k)}(G) \le \frac{k(1 - \Delta(G)) + n\Delta(G)}{2}.$$

Furthermore, this bound is sharp for C_n if k is even and P_n when n is odd.

Theorem D. *Let G be a graph of order $n \geq 2$, size m, minimum degree δ, maximum degree Δ and without isolated vertices . Then*

$$\beta'_{(1,k)}(G) \leq \frac{(2m+n)\Delta + (n-k)\Delta^2}{2\delta} - m.$$

Theorem E. *For $n \geq 2$ and positive integer $1 \leq k \leq n$,*

$$\beta'_{(1,k)}(P_n) = n + 1 - 2\lceil \frac{k}{2} \rceil.$$

Theorem F. *For $n \geq 3$ and any positive integer $1 \leq k \leq n$,*

$$\beta'_{(1,k)}(C_n) = n - 2\lceil \frac{k}{2} \rceil.$$

Theorem G. *For $n \geq 2$,*

$$\beta'_{(1,k)}(K_{1,n-1}) = \begin{cases} n & \text{if } k \leq n-1 \\ 0 & \text{if } k = n \text{ and } n \text{ is odd} \\ 1 & \text{if } k = n \text{ and } n \text{ is even.} \end{cases}$$

2 Lower bounds on SbkMN of trees and cactus graphs

In this section we give a lower bound for signed (b,k)-matching number of trees and connected cactus graphs. The proof of the first proposition is clear and therefore omitted.

Proposition 1. *For any $n \geq 2$, $b \leq n-1$ and $1 \leq k \leq n$,*

$$\beta'_{(b,k)}(K_{1,n-1}) = \begin{cases} n-1 & \text{if } k \leq n-1 \\ b-1 & \text{if } k = n \text{ and } n-b \equiv 0 \pmod 2 \\ b & \text{if } k = n \text{ and } n-b \equiv 1 \pmod 2. \end{cases}$$

If G is a graph, then let $L(G)$ denote the set of leaves.

Proposition 2. *Let G be a graph of order n such that $k \leq |L(G)|$ for an integer $1 \leq k \leq n$. If b is a positive integer, then $\beta'_{b,k}(G) = |E(G)|$.*

Proof. If we define $f : E(G) \to \{-1, 1\}$ by $f(e) = 1$ for each $e \in E(G)$, then we observe that f is a SbkM. □

Theorem 3. *For any tree T of order $n \geq 2$, positive integer $b < \Delta(T)$ and each positive integer $1 \leq k \leq n$,*

$$\beta'_{(b,k)}(T) \geq \min\{n - k + b - 1, n - 1\}.$$

Proof. First assume that $b > k$. Then it is well-known that $|L(T)| \geq \Delta(T)$. Thus it follows that

$$|L(T)| \geq \Delta(T) > b > k.$$

Applying Proposition 2, we obtain the desired result

$$\beta'_{(b,k)}(T) = |E(T)| = n - 1 \geq \min\{n - k + b - 1, n - 1\}.$$

Next assume that $b \leq k$. Then $\min\{n-k+b-1, n-1\} = n-k+b-1$. The proof is by induction on n. If $n = 2, 3, 4$, then the result follows by Theorem E and Proposition 1. Suppose $n \geq 5$ and that the statement is true for any nontrivial T' of order $n' < n$ and any integer k' with $1 \leq k' \leq n'$. Let T be a tree of order n and $1 \leq k \leq n$.

If T is a star, then the result is true by Proposition 1. Thus we may assume $\mathrm{diam}(T) \geq 3$. Let T be rooted at a leaf v_0 of a longest path. Let v be a vertex at distance $\mathrm{diam}(T) - 1$ from v_0 on a longest path starting from v_0 and w the parent of v. Suppose c is the number of v's children. Then $c \geq 1$. If $k \leq c + 1$, then assign $+1$ to all edges of T to obtain a SbkM of T with weight $n-1$ which follows the result. Hence, we may assume $k > c+1$.

Case 1. $c = 1$.
Let u be the leaf adjacent to v and $T' = T - T_v$. Then T' has order $n' = n-2$. Assume $k' = k-2$. Since $c+2 \leq k \leq n$, we have $c \leq k' \leq n'$. By the inductive hypothesis, $\beta'_{(b,k')}(T') \geq n' - k' + b - 1$. Let f' be a $\beta'_{(b,k')}(T')$-matching. Define $f : E(T) \to \{-1, 1\}$ by $f(wv) = -1, f(vu) = 1$ and $f(e) = f'(e)$ for $e \in E(T')$. Obviously, f is a SbkM of T and we have with $\beta'_{(b,k)}(T) \geq f(E(T)) = f'(E(T')) \geq n' - k' + b - 1 = n - k + b - 1$.

Case 2. $c \geq 2$.
Let u_1 and u_2 be two leaves adjacent to v and let $T' = T - \{u_1, u_2\}$. Then T' has order $n' = n - 2$. Assume $k' = k - 2$. Since $c + 2 \leq k \leq n$, we have $c \leq k' \leq n'$. By the inductive hypothesis, $\beta'_{(b,k')}(T') \geq n' - k' + b - 1$. Let f' be a $\beta'_{(b,k')}(T')$-matching. Define $f : E(T) \to \{-1, 1\}$ by $f(vu_1) = -1, f(vu_2) = 1$ and $f(e) = f'(e)$ otherwise. Obviously f is a SbkM of G with $\beta'_{(b,k)}(T) \geq f(E(T)) = f'(E(T')) \geq n' - k' + b - 1 = n - k + b - 1$. This completes the proof. □

Theorem 4. *Let G be a connected cactus graph of order $n \geq 3$ with exactly $p \geq 0$ cycles. If $b < \Delta(G)$ and $k \leq n$ are positive integers, then*

$$\beta'_{(b,k)}(G) \geq \min\{n-k+b-(p+1), n+p-1\}.$$

Proof. We proceed by induction on the number p of cycles. If $p = 0$, then Theorem 3 shows that the desired bound is valid. Let now $p \geq 1$. Note that $|E(G)| = n+p-1$ and $|L(G)| \geq \Delta(G) - 2p$.

First assume that $b \geq k+2p$. Then $|L(G)| \geq \Delta(G)-2p > b-2p \geq k$, and hence Proposition 2 implies that

$$\beta'_{(b,k)}(G) = |E(G)| = n+p-1 = \min\{n-k+b-(p+1), n+p-1\}.$$

If $b = 2p+k-1$, then $\min\{n-k+b-(p+1), n+p-1\} = n+p-2$ and $|L(G)| \geq \Delta(G)-2p \geq b+1-2p = k$. Thus again Proposition 2 leads to the desired bound

$$\beta'_{(b,k)}(G) = |E(G)| = n+p-1 \geq n+p-2.$$

Next assume that $b \leq 2p+k-2$. Let C be a cycle of G and $e_1 \in E(C)$. Since all cycles of G are edge disjoint, $H = G - e_1$ is also a connected cactus graph with exactly $p-1$ cycles. Hence the induction hypothesis leads to

$$\beta'_{(b,k)}(H) \geq \min\{n-k+b-p, n+p-2\} = n-k+b-p.$$

If f is a $\beta'_{(b,k)}(H)$-matching, then define define $h : E(G) \to \{-1,1\}$ by $h(e) = f(e)$ for $e \in E(H)$ and $h(e_1) = -1$. Obviously, h is a signed (b,k)-matching of G such that

$$\beta'_{(b,k)}(G) \geq \beta'_{(b,k)}(H)-1 \geq n-k+b-(p+1) \geq \min\{n-k+b-(p+1), n+p-1\},$$

and the proof is complete. □

For the special family of Eulerian cactus graphs, the next result presents a much better lower bound when $b \geq 2$.

Theorem 5. *Let G be an Eulerian cactus graph of order n and girth g with exactly $p \geq 1$ cycles. If $b \geq 2$ is an integer, then*

$$\beta'_{(b,n)}(G) \geq \begin{cases} p(g-2) + \min\{b, 2p\} & \text{if } b \text{ is even} \\ p(g-2) + \min\{b-1, 2p\} & \text{if } b \text{ is odd.} \end{cases}$$

Proof. We proceed by induction on the number p of cycles. If $b \geq 2p$, then $b \geq 2p \geq \Delta(G)$ and thus

$$\beta'_{(b,n)}(G) = |E(G)| \geq pg = p(g-2) + 2p \geq p(g-2) + \min\{b, 2p\}.$$

Assume now that $b < 2p$. Let $B = u_1u_2 \ldots u_ku_1$ be an end-block of G with the cut-vertex u_1 of G. Then $H = G - (V(B) - u_1)$ is also an Eulerian cactus graph with exactly $p-1$ cycles and girth at least g. Therefore the induction hypothesis leads to

$$\beta'_{(b,n(H))}(H) \geq \begin{cases} (p-1)(g-2) + \min\{b, 2p-2\} & \text{if } b \text{ is even} \\ (p-1)(g-2) + \min\{b-1, 2p-2\} & \text{if } b \text{ is odd.} \end{cases} \tag{3}$$

If f is a $\beta'_{(b,n(H))}(H)$-matching, then define $h : E(G) \to \{-1, 1\}$ by $h(e) = f(e)$ for $e \in E(H)$, $h(u_iu_{i+1}) = 1$ for $i = 1, 2, \ldots, k-1$ and $h(u_ku_1) = -1$. Obviously, h is a signed (b,n)-matching of G such that

$$\beta'_{(b,n)}(G) \geq \beta'_{(b,n(H))}(H) + k - 2 \geq \beta'_{(b,n(H))}(H) + g - 2.$$

Combining this with (3), we obtain the desired result. □

The next example will demonstrate that Theorem 5 is best possible.

Example Let the cactus graph G of order n consists of a vertex u and exactly $p \geq 2$ edge-disjoint cycles $C_1, C_2, \ldots, C_p$ of length g containing u. If $2 \leq b < 2p$, then it is straightforward to verify that

$$\beta'_{(b,n)}(G) = \begin{cases} p(g-2) + b & \text{if } b \text{ is even} \\ p(g-2) + b - 1 & \text{if } b \text{ is odd.} \end{cases}$$

3 Upper bounds for SbkMN of graphs

First note that for any graph G, when b is at least $\Delta(G)$, $\beta'_{(b,k)}(G) = |E(G)|$ for any $1 \leq k \leq |V(G)|$. Then hereafter we may assume for any graph G, $b < \Delta(G)$. In this section we present some upper bounds on $\beta'_{(b,k)}(G)$ in terms of order of G, maximum and minimum degree and degree sequence of G. The first theorem generalizes Theorems B and C.

Theorem 6. *For any graph G of order $n \geq 2$ and without isolated vertices, positive integer $b < \Delta(G)$ and $1 \leq k \leq n$,*

$$\beta'_{(b,k)}(G) \leq \frac{k(b - \Delta(G)) + n\Delta(G)}{2}.$$

Proof. Let f be a $\beta'_{(b,k)}(G)$-matching. We have

$$\begin{aligned}
\beta'_{(b,k)}(G) &= \textstyle\sum_{e\in E(G)} f(e) = \frac{1}{2}\sum_{v\in V(G)}\sum_{e\in E(v)} f(e)\\
&= \frac{1}{2}\textstyle\sum_{v\in B_f}\sum_{e\in E(v)} f(e) + \frac{1}{2}\sum_{v\in V(G)\setminus B_f}\sum_{e\in E(v)} f(e)\\
&\leq \frac{b|B_f|}{2} + \frac{|V(G)\setminus B_f|\Delta(G)}{2}\\
&= \frac{|B_f|(b-\Delta(G)) + n\Delta(G)}{2}.
\end{aligned}$$

The result follows from $k \leq |B_f|$. □

Theorem 7. *Let G be a graph of order $n \geq 2$, size m, without isolated vertices, and with degree sequence $(d_1, d_2, \ldots, d_n)$ where $d_1 \leq d_2 \leq \ldots \leq d_n$. For each positive integer $b < \Delta(G)$ and $1 \leq k \leq n$,*

$$\beta'_{(b,k)}(G) \leq \frac{2bm - \sum_{i=1}^{k} d_i^2}{2d_1} + m.$$

Proof. Let g be a $\beta'_{(b,k)}(G)$-matching and let $B_g = \{v_{j_1}, v_{j_2}, \ldots, v_{j_{|B_g|}}\}$. Define $f : E(G) \longrightarrow \{-1, 0\}$ by $f(e) = (g(e)-1)/2$ for each $e \in E(G)$. We have

$$\sum_{e\in E(G)} f(N_G[e]) = \sum_{e=uv\in E(G)} \frac{g(N_G[e]) - \deg(u) - \deg(v) + 1}{2}.$$

Since

$$\sum_{e\in E(G)} (g(N_G[e]) + g(e)) = \sum_{v\in V(G)} g(E(v))\deg(v)$$

and

$$\sum_{e=uv\in E(G)} (\deg(u) + \deg(v)) = \sum_{v\in V(G)} \deg(v)^2,$$

we have

$$\begin{aligned}
\textstyle\sum_{e\in E(G)} f(N_G[e]) &= \frac{1}{2}\textstyle\sum_{v\in V(G)}(g(E(v))\deg(v)-\deg(v)^2)-\\
&- \frac{1}{2}\textstyle\sum_{e\in E(G)} g(e)+\frac{m}{2}\\
&\le \frac{1}{2}\textstyle\sum_{v\in V(G)\setminus B_g}(g(E(v))\deg(v)-\deg(v)^2)+\\
&\quad \frac{1}{2}\textstyle\sum_{i=1}^{|B_g|}(bd_{j_i}-d_{j_i}^2)-\frac{1}{2}\beta'_{(b,k)}(G)+\frac{m}{2}\\
&\le \frac{1}{2}\textstyle\sum_{i=1}^{|B_g|} bd_{j_i}-\frac{1}{2}\textstyle\sum_{i=1}^{|B_g|} d_{j_i}^2-\frac{1}{2}\beta'_{(b,k)}(G)+\frac{m}{2}\\
&\le \frac{1}{2}\textstyle\sum_{i=1}^{n} bd_i-\frac{1}{2}\textstyle\sum_{i=1}^{k} d_i^2-\frac{1}{2}\beta'_{(b,k)}(G)+\frac{m}{2}\\
&= bm-\frac{1}{2}\textstyle\sum_{i=1}^{k} d_i^2-\frac{1}{2}\beta'_{(b,k)}(G)+\frac{m}{2}\\
&= \frac{(2b+1)m}{2}-\frac{1}{2}\textstyle\sum_{i=1}^{k} d_i^2-\frac{1}{2}\beta'_{(b,k)}(G) \qquad (1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\textstyle\sum_{e\in E(G)} f(N_G[e]) &= \textstyle\sum_{v\in V(G)} f(E(v))\deg(v)-\textstyle\sum_{e\in E(G)} f(e)\\
&\ge \textstyle\sum_{v\in V(G)} f(E(v))d_1-\textstyle\sum_{e\in E(G)} f(e)\\
&= d_1(2\textstyle\sum_{e\in E(G)} f(e))-\textstyle\sum_{e\in E(G)} f(e)\\
&= (2d_1-1)\textstyle\sum_{e\in E(G)} f(e). \qquad (2)
\end{aligned}$$

By (1) and (2)

$$\sum_{e\in E(G)} f(e)\le \frac{\frac{(2b+1)m}{2}-\frac{1}{2}\sum_{i=1}^{k} d_i^2-\frac{1}{2}\beta'_{(b,k)}(G)}{2d_1-1}.$$

Since $g(E(G))=2f(E(G))+m$, we have

$$\beta'_{(b,k)}(G)=\sum_{e\in E(G)} g(e)\le \frac{1}{2d_1-1}((2b+1)m-\sum_{i=1}^{k} d_i^2-\beta'_{(b,k)}(G))+m.$$

Thus,

$$\beta'_{(b,k)}(G) \le \frac{2bm - \sum_{i=1}^{k} d_i^2}{2d_1} + m,$$

as desired. □

An immediate consequence of Theorem 7 now follows.

Corollary 8. *For every r-regular graph G of order n,*

$$\beta'_{(b,k)}(G) \le \frac{(n-k)r + nb}{2}.$$

Theorem 9. *Let G be a r-regular and 1-factorable graph of order $n \ge 4$, and let $b < r$ be a positive integer. Then*

$$\beta'_{(b,n)}(G) = \begin{cases} \frac{bn}{2} & \text{if } r-b \equiv 0 \pmod 2 \\ \frac{(b-1)n}{2} & \text{if } r-b \equiv 1 \pmod 2. \end{cases}$$

Proof. Let $\{M_1, M_2, \ldots, M_r\}$ be a decomposition of the edge set $E(G)$ into perfect matchings.

If $r-b \equiv 0 \pmod 2$, then define $f(e) = 1$ when $e \in \bigcup_{i=1}^{\frac{r+b}{2}} M_i$ and $f(e) = -1$ when $e \in \bigcup_{\frac{r+b}{2}+1}^{r} M_i$. This implies $f(v) = b$ for each vertex $v \in V(G)$ and thus $\beta'_{(b,n)}(G) = \frac{bn}{2}$.

If $r-b \equiv 1 \pmod 2$, then define $f(e) = 1$ when $e \in \bigcup_{i=1}^{\frac{r+b-1}{2}} M_i$ and $f(e) = -1$ when $e \in \bigcup_{\frac{r+b-1}{2}+1}^{r} M_i$. This implies $f(v) = b-1$ for each vertex $v \in V(G)$. Since $f(v) = b$ is not possible in this case, we obtain $\beta'_{(b,n)}(G) = \frac{(b-1)n}{2}$. □

Theorem 9 shows that Corollary 8 is best possible when b and n are both even and $k = n$.

Applying Theorem 9 and the well-known classical Theorem of König that a r-regular bipartite graph is 1-factorable, we obtain the next result immediately.

Theorem 10. *Let G be a r-regular bipartite graph of order $n \ge 4$, and let $b < r$ be a positive integer. Then*

$$\beta'_{(b,n)}(G) = \begin{cases} \frac{bn}{2} & \text{if } r-b \equiv 0 \pmod 2 \\ \frac{(b-1)n}{2} & \text{if } r-b \equiv 1 \pmod 2. \end{cases}$$

The following theorem is a generalization of Theorem D.

Theorem 11. *Let G be a graph of order $n \geq 2$, size m, minimum degree δ, maximum degree Δ and without isolated vertices . For each positive integer $b < \Delta(G)$ and $1 \leq k \leq n$,*

$$\beta'_{(b,k)}(G) \leq \frac{(2m+bn)\Delta + (\Delta^2 - (b-1)\Delta)(n-k)}{2\delta} - m.$$

Proof. Let f be a $\beta'_{(b,k)}(G)$-matching. First note that for each vertex $v \in B_f$, $|E(v) \cap P| \leq \lceil \frac{\deg(v)+b}{2} \rceil$. We have

$$\begin{aligned}
(2\delta)|P| &\leq \textstyle\sum_{e=uv\in P}(\deg(u)+\deg(v)) \\
&\leq \textstyle\sum_{v\in V(G)} |P \cap E(v)| \deg(v) \\
&\leq \textstyle\sum_{v\in B_f} |P \cap E(v)| \deg(v) + \sum_{v\in V(G)\setminus B_f} |P \cap E(v)| \deg(v) \\
&\leq \textstyle\sum_{v\in B_f} (\lfloor \frac{\deg(v)+b}{2} \rfloor) \deg(v) + \sum_{v\in V(G)\setminus B_f} \deg(v)^2 \\
&\leq \textstyle\sum_{v\in B_f} \lfloor \frac{\deg(v)+b}{2} \rfloor \deg(v) + \\
&\quad \textstyle\sum_{v\in V(G)\setminus B_f} \lfloor \frac{\deg(v)+b}{2} \rfloor \deg(v) + \\
&\quad \textstyle\sum_{v\in V(G)\setminus B_f} (\frac{\deg(v)^2}{2} - \frac{(b-1)\deg(v)}{2}) \\
&\leq \textstyle\sum_{v\in V(G)} \lfloor \frac{\deg(v)+b}{2} \rfloor \Delta + \frac{\Delta^2-(b-1)\Delta}{2} |V(G) \setminus B_f| \\
&\leq \Delta \textstyle\sum_{v\in V(G)} \frac{\deg(v)+b}{2} + \frac{\Delta^2-(b-1)\Delta}{2}(n-k) \\
&= \Delta(m + \frac{bn}{2}) + \frac{\Delta^2-(b-1)\Delta}{2}(n-k).
\end{aligned}$$

It follows that

$$|P| \leq \frac{\Delta(m+\frac{bn}{2}) + \frac{\Delta^2-(b-1)\Delta}{2}(n-k)}{2\delta}.$$

Now the result follows from $\beta'_{(b,k)}(G) = 2|P| - m$. □

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