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CONFORMAL JACOBI OPERATORS OF HYPERSURFACES IN THE DE SITTER SPACE

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Abstract

Classification of the complete spacelike hypersurfaces in the de Sitter space S_1^{n+1} $(n \geq 4)$ with commuting Jacobi operators and non-zero index of nullity.

1 Introduction

The spacelike hypersurfaces in the de Sitter space S_1^{n+1} have been of increasing interest in the resent years from different points of view. For instance, Ramanathan [12] proved that every compact spacelike surface in S_1^3 with constant mean curvature is totally umbilical. This result was generalized to hypersurfaces of any dimension by Montiel [11]. Li [9] obtained the same conclusion when the compact spacelike surface has constant Gaussian curvature(see also [2]). Akutagawa [1] and Ramanathan [12] investigated spacelike hypersurface in a de Sitter space and proved independently that a complete spacelike hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfied $H^2 \leq 1$ when $n = 2$ and $n^2H^2 \leq 4(n-1)$ when $n \geq 3$. Later, Cheng [8] generalized this result to general submanifolds in a de Sitter space.

On the other hand, the well-known examples with $H^2 = \frac{4(n-1)}{n^2}$ when $n > 2$ are the umbilical sphere $S^n(\frac{(n-2)^2}{n^2})$ and the hyperbolic cylinder $H^1(c_1)\times$

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$S^{n-1}(c_2)$, $c_1 = 2-n$ and $c_2 = \frac{n-2}{n-1}$. In [10], Liu proved that only n-dimensional ($n \geq 3$) complete submanifolds in a de Sitter space S_p^{n+p} with parallel mean curvature vector and $H^2 = \frac{4(n-1)}{n^2}$ are totally umbilical or hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ in S_1^{n+1} or has unbounded volume and positive Ricci curvature.

As a natural generalization of Ramanathan and Li results, Aledo and Galvez [4] characterized the totally umbilical round spheres of S_1^3 as the only compact linear Wingarten spacelike surface. In [3], Aledo and Espinar classified the spacelike surfaces in S_1^3 with the mean curvature H and the Gauss-Kronecker curvature K verifying the relationship $2a(H-1)+b(K-1)=0$ with $a+b \neq 0$.

On the other hand, conformally flat manifolds represent a classical field of investigation in Riemannian geometry. Brozos-Vazquez and Gilkey [6], characterized Riemannian manifolds of constant sectional curvature in tensor of commutation properties of their Jacobi operators. Ryan [13] classified conformally flat spaces with parallel Ricci tensor and who proved the following

Theorem 1.1. *Let M be an n-dimensional connected, complete and conformally flat space with parallel Ricci tensor. Then M is locally isometric to one following spaces:*

$$\mathbb{R}^n, S^n(k), H^n(-k), \mathbb{R} \times S^{n-1}(k), \mathbb{R} \times H^{n-1}(-k), S^p(k) \times H^{n-p}(-k)$$

In this paper, firstly we show that, if M^n is a manifold with commuting conformal Jacobi operators, then M^n is conformally flat. Then we prove that, if M^n is a connected spacelike hypersurface of the de Sitter space S_1^{n+1} with a non-zero index of nullity, then M^n is totally umbilical or isometric to the $\mathbb{R} \times S^{n-1}(c)$, $S^1(c_1) \times H^{n-1}(c_2)$ or $S^{n-1}(c_1) \times H^1(c_2)$.

2 Preliminaries

2.1 The de Sitter and Hyperbolice Space

Let L^{n+2} be the $(n+2)$-dimensional Lorentz-Minkowski space given as the vectorial space $\mathbb{R}^{n+2}$ with the Lorentzian metric $\langle , \rangle$ induced by the quadratic form $-x_0^2 + x_1^2 + ... + x_{n+1}^2$, and consider the *de Sitter space* realized as the Lorentzian submanifold

$$S_1^{n+1} = \{(x_0, x_1, ..., x_{n+1}) \in L^{n+2} : -x_0^2 + x_1^2 + ... + x_{n+1}^2 = 1\}.$$

As it is well-known, S_1^{n+1} inherits from L^{n+2} a time orientable Lorentzian metric which makes it the standard model of a Lorentzian space of constant sectional curvature 1. We will also consider the *hyperbolic space*

$$H^{n+1} = \{(x_0, x_1, ..., x_{n+1}) \in L^{n+2} : -x_0^2 + x_1^2 + ... + x_{n+1}^2 = -1, x_0 > 0\}.$$

A smooth immersion $\psi : M^n \longrightarrow S_1^{n+1}$ of a n-dimensional connected manifold M^n is called a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on M^n, which, as usual, is also denoted by $\langle , \rangle$.

2.2 The Weyl Curvature

Let ∇ be the Levi-Civita connection of M. The curvature operator $R(x, y)$ and curvature tensor $R(x, y, z, w)$ are defined by setting

$$R(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}$$

and

$$R(x, y, z, w) := \langle R(x, y)z, w\rangle.$$

Let e_i be a local orthonormal frame field for the tangent bundle. We sum over repeated indices to define the *Ricci tensor* ρ and *scalar curvature* τ by setting:

$$\rho_{ij} := \textstyle\sum_k R_{ikkj}$$

and

$$\tau := \textstyle\sum_i \rho_{ii}$$

where, $R_{ijkl} = R(e_i, e_j, e_k, e_l)$.

The associated *Ricci operator* is defined by $\rho(e_i) = \sum_j \rho_{ij} e_j$. We introduce additional tensors by

$$L(x, y)z := \langle \rho(y), z\rangle x - \langle \rho(x), z\rangle y + \langle y, z\rangle \rho(x) - \langle x, z\rangle \rho(y)$$

and

$$R_0(x, y)z := \langle y, z\rangle x - \langle x, z\rangle y.$$

The *Weyl conformal curvature operator* $W(x, y)$ is defined by

$$W(x, y) := R(x, y) + \tfrac{\tau}{(n-1)(n-2)} R_0(x, y) - \tfrac{1}{n-2} L(x, y).$$

2.3 Conformal Geometry

We say that two Riemannian metrics g_1 and g_2 on Riemannian manifold of M are *conformally equivalent* if $g_1 = \alpha.g_2$ where α is a smooth positive scaling function. The Weyl conformal curvature operator is invariant on a conformal class as

$$W_{g_1} = W_{g_2} \qquad \text{if} \qquad g_1 = \alpha.g_2$$

Conformal analogues of notions in Riemannian geometry can be obtained by replacing the full curvature operator R by the Weyl operator W; we add the name "conformal" in doing this. For example, one says that (M^n, g) is *conformally flat* if the Weyl tensor W vanishes identically; this implies that (M^n, g) is conformally equivalent to flat space. Then for a conformally flat space we have

$$\begin{aligned} R_{ijkl} = \tfrac{1}{n-2}&(g_{il}\rho_{jk} + g_{jk}\rho_{il} - g_{ik}\rho_{jl} - g_{jl}\rho_{ik}) - \\ &- \tfrac{\tau}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl}). \end{aligned}$$

2.4 Jacobi operator and conformal Jacobi operator

The *Jacobi operator* of the tangent bundle is a self-adjoint map $J_R(x)$ defined by

$$J_R(x)y := R(y, x)x.$$

Similarly, we follow the discussion of [5], the *conformal Jacobi operator* J_W is defined by

$$J_W(x)y := W(y, x)x.$$

2.5 Index of Nullity and Conullity

The *nullity vector space* of the curvature tensor at a point p of a Riemannian manifold (M^n, g) is given by

$$E_{0p} = \{x \in T_pM : R(x, y)z = 0 \text{ for all } y, z \in T_pM\}.$$

The *index of nullity* at p is the number $v(p) = \dim E_{op}$. The *index of conullity* at p is the number $u(p) = n - v(p)$.

3 Manifolds with commuting conformal Jacobi operators

Theorem 3.1. *Let M be a Riemannian manifold. If $J_W(x)J_W(y) = J_W(y)J_W(x)$ for all x, y, then M is conformally flat.*

Proof. Since $J_W(x)J_W(y) = J_W(y)J_W(x)$ for all $x, y \in T_pM$, so the conformal Jacobi operators $\{J_W(x)\}|_{x\in T_pM}$ form a commuting family of self-adjoint operators. Such a family can be simultaneously diagonalized - i.e. we can decompose

$$T_pM = \oplus(T_pM)_\lambda,$$

where $J_W(x)y = \lambda(x)y$ for all $y \in (T_pM)_\lambda$.
Choose $y \in T_pM$ such that $y = \sum_\lambda y_\lambda$ $(y_\lambda \in (T_pM)_\lambda)$ and $y_\lambda \neq 0$ for all λ. We have

$$0 = J_W(y)y = \sum_\lambda \lambda(y)y_\lambda.$$

Since $\{y_\lambda\}$ is a linearly independent set, this implies $\lambda(y) = 0$ for all λ. We have $\lambda(y) = \langle J_W(y)z_\lambda, z_\lambda\rangle$, for a fix unit vector $z_\lambda \in (T_pM)_\lambda$, so λ is a continuous function. Since the set of the $\{y = \sum_\lambda y_\lambda : y_\lambda \neq 0 \forall\lambda\}$ is a dense open subset of T_pM, and λ vanishes on this subset, then λ vanishes identically. Thus $J_W(x) = 0$ for all $x \in T_pM$; it now follows that $W = 0$ and so M is conformally flat. $\square$

4 The hypersurfaces of S_1^{n+1} with non-zero index of nullity

We begin this section with two examples:
Example 1. Let $(\mathbb{CP}^n, g_{FS})$ denote the complex projective space with Fubini-Study metric. We know that the sectional curvature of $(\mathbb{CP}^n, g_{FS})$ satisfies the inequality $\frac{1}{4} < K \leq 1$. So the complex projective space is a manifold with the index of nullity zero, therefore it is not conformally flat.
Example 2. Let $(\overline{M}, \overline{g})$ be an (n-1)-dimensional space form with constant curvature of c. Suppose $\lambda(t) = \frac{1}{1+t}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x > -1\}$. Consider the product manifold $M = \mathbb{R}_+ \times \overline{M}$, with the Riemannian metric

$$g = dx^0 \otimes dx^0 + \pi^*\overline{g},$$

where x^0 is the natural coordinate on $\mathbb{R}_+$ and $\pi : \mathbb{R}_+ \times \overline{M} \longrightarrow \overline{M}$ is the projection on the second factor. The manifold (M, g) is called a *Riemannian cone* over $(\overline{M}, \overline{g})$. Let T denote the unit vector field tangent to $\mathbb{R}_+$ in $\mathbb{R}_+ \times \overline{M}$. The curvature tensor of $M = \mathbb{R}_+ \times \overline{M}$ is described by (see [7])

$$R(X, Y)Z = g(B_0(Y), Z)B_0(X) - g(B_0(X), Z)B_0(Y) + (\pi^*\overline{R})(X, Y)Z$$

for all tangent vector field X, Y, Z to $\overline{M}$, where $B_0(X) := \bigtriangledown_X T = \lambda(X - -g(X,T)T)$. At any point $p \in M$, fix an orthonormal basis of tangent vectors $\{e_0, e_1, ..., e_{n-1}\}$, with $e_0 = T_p$ and $e_1, ..., e_{n-1}$ tangent to the space form $(\overline{M}, \overline{g})$. Then, we have

$$\begin{array}{lll} R_{ijkh} = 0, & \text{if} & 0 \in \{ijkh\} \\ R_{ijkh} = \lambda^2(c-1)(\delta_{ik}\delta_{jh} - \delta_{jk}\delta_{ih}), & \text{otherwise} & \end{array}$$

This shows that, if $c \neq 1$, then (M, g) has the index of nullity equal to 1. Also we show easily that (M, g) is conformally flat.

Theorem 4.1. *Let $M^n (n \geq 3)$ be a complete and connected spacelike hypersurface in the de Sitter space S_1^{n+1}. If the Weyl conformal operators are commuting and the index of nullity M^n is non-zero, then M^n is totally umbilical or isometric to one of product spaces $\mathbb{R} \times S^{n-1}(c)$, $S^1(c_1) \times H^{n-1}(c_2)$ or $S^{n-1}(c_1) \times H^1(c_2)$.*

Proof. If, for any $p \in M$, we have $v(p) = n$, then M is flat therefore it is a totally umbilical hypersurface. Now let $v(p) < n$. Since $v(p) \neq 0$, then there exists a locally orthonormal frame field $\{e_0, e_1, ..., e_{n-1}\}$ of M^n, such that $e_0(p) \in E_{0p}$. Hence, we have $R_{0jkl} = 0$ for all j, k, l, and it follows that $\rho_{0k} = -\sum_j R_{0jkj} = 0$ for all k. Since M^n is conformally flat, then we have

$$0 = R_{0jkl} = \frac{1}{n-2}(\delta_{0l}\rho_{jk} - \delta_{0k}\rho_{jl}) - \frac{\tau}{(n-1)(n-2)}(\delta_{0l}\delta_{jk} - \delta_{jl}\delta_{0k}), \quad (1)$$

for any choice of j, k, l. Choosing $k = 0$ and $l \neq 0$, we then get

$$\rho_{jl} - \frac{\tau}{n-1}\delta_{jl} = 0. \quad (2)$$

Hence, the Ricci tensor is described by

$$\rho_{ii} = \frac{\tau}{n-1} \qquad if \qquad i \geq 1, \quad (3)$$

and $\rho_{ij} = 0$ in all the other cases.

Clearly, if the index of nullity is greater than 1, we can choose at least two locally vector field e_0, e_1, such that $e_0(p)$ and $e_1(p)$ belong to E_{0p}. Let $\{e_0, e_1, ..., e_{n-1}\}$ be a locally orthonormal frame field. But then, since e_1 is a nullity vector field, from equation (4.3), we have

$$0 = \rho_{11} = \frac{\tau}{n-1}, \quad (4)$$

that is, $\tau = 0$ and so, again by (4.3), $\rho_{ij} = 0$ for alli, j. Then $v(p) = 0$ and this is a contradiction.

Now suppose $\overline{M}$ be the conullity integral submanifold of M^n; that is, for any $p \in \overline{M}$ and any $x \in T\mathbb{O}_p\overline{M}$ then x does not belong to the nullity vector space of E_{0p}. Clearly $\overline{M}$ is (n-1)-dimensional hypersurface of M^n. By equation (4.3), the Ricci tensor of $\overline{M}$ is

$$Ric = \frac{\tau}{n-1}g, \tag{5}$$

then $\overline{M}$ is an Einstein manifold. Since $n \geq 3$, it results that τ is constant. Therefore by (4.3), the Ricci tensor of M^n is parallel. Now by Theorem 1.1, we get that M^n is isometric to one of product spaces $\mathbb{R} \times S^{n-1}(c)$, $S^1(c_1) \times H^{n-1}(c_2)$ or $S^{n-1}(c_1) \times H^1(c_2)$. $\square$

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