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VOLTERRA INTEGRAL EQUATIONS GOVERNED BY HIGHLY OSCILLATORY FUNCTIONS ON TIME SCALES

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Abstract

In this paper, we present an existence result for Volterra integral equations on time scales. Since the theory of time scales unifies the cases of difference and differential problems, our result encompasses both situations and not only these ones. Moreover, we use the Henstock-Δ-integral, therefore the situation where the equation is governed by an oscillatory function is also covered.

1 Introduction

It is well known that there are many analogies between the theories of difference equations and differential equations. The concept of time scales unifies these two situations but gives also solutions for problems on discrete sets with non-uniform step size or combinations of real and discrete intervals and many others.

On the other hand, as it was seen in literature (beginning with [2]), the study of dynamical systems leads, in a very natural way, to integrals of highly oscillatory functions, such as the Henstock-type integrals. On time scales, integrals of this type were introduced on the real line in [16] and [3] and in general Banach spaces in [8] and, recently, used in applications (see [9]).

Combining methods used in both theories (time scales and non-absolute

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integrability in Banach spaces) we study, in the present paper, a Banach space Volterra integral equation on time scales with a highly oscillatory function on the right hand side:

$$x(t) = (H)\int_0^t f(t,s,x(s))\Delta s,\ \forall t \in \mathbb{T}. \tag{1}$$

To obtain the existence result, we apply a generalization of Darbo's fixed point theorem given in [14].

Let us first recall some basic concepts related to the theory of time scales. A time scale $\mathbb{T}$ is a closed nonempty subset of the real numbers $\mathbb{R}$ endowed with the usual topology (as, for example, the set of nonnegative integers, any closed interval or finite union of closed intervals, but not only these).
The notion of a time scale was introduced by [12]. See also [5], [6] and the references therein.

Definition 1 *Let $\mathbb{T}$ be a time scale. Then the forward jump operator at the point t, denoted by $\sigma(t)$, is $\sigma(t) = \inf\{s > t, s \in \mathbb{T}\}$ and, similarly, the backward jump operator is $\rho(t) = \sup\{s < t, s \in \mathbb{T}\}$. We make the convention that* $\inf\emptyset = \sup\mathbb{T}$ *and* $\sup\emptyset = \inf\mathbb{T}$.

If $\sigma(t) > t$, then t is called right-scattered; if $\rho(t) < t$, we say that t is left-scattered. If $t < \sup\mathbb{T}$ and $\sigma(t) = t$, we say t is right-dense and, likewisely, if $t > \inf\mathbb{T}$ and $\rho(t) = t$, then t is said to be left-dense.
Let $a, b \in \mathbb{T}$. Define the time scale interval by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T}, a \leq t \leq b\}$.

Let X be a Banach space and consider a function $f : \mathbb{T} \to X$. In the sequel, we present the Henstock-Δ-integral, following the method used in [16] to introduce the real Henstock-Kurzweil integral on time scales. A pair of functions $\delta = (\delta_L, \delta_R)$ is a Δ-gauge on $[a,b]_{\mathbb{T}}$ if $\delta_L(t) > 0$ on $(a,b]_{\mathbb{T}}$, $\delta_L(a) \geq 0$, $\delta_R(t) > 0$ on $[a,b)_{\mathbb{T}}$, $\delta_R(b) \geq 0$ and $\delta_R(t) \geq \sigma(t) - t, \forall t \in [a,b)_{\mathbb{T}}$.

Definition 2 *A partition of $[a,b]_{\mathbb{T}}$ is a finite family $\{(t_{i-1}, t_i), \xi_i\}_{i=1}^n$ of non-overlapping intervals covering $[a,b]_{\mathbb{T}}$ with the tag points $\xi_i \in (t_{i-1}, t_i)$; a partition is said to be δ-fine if for each $i = 1, ..., n$,*

$$\xi_i - \delta_L(\xi_i) \leq t_{i-1} < t_i \leq \xi_i + \delta_R(\xi_i).$$

Definition 3 *A function $f : [a,b]_{\mathbb{T}} \to X$ is Henstock-Δ-integrable (shortly, H-Δ-integrable) on $[a,b]_{\mathbb{T}}$ if there exists an element* (H) $\int_a^b f(s)\Delta s \in X$ *such that, for every $\varepsilon > 0$, there is a Δ-gauge δ_ε with*

$$\left\| \sum_{i=1}^n f(\xi_i)(t_{i-1} - t_i) - (\mathrm{H})\int_a^b f(s)\Delta s \right\| < \varepsilon$$

for every δ_ε-fine partition. We call this element the Henstock-Δ-integral of f on $[a,b]_{\mathbb{T}}$.

The definition of Henstock delta integral makes sense since in [16] the following lemma (Cousin's Lemma for time scale theory) was proved:

Lemma 4 *If δ is a Δ-gauge for $[a,b]_{\mathbb{T}}$, then one can find a δ-fine partition of $[a,b]_{\mathbb{T}}$.*

Remark 5 *In particular, if the time scale is a real closed interval, then the previous Definition* 3 *gives the Henstock integral for Banach-valued functions* (*see* [7])*. Moreover, if X is the real line, one obtains the real Henstock-Kurzweil integral, for which the reader is referred to the monograph* [11].

The space of Henstock-Δ-integrable X-valued functions will be denoted by $\mathcal{H}(\mathbb{T},X)$ and will be considered provided with the Alexiewicz norm:

$$\|f\|_A = \sup_{a,b\in\mathbb{T}} \left\| (\mathrm{H}) \int_a^b f(s)\Delta s \right\| .$$

As for the usual Henstock integral, it can be seen ([8]) that

Lemma 6 *If $f : [a,b]_{\mathbb{T}} \to X$ is Henstock-Δ-integrable on $[a,b]_{\mathbb{T}}$, then it is Henstock-Δ-integrable on any time scale interval $[a',b']_{\mathbb{T}}$, where $a \le a' < b' \le b$. Besides, the primitive in Henstock-Δ sense $t \mapsto (H)\int_a^t f(s)\Delta s$ is continuous on $[a,b]_{\mathbb{T}}$.*

The space of continuous functions endowed with the usual norm is denoted by $C(\mathbb{T},X)$ and let B_R be its closed ball centered in the constant null function with radius R.

2 An existence result for Volterra integral equations on time scale

We prove in this section an existence result for the Volterra integral equation (1) on a bounded time scale (suppose that 0 is the minimum of the time scale and denote by b its maximum).

Our main theorem will be proved by applying the following generalization of the Darbo's fixed point Theorem given in [14]:

Lemma 7 *Let F be a closed convex subset of a Banach space and the operator $A : F \to F$ be continuous with $A(F)$ bounded. For any bounded $B \subset F$ set*

$$\tilde{A}^1(B) = A(B) \quad \textit{and} \quad \tilde{A}^n(B) = A\left(\overline{co}\left(\tilde{A}^{n-1}(B)\right)\right), \forall n \ge 2.$$

If there exist a positive constant $0 \leq k < 1$ and a natural number n_0 such that $\alpha(\tilde{A}^{n_0}(B)) \leq k\alpha(B)$ for every bounded $B \subset F$, then A has a fixed point.

Here α is the Hausdorff measure of noncompactness (for which the reader is referred to [4]). A result of Ambrosetti-type ([1]) proved in [9] will be useful:

Lemma 8 *Let $\mathcal{K} \subset C(\mathbb{T}, X)$ be bounded and equi-continuous. Then*

$$\alpha(\mathcal{K}) = \sup_{t \in \mathbb{T}} \alpha(\mathcal{K}(t)).$$

Theorem 9 *Let $f : \mathbb{T} \times \mathbb{T} \times X \to X$ satisfy:*
i) for every $t \in \mathbb{T}$ and every continuous $x : \mathbb{T} \to X$, the function $f(t, \cdot, x(\cdot))$ is H-Δ-integrable;
ii) the H-Δ-primitives $(H) \int_0^{\cdot} f(\cdot, s, x(s))\Delta s$ are uniformly continuous on $\mathbb{T}$, uniformly with respect to x in any ball of the space of continuous functions: for every $R > 0$ and every $\varepsilon > 0$, one can find $0 < \delta_{\varepsilon,R} < 1$ such that

$$\left\| (H) \int_{t'}^{t"} f(t, s, x(s))\Delta s \right\| < \varepsilon, \ \forall |t' - t"| < \delta_{\varepsilon,R}, \ \forall x \in B_R.$$

iii) the map $x \mapsto f(t, \cdot, x(\cdot))$ from $C(\mathbb{T}, X)$ to $H(\mathbb{T}, X)$ is $\| \cdot \|_A$-uniformly continuous, uniformly with respect to $t \in \mathbb{T}$;
iv) $\limsup_{R \to \infty} \dfrac{b+1}{R\delta_{1,R}} < 1$;
v) there exists a positive constant c such that

$$\alpha(F(t, [0, t], D)) \leq c\alpha(D), \ \forall t \in \mathbb{T}, \ \forall D \subset X \text{ bounded}.$$

Then the Volterra integral equation possess continuous solutions.

Proof. By hypothesis *iv*), one can find $R_0 > 0$ such that for any $R \geq R_0$,

$$\frac{b+1}{\delta_{1,R}} < R.$$

Let $A : C(\mathbb{T}, X) \to C(\mathbb{T}, X)$ be defined by

$$Ax(t) = (H) \int_0^t f(t, s, x(s))\Delta s.$$

We prove that A is a continuous operator mapping the closed ball B_{R_0} of $C(\mathbb{T}, X)$ into itself. First of all, for every $t \in \mathbb{T}$ and for all $x \in B_{R_0}$, let $N \in \mathbb{N}$

be the integer part of $\frac{t}{\delta_{1,R_0}}$. Then

$$\begin{aligned}
&\left\|(H)\int_0^t f\left(t,s,x(s)\right)\Delta s\right\| \leq \left\|(H)\int_0^{\delta_{1,R_0}} f\left(t,s,x(s)\right)\Delta s\right\| + ...\\
&+\left\|(H)\int_{(N-1)\delta_{1,R_0}}^{N\delta_{1,R_0}} f\left(t,s,x(s)\right)\Delta s\right\| + \left\|(H)\int_{N\delta_{1,R_0}}^{t} f\left(t,s,x(s)\right)\Delta s\right\|\\
&\leq N+1\\
&\leq \frac{b}{\delta_{1,R_0}}+1 \leq \frac{b+1}{\delta_{1,R_0}}.
\end{aligned}$$

It follows that, for any $x \in B_{R_0}$,

$$\|Ax\|_C \leq \frac{b+1}{\delta_{1,R_0}} < R_0.$$

The continuity immediately comes from hypothesis $iii)$.
Next, we prove that $F = \overline{co}A(B_{R_0})$ is equi-continuous. By Lemma 2.1 in [14], it suffices to show that $A(B_{R_0})$ is equi-continuous. For all $u \in B_{R_0}$ and all $t' < t" \in \mathbb{T}$, we have

$$\|Ax(t') - Ax(t")\| = \left\|(H)\int_{t'}^{t"} f(t,s,x(s))\Delta s\right\|$$

and, by assumption $ii)$, this can be made less than some fixed ε if $t', t"$ are close enough. So, the equi-continuity follows.
Obviously, $A : F \to F$ is bounded and continuous. Let us prove, by mathematical induction, that for every $B \subset F$ and any $n \in \mathbb{N}$, $\tilde{A}^n(B) \subset A(B_{R_0})$, so $\tilde{A}^n(B)$ is bounded and equi-continuous. For $n = 1$ it is true, since $A(B) \subset A(F) \subset A(B_{R_0})$. Suppose now that this is true for $n-1$ and prove it for n:

$$\tilde{A}^n(B) = A(\overline{co}(\tilde{A}^{n-1}(B))) \subset A(\overline{co}(A(B_{R_0}))) \subset A(\overline{co}(B_{R_0})) = A(B_{R_0}).$$

Then, by Lemma 8,

$$\alpha\left(\tilde{A}^n(B)\right) = \sup_{t\in\mathbb{T}} \alpha\left(\tilde{A}^n(B)(t)\right), \quad \forall n \in \mathbb{N}.$$

As in the second part of the proof of Theorem 3.1 in [14], one can show that there exist a constant $0 \leq k < 1$ and a positive integer n_0 such that for any $B \subset F$, $\alpha(\tilde{A}^{n_0}(B)) \leq k\alpha(B)$.
Let $(v_n)_n$ be an arbitrary countable subset of $\tilde{A}^1(B) = A(B)$. There exists a sequence $(u_n)_n \subset B$ such that $v_n = Au_n$. Using now a mean value result

proved in [9] for the Henstock-Kurzweil-Pettis Δ-integral (the class of functions integrable in this sense is larger than that of Henstock-Δ-integrable functions) one gets that

$$\begin{aligned}\alpha\left(\{v_n(t), n \in \mathbb{N}\}\right) &= \alpha\big(\{Au_n(t), n \in \mathbb{N}\}\big)\\ &= \alpha\Big((H)\int_0^t f(t, s, \{u_n(s), n\})\Delta s\Big)\\ &\leq \alpha\left(t\overline{conv}(f(t, [0, t], \{u_n([0, t]), n\}))\right).\end{aligned}$$

By hypothesis $v)$ it follows that

$$\alpha(\{v_n(t), n \in \mathbb{N}\}) \leq ct\alpha(\{u_n([0, t])\})).$$

Since the Banach space is separable, this implies that

$$\alpha\left(\tilde{A}^1(B)(t)\right) \leq ct\alpha(B).$$

It can be shown, by mathematical induction, that for every $m \in \mathbb{N}$,

$$\alpha\left(\tilde{A}^{m+1}(B)\right) \leq ct\alpha\left(\tilde{A}^m(B)\right),$$

and so $\alpha\left(\tilde{A}^{m+1}(B)\right) \leq (ct)^{m+1}\alpha(B)$. For some integer n_0 the evaluation term $(ct)^{n_0}$ can be made less than 1 and so, by Lemma 7, A has a fixed point, which is a global solution to our equation. ■

Remark 10 *Our assumptions are less restrictive than similar results on time scale* (*see* [5] *or* [6]) *where absolutely convergent integrals are considered. Moreover, since the time scale theory encompasses the classical theory* (*the case* $\mathbb{T} = \mathbb{R}$), *our main result extends those already given on real intervals, as the existence theorem in* [10]. *Related existence results are proved as particular, the single-valued, case in* [17] *and* [18]. *Finally, the result presented in* [15] *under Carathéodory type assumptions is also contained.*

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