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SOLUTION OF MULTI - DELAY SYSTEMS VIA COMBINED BLOCK-PULSE FUNCTIONS AND LEGENDRE POLYNOMIALS

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Abstract

A method for finding the solution of a linear time varying multi-delay systems using a hybrid function is proposed. The properties of the hybrid functions which consists of block-pulse functions plus Legendre polyno- mials are presented. The method is based upon expanding various time functions in the system as their truncated hybrid functions. Operational matrices of integration, delay and product are presented and are utilized to reduce the solution of multi-delay systems to the solution of algebraic equations. An Illustrative examples are included to demonstrate the va- lidity and applicability of the technique. Key words: Orthogonal functions, hybrid functions, multi-delay sys- tems, block-pulse functions, Legendre polynomials.

1 Introduction

Delays occur frequently in biological, chemical, transportation, electronic, com- munication, manufacturing and power systems [1]. Time-delay and multi-delay systems are therefore very important classes of systems whose control and op- timization have been of interest to many investigators [2-5]. The

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available sets of orthogonal functions can be divided into three classes. The first includes set of piecewise constant basis functions (PCBFs)(e.g., Walsh, block-pulse,etc.). The second consists of a set of orthogonal polynomials (e.g., Laguerre, Legendre, Chebyshev, etc.). The third is the widely used set of sine-cosine functions in Fourier series. While orthogonal polynomials and sine-cosine functions together form a class of continuous basis functions, PCBFs have in- herent discontinuities or jumps. It is worth mentioning that approximating a continuous function with PCBFs, results in an approximation that is piecewise constant. On the other hand if a discontinuous function is approximated by continuous basis functions, the discontinuities are not properly modeled. Signals frequently have mixed features of continuity and jumps. These signals are continuous over certain segments of time, with discontinuities or jump occurring at the transitions of the segments. In such situations, neither the continuous basis functions nor PCBFs taken alone would form an efficient basis in the representation of such signals.

Orthogonal functions have received considerable attention in dealing with various problems of dynamic systems. Much progress has been made towards the solution of delay systems. The approach is to convert the delay-differential equation to an algebraic form through the use of operational matrices of integration and delay. These matrices can be uniquely determined based on the particular choices of basis functions. Special attention has been given to applica- tions of Walsh functions [6], block pulse functions [7], Laguerre polynomials [8], Legendre polynomials [9], and Chebyshev polynomials [10]. Moreover, Walsh functions was used for the solution of multi-delay systems in [11]. Due to the nature of these functions, the solution obtained were piecewise constant. In general, the computed response of the delay systems via orthogonal functions and Taylor series is not in good agreement with the exact response of the system [12].

In the present paper we introduce a new direct computational method to solve linear time varying multi-delay systems. The method consists of reducing the multi-delay problem to a set of algebraic equations by first expanding the candidate function as a hybrid function with unknown coefficients. These hybrid functions, which consists of block-pulse functions plus Legendre polynomials are first introduced. The operational matrices of integration and delay are given. These matrices are then used to evaluate the coefficients of the hybrid function for the solution of multi- delay systems. The paper is organized as follows: In Section 2 we describe the basic properties of the hybrid functions of block-pulse and Legendre Polynomials required for our subsequent development. Section 3 is devoted to the formulation of linear time varying multi-delay systems. In Section 4 we apply the proposed numerical method to multi-delay systems, and in Section 5, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering a numerical example.

2 Properties of hybrid functions

2.1 Hybrid functions of block-pulse and Legendre Polynomials.

Hybrid functions $b_{nm}(t), n = 1, 2, \cdots, N, m = 0, 1, \cdots, M-1,$ have three arguments; n and m are the order of block-pulse functions and Legendre polynomials respectively, and t is the normalized time. They are defined on the interval $[0, t_f)$ as

$$b_{nm}(t) = \begin{cases} P_m(\frac{2N}{t_f}t - 2n + 1), t \in \left[\frac{n-1}{N}t_f, \frac{n}{N}t_f\right] \\ 0, \text{otherwise.} \end{cases} \tag{1}$$

Here, $P_m(t)$ are the well-known Legendre polynomials of order m which satisfy the following recursive formula.

$$P_0(t) = 1,\ P_1(t) = t$$

$$P_{m+1}(t) = \frac{2m+1}{m+1} t P_m(t) - \frac{m}{m+1} P_{m-1}(t),\ m = 1, 2, 3, \cdots$$

2.2 Function approximation

A function $f(t)$, defined over the interval 0 to t_f may be expanded as

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} b_{nm}(t) = C^T B(t),$$

where

$$C = [c_{10}, \cdots, c_{1M-1}, c_{20}, \cdots, c_{2M-1}, \cdots, c_{N0}, \cdots, c_{NM-1}]^T, \tag{2}$$

and

$$B(t) = [b_{10}(t), \cdots, b_{1M-1}(t), b_{20}(t), \cdots, b_{2M-1}(t), \cdots, b_{N0}(t), \cdots, b_{NM-1}(t)]^T. \tag{3}$$

The integration of the vector $B(t)$ defined in (3) can be approximated by

$$\int_0^t B(t')dt' \simeq PB(t), \tag{4}$$

where P is the $MN \times MN$ operational matrix of integration and is given in [13]. The product of two hybrid function vectors can be approximated as

$$B(t)B^T(t)C \simeq \tilde{C}B(t), \tag{5}$$

where $\tilde{C}$ is a $MN \times MN$ where $\tilde{C}$ is the operational matrix of product and is given in [14].

2.3 The multi-delay operational matrix of delay for the hybrid of block-pulse and Legendre polynomials

The delay function $B(t-k_j)$, $j = 1, 2, \ldots, r$ is the shift of the function $B(t)$ defined in (3), along the time axis by k_j, where $k_1, k_2, \ldots, k_r$ are rational numbers in $(0,1)$. It is assumed without loss of generality that $k_1 < k_2 < \cdots < k_r$.

The general expression is given by

$$B(t-k_j) = D_j B(t), t > k_j \tag{6}$$

where D_j is the delay operational matrix of hybrid functions corresponding to k_j.

To find D_j for $j = 1, 2, \ldots, r$, we first choose N the order of block-pulse functions in the following manner:

We define w as the smallest positive integer number for which $wk_j \in \mathbf{Z}$ for $j = 1, 2, \ldots, r$. Next we choose λ as the greatest common divisor of the integers wk_j, $j = 1, 2, \ldots, r$, that is

$$\lambda = \text{g.c.d}(wk_1, wk_2, \ldots, wk_r).$$

Let

$$N = \begin{cases} \dfrac{w}{\lambda}, & \text{if } \frac{w}{\lambda} \in \mathbf{Z}, \\ \left[\dfrac{w}{\lambda}\right] + 1, & \text{otherwise,} \end{cases}$$

where [.] denotes greatest integer value.

With the aid of(1), it is noted that for the case $k_j < t < k_j + \frac{\lambda}{w}$, the only terms with nonzero values are $b_{1m}(t-k_j)$ for $m = 0, 1, 2, \ldots, M-1$. If we set $\beta_j = \frac{wk_j}{\lambda} + 1$, and expand $b_{1m}(t-k_j)$ in terms of $b_{\beta_j m}(t)$, since $b_{1m}(t-k_j) = b_{\beta_j m}(t)$, then the coefficient (element) of the delay matrix is an $M \times M$ identity matrix.

In a similar manner, for $k_j + \frac{\lambda}{w} < t < k_j + \frac{2\lambda}{w}$, only $b_{2m}(t-k_j)$ for $m = 0, 1, 2, \ldots, M-1$ has nonzero values. If we set $\gamma_j = \beta_j + 1$, and expand $b_{2m}(t-k_j)$ in terms of $b_{\gamma_j m}(t)$, since $b_{2m}(t-k_j) = b_{\gamma_j m}(t)$, then the element of the delay matrix is $M \times M$ identity matrix. Thus, if we expand $B(t-k_j)$ in terms of $B(t)$ we find $NM \times NM$ matrix D_j as

$$D_j = \begin{pmatrix} 0 & 0 & \cdots & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is noted that the first identity matrix in the first row is located at the β_jth column.

3 Problem statement

Consider the following linear time-varying multi-delay system:

$$\dot{X}(t) = E(t)X(t) + \sum_{j=1}^{r} F_j(t)X(t-k_j) + G(t)U(t), \quad 0 \le t \le 1 \tag{7}$$

$$X(0) = X_0, \tag{8}$$

$$X(t) = \Phi(t), \quad t < 0 \tag{9}$$

where $X(t) \in \mathbf{R}^l$, $U(t) \in \mathbf{R}^q$, $E(t)$, $G(t)$, and $F_j(t)$, $j = 1, 2, \ldots, r$, are matrices of appropriate dimensions, X_0 is a constant specified vector, and $\Phi(t)$ is an arbitrary known function. The problem is to find $X(t)$, $0 \le t \le 1$, satisfying (7)–(9).

4 Approximation using hybrid functions

Let

$$X(t) = [x_1(t), x_2(t), \ldots, x_l(t)]^T, \qquad U(t) = [u_1(t), u_2(t), \ldots, u_q(t)]^T, \tag{10}$$

$$\hat{B}(t) = I_l \otimes B(t), \qquad \hat{B}_1(t) = I_q \otimes B(t) \tag{11}$$

where I_l and I_q are the l and q dimensional identity matrices, $B(t)$ is $MN \times 1$ vector and $\otimes$ denotes Kronecker product [14]. Using the property of the Kronecker product, $\hat{B}(t)$ and $\hat{B}_1(t)$ are matrices of order $lMN \times l$ and $qMN \times q$ respectively.

Assume that each $x_i(t)$ and each of $u_j(t)$, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, q$, can be written in terms of hybrid functions as

$$x_i(t) = B^T(t)X_i, \qquad u_j(t) = B^T(t)U_j.$$

Then using (16) and (17) we have

$$X(t) = \hat{B}^T(t)X, \qquad U(t) = \hat{B}_1^T(t)U, \tag{12}$$

where X and U are vectors of order $lMN \times 1$ and $qMN \times 1$, respectively, given by

$$X = [X_1, X_2, \ldots, X_l]^T, \qquad U = [U_1, U_2, \ldots, U_q]^T.$$

Similarly we have

$$X(0) = \hat{B}^T(t)d, \qquad \Phi(t - k_j) = \hat{B}^T(t)R_j, \tag{13}$$

where d and R_j, $j = 1, 2, \ldots, r$, are vectors of order $lMN \times 1$ given by

$$d = [d_1, d_2, \ldots, d_l]^T, \qquad R_j = [\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jl}]^T.$$

We now expand $E(t)$, $F_j(t)$, $j = 1, 2, \ldots, r$, and $G(t)$ by hybrid functions as follows:

$$E(t) = [E_{10}, E_{11}, \ldots, E_{1M-1}, \ldots, E_{N0}, E_{N1}, \ldots, E_{NM-1}]^T \hat{B}(t) = E^T \hat{B}(t),$$

$$F_j(t) = \left[F_{j10}, F_{j11}, \ldots, F_{j1(M-1)}, \ldots, F_{jN0}, F_{jN1}, \ldots, F_{jN(M-1)}\right]^T \hat{B}(t) = F_j^T \hat{B}(t),$$

$$G(t) = [G_{10}, G_{11}, \ldots, G_{1M-1}, \ldots, G_{N0}, G_{N1}, \ldots, G_{NM-1}]^T \hat{B}_1(t) = G^T \hat{B}_1(t),$$

where E^T, F_j^T, $j = 1, 2, \ldots, r$, and G^T are of dimensions $l \times lMN$, $l \times lMN$ and $l \times qMN$, respectively.

We can also write $X(t - k_j)$, $j = 1, 2, \ldots, r$, in terms of hybrid functions as

$$X(t - k_j) = \begin{cases} \hat{B}^T(t)R_j, 0 \le t \le k_j \\ \hat{B}^T(t)\hat{D}_j^T X, k_j < t \le 1 \end{cases}$$

where

$$\hat{D}_j = I_l \otimes D_j$$

and D_j is the delay operational matrix given in (6).

Now we have

$$E(t)X(t) = E^T \hat{B}(t)\hat{B}^T(t)X = \hat{B}^T(t)\tilde{E}^T X,$$

$$G(t)U(t) = G^T \hat{B}_1(t)\hat{B}_1^T(t)U = \hat{B}^T(t)\tilde{G}^T U, \tag{14}$$

where $\tilde{E}$ and $\tilde{G}$ can be calculated similarly to matrix $\tilde{C}$ in (5). Moreover

$$\int_0^t \hat{B}^T(t')\, dt' = (I_l \otimes B^T(t))(I_l \otimes P^T) = \hat{B}^T(t)\hat{P}^T, \tag{15}$$

$$\int_0^t F_j(t')X(t'-k_j)\,dt' = \begin{cases} \hat{B}^T(t)\hat{P}^T\tilde{F}_j^T R_j, & 0 \le t \le k_j \\ \hat{B}^T(t)Z_j\tilde{F}_j^T R_j + \hat{B}^T(t)\hat{P}^T\tilde{F}_j^T\hat{D}_j^T X, & k_j < t \le 1 \end{cases} \tag{16}$$

where

$$\hat{P} = I_l \otimes P,$$

and P is the operational matrix of integration given in (4) and

$$\int_0^{k_j} \hat{B}^T(t)\,dt = \hat{B}^T(t)Z_j,$$

where Z_j, $j = 1, 2, \ldots, r$, is a constant matrix of order $lMN \times lMN$.
By integrating (7) from 0 to t and using (12)–(16) we have

$$\hat{B}^T(t)X - \hat{B}^T(t)d = \hat{B}^T(t)\hat{P}^T\tilde{E}^T X +$$

$$+\sum_{j=1}^{r}\left[\hat{B}^T(t)\hat{P}^T\tilde{F}_j^T R_j + \hat{B}^T(t)Z_j\tilde{F}_j^T R_j + \hat{B}^T(t)\hat{P}^T\tilde{F}_j^T\hat{D}_j^T X\right] + \hat{B}^T(t)\hat{P}^T\tilde{G}^T U, \tag{17}$$

using (17) we get

$$X = \left[I - \hat{P}^T\tilde{E}^T - \sum_{j=1}^{r}\hat{P}^T\tilde{F}_j^T\hat{D}_j^T\right]^{-1}\left[d + \sum_{j=1}^{r}\left(\hat{P}^T\tilde{F}_j^T R_j + Z_j\tilde{F}_j^T R_j\right) + \hat{P}^T\tilde{G}^T U\right].$$

5 An illustrative example

Consider the time-varying multi-delay system described by

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix} t & 1\\ t & 2t\end{pmatrix}\begin{pmatrix}x_1(t-\frac{1}{3})\\ x_2(t-\frac{1}{3})\end{pmatrix} + \begin{pmatrix}2 & t\\ t^2 & 0\end{pmatrix}\begin{pmatrix}x_1(t-\frac{2}{3})\\ x_2(t-\frac{2}{3})\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u(t), \tag{18}$$

with

$$x_1(t) = x_2(t) = u(t) = 0, \qquad t \in \left[-\frac{2}{3}, 0\right] \tag{19}$$

and

$$u(t) = 2t + 1, \qquad t > 0.$$

The exact solutions are [12]

$$x_1(t) = \begin{cases} 0, & 0 \le t < \frac{1}{3} \\ \frac{7}{162} - \frac{2}{9}t + \frac{1}{6}t^2 + \frac{1}{3}t^3, & \frac{1}{3} \le t < \frac{2}{3} \\ \frac{11}{162} - \frac{58}{243}t + \frac{31}{162}t^2 + \frac{1}{9}t^3 + \frac{7}{72}t^4 + \frac{1}{6}t^5, & \frac{2}{3} \le t \le 1 \end{cases}$$

and

$$x_2(t) = \begin{cases} t + t^2, & 0 \le t < \frac{1}{3} \\ \frac{5}{486} + t + \frac{7}{9}t^2 + \frac{2}{9}t^3 + \frac{1}{2}t^4, & \frac{1}{3} \le t < \frac{2}{3} \\ \frac{1}{486} + t + \frac{200}{243}t^2 + \frac{20}{81}t^3 + \frac{29}{72}t^4 - \frac{1}{9}t^5 + \frac{1}{6}t^6, & \frac{2}{3} \le t \le 1. \end{cases}$$

Here, we solve this problem by choosing $N = 3$ and $M = 7$. Let

$$x_1(t) = C_1^T B(t), \qquad x_2(t) = C_2^T B(t), \tag{20}$$

where C_1, C_2 and $B(t)$ can be obtained similarly to (2) and (3). By expanding t and t^2 in terms of hybrid functions we get

$$t = K_1^T B(t), \quad t^2 = K_2^T B(t).$$

We also have

$$tx_1(t - \frac{1}{3}) = C_1^T D_1 \tilde{K}_1 B(t), \qquad tx_2(t - \frac{1}{3}) = C_2^T D_1 \tilde{K}_1 B(t), \tag{21}$$

$$t^2 x_1(t - \frac{2}{3}) = C_1^T D_2 \tilde{K}_2 B(t), \qquad tx_2(t - \frac{2}{3}) = C_2^T D_2 \tilde{K}_1 B(t), \tag{22}$$

where $\tilde{K}_1$ and $\tilde{K}_2$ can be calculated similarly to matrix $\tilde{C}$ in Eq. (5). By integrating Eq. (18) from 0 to t and using (19)–(22) we can calculate the values of C_1^T and C_2^T and using (20) the same values as the exact $x_1(t)$ and $x_2(t)$ would be obtained.

6 Conclusion

The hybrid of block-pulse functions and Legendre polynomials and the associated operational matrices of integration P, product C, and delay D are applied to solve the linear time varying multi-delay systems. The method is based upon reducing the system into a set of algebraic equations. The matrices P, C, and D have many zeros; hence, the method is computationally very attractive. It is also shown in the example that the hybrid of block-pulse functions and Legendre polynomials provides an exact solution when the exact solutions in each subintervals are polynomials.

References

[1] Malek-Zavarei, M., Jamshidi, M., *Time-Delay Systems: Analysis, Optimization and Applications*, Elsevier Science Ltd, New York, 1987

[2] Harrison, R. F., Optimal control of vehicle suspension dynamics incorporating front-to-rear excitation delays: an approximate solution, *J. Sound Vibration*, **168**, 1993, pp. 339 354

[3] Lee, Y.S., Maurer, P., A Compiled event-driven multi delay simulator, Tech- nical Report Number DA-30, Department of Computer Science and Engi- neering, University of South Florida, Tampa, FL, 1995

[4] Cai, G., Huang, J., Optimal control method with time delay in control, *J. Sound Vibration*, **251**, 2002, pp. 383394

[5] Ji, J. C., Leung, A. Y. T., Resonances of a non-linear s.d.o.f. system with two time-delays in linear feedback control, *J. Sound Vibration*, **253**, 2002, 9851000

[6] Chen, W.L., Shih, Y. P., Shift Walsh matrix and delay differential equations, *IEEE Trans. Automatic Control*, **AC-23**, 1978, J. Franklin Inst., pp. 265280

[7] Shih, Y.P., Hwang, C., Chia, W.K., Parameter estimation of delay systems via block pulse functions, *Trans. the ASME J. Dynamic Systems, Measurement and Control*, **102**, 1980, pp. 159162

[8] Kung, F.C., Lee, H., Solution and parameter estimation of in linear time-invariant delay systems using Laguerre polynomial expansion, *Trans. the ASME J. Dynamic Systems, Measurement and Control*, **105**, 1983, pp. 297301

[9] Lee, H., Kung, F.C., Shifted Legendre series solution and parameter estimation of linear delayed systems, *International Journal of Systems Science*, **16**, 1985, pp. 12491256

[10] Horng, I.R., Chou, J.H., Analysis, parameter estimation and optimal control of time-delay systems via Chebyshev series, *International Journal of Control*, **441**, 1985, pp. 12211234

[11] Chen, W.L., Walsh series analysis of multi-delay systems, Journal of the Franklin Institute, **313**, 1982, pp. 185246

[12] Datta, K.B., Mohan, B.M., *Orthogonal Functions in Systems and Control*, World Scientific,1995

[13] Marzban, H., Razzaghi, M., Optimal Control of Linear Delay Systems via Hybrid of Block-Pulse and Legendre Polynomials, *Journal of the Franklin Institute*, **341**, 2004, pp. 279-293

[14] P. Lancaster, *Theory of Matrices*, Academic Press, New York, 1969

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