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CHAOS IN NONAUTONOMOUS DYNAMICAL SYSTEMS

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Abstract

We introduce a notion of topological entropy in nonautonomous continuous dynamical systems (or more precisely process) acting on a not necessarily compact space. It is a generalisation of the one introduced in [1] for nonautonomous discrete dynamical systems.

1 Introduction

Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. We say that a subset $E \subset X$ is an (n, ε, f)-*spanning set* if for every $x \in X$ there is $y \in E$ such that $d(f^i(x), f^i(y)) < \varepsilon$ for all $i = 0, 1, \ldots, n-1$. We denote by $S_f(n, \varepsilon)$ the minimal cardinal among all possible (n, ε, f)-separated subsets of X. It is well known that the following limit always exists

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log S_f(n, \varepsilon)}{n}.$$

and that $h(f) \in [0, +\infty]$. We call $h(f)$ *the topological entropy* of the map f

Replacing n by t and f by a flow, one can define spanning sets and topological entropy in the case of continuous dynamical systems. In this article we go a step further and define this notion for flows (see Definition 5). The only difference is that the structure of spanning set will additionally depend on the time of the observation that observation has started.

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In most cases, local processes arise in applications, as solutions of (nonautonomous) differential equations, let say on $\mathbb{R}^n$. Consequence of this approach is that X is usually not compact itself; however, sometimes it is possible to find a subset of the space, where interesting dynamics can take place. In our opinion, every closed and bounded set K with $h_K(\varphi, s) > 0$ belongs to the class of such sets. Although we deal with the closed and bounded subsets, our approach coincides with the standard ones in the case of compact spaces and $\mathbb{R}^n$ (cf. [1]).

Our approach is motivated by applications, so entropy of a nonautonomous dynamical system (induced by numerous Poincaré sections) should reflect complicated dynamics of related (nonautonomous) differential equations. In particular, case when differential equation is autonomous our definition should coincide with well known definition for flows. Furthermore we should obtain a tool, which by analysis of appropriate Poincaré sections answers (or at least provides lower bound) what is the topological entropy of the system (in a properly chosen closed and bounded subset). So in practice we should calculate the entropy of an induced discretisation and then relate it to the entropy of the process.

2 Some definitions

We denote by $\mathbb{N}$ the set of nonnegative integers.

We say that a strictly increasing sequence $\Upsilon = (t_i)_{i\in\mathbb{Z}} \subset \mathbb{R}$ is *forward syndetic* if there exist $k \in \mathbb{Z}$ and $N > 0$ such that $t_{m+1} - t_m < N$ for every $m > k$.

2.1 Processes

Let (Y, d) be a topological (not necessary compact) metric space and $\Omega \subset \mathbb{R} \times \mathbb{R} \times Y$ be an open set.

By a *local process* on Y we mean a continuous map $\varphi : \Omega \longrightarrow Y$, such that the following three conditions are satisfied:

i) $\forall \sigma \in \mathbb{R},\ x \in Y,\ (t^-_{(\sigma,x)}, t^+_{(\sigma,x)}) = \{t \in \mathbb{R} :\ (\sigma, t, x) \in \Omega\}$ is an open interval containing 0,

ii) $\forall \sigma \in \mathbb{R},\ \varphi(\sigma, 0, \cdot) = \mathrm{id}_Y$,

iii) $\forall x \in X,\ \sigma, s \in \mathbb{R},\ t \in \mathbb{R}$ if $(\sigma, x, s) \in \Omega$, $(\sigma + s, t, \varphi(\sigma, s, x)) \in \Omega$ then $(\sigma, s + t, x) \in \Omega$ and $\varphi(\sigma, s + t, x) = \varphi(\sigma + s, t, \varphi(\sigma, s, x))$.

For abbreviation, we write $\varphi_{(\sigma,t)}(x)$ instead of $\varphi(\sigma, t, x)$ and $\varphi(\sigma, S, x)$ instead of $\varphi(\{\sigma\} \times S \times \{x\})$ for any $S \subset (t^-_{(\sigma,x)}, t^+_{(\sigma,x)})$.

Given a local process φ on Y one can define a local flow ϕ on $\mathbb{R} \times Y$ by

$$\phi(t, (\sigma, x)) = (t + \sigma, \varphi(\sigma, t, x)).$$

Unfortunately, in contrast to Y, the space $\mathbb{R} \times Y$ is never compact (we assume that $\mathbb{R}$ is always endowed with the standard metric induced by $|\cdot|$).

2.2 Nonautonomous discrete dynamical systems

Let $X_\infty = \{(X_i, d_i)\}_{i \in \mathbb{Z}}$ be a sequence of (not necessary compact) metric spaces and $f_\infty = \{f_i\}_{i \in \mathbb{Z}}$ be sequence of continuous maps, where $f_i : X_i \longrightarrow X_{i+1}$. For every $i \in \mathbb{Z}$, $n \in \mathbb{N} \setminus \{0\}$ we write $f_i^n = f_{i+(n-1)} \circ \ldots \circ f_{i+1} \circ f_i$, $f_i^0 = \mathrm{id}_{X_i}$ and additionally $f_i^{-n} = (f_i^n)^{-1}$ where the righthand side denotes preimage and can be applied only to sets. Note that we do not assume that the maps f_i are invertible or even onto. We call the pair (X_∞, f_∞) *a nonautonomous discrete dynamical system (abbreviated NDDS).*

Let $k \in \mathbb{Z}$. The *positive trajectory* of a point $x \in X_k$ is the sequence $(f_k^n(x))_{n \in \mathbb{N}}$. A *maximal trajectory* of a point $x \in X_k$ is any sequence $(y_i)_{i \in C}$ such that $y_k = x$, $f_i(y_i) = y_{i+1}$ for every $i \in C$ where $C = \mathbb{Z}$ or $C = [l, \infty) \cap \mathbb{Z}$ for some $l \leq k$ and $f_{l-1}^{-1}(\{y_l\}) = \emptyset$. If $C = \mathbb{Z}$ then we call $(y_i)_{i \in C}$ a *full trajectory* of x.

We say that a NDDS (X_∞, f_∞) is *proper (with respect to a metric space (Y, d)) (denoted PNDDS)* iff for every $i \in \mathbb{Z}$ the following two conditions are satisfied

$$X_i \subset Y \text{ and } d_i = d\,|_{X_i \times X_i}, \tag{2.1}$$

$$f_i \text{ is injective.} \tag{2.2}$$

Remark 1 *If NDDS (X_∞, f_∞) is proper then every maximal trajectory is uniquely determined, however it may happen that it is unbounded.*

2.3 Discretisation

Let φ be a local process on a metric space (Y, d) and let $\Upsilon = (t_i)_{i \in \mathbb{Z}}$ be a strictly increasing sequence such that

$$\lim_{i \to \pm\infty} t_i = \pm\infty \tag{2.3}$$

holds. The Υ-*discretisation of* φ is the nonautonomous discrete dynamical system $\varphi_\Upsilon = (X_\infty, f_\infty)$ given by

$$X_i = \left\{ x \in Y \ : \ t^+_{(t_i, x)} = \infty \right\}, \tag{2.4}$$

$$f_i = \varphi_{(t_i, t_{i+1} - t_i)}. \tag{2.5}$$

Proposition 2 *Let φ be a local process on a metric space (Y, d) and $\Upsilon = (t_i)_{i\in\mathbb{Z}}$ be a strictly increasing sequence such that* (2.3) *holds. Then the Υ-discretisation of φ is a proper nonautonomous discrete dynamical system with respect to Y.*

Proof It is easy to observe that (2.1) follows by (2.4). Since $\varphi_{(\sigma,t)}$ is continuous and injective for every σ and t whenever it is defined and, by (2.5), $f_i : X_i \longrightarrow X_{i+1}$ holds for every $i \in \mathbb{Z}$, it follows that f_i is continuous and injective for every $i \in \mathbb{Z}$.

3 Entropy

3.1 Entropy for processes

Let φ be a local process acting on a metric space (X, d), $s \in \mathbb{R}$ and K be a closed and bounded subset of X. We define sets

$$\Lambda_K^+(\varphi, s) = \{x \in K \ : \ \varphi(s, t, x) \in K \text{ for every } t \geq 0\},$$
$$\Lambda_K(\varphi, s) = \{x \in K \ : \ \varphi(s, t, x) \in K \text{ for every } t \in \mathbb{R}\}.$$

Proposition 3 *Let φ be a local process acting on a metric space (X, d), let $s \in \mathbb{R}$ and let K be a closed and bounded subset of X. Sets $\Lambda_K^+(\varphi, s)$ and $\Lambda_K(\varphi, s)$ are closed and bouded.*

Proof Let $\{x_j\}_{j\in\mathbb{N}} \subset \Lambda_K^+(\varphi, s)$ be such that the limit $\lim_{j\to\infty} x_j = x$ exists. We show that $x \in \Lambda_K^+(\varphi, s)$. Let us fix $t \geq 0$. Then $\varphi(s, t, x_j) \in K$ for every $j \in \mathbb{N}$. Thus $K \ni \lim_{j\to\infty} \varphi(s, t, x_j) = \varphi(s, t, x)$. By an arbitrariness of t, $x \in \Lambda_K^+(\varphi, s)$.

The case of $\Lambda_K(\varphi, s)$ is analogous.

Fix $\varepsilon > 0$, $s \in \mathbb{R}$, $T > 0$ and a closed and bounded set $K \subset X$. We say that a subset $E \subset K$ is a $(s, T, \varepsilon, K, \varphi)$*-spanning set (with respect to the set K)* if for every $y \in \Lambda_K(\varphi, s)$ there is $x \in E$ such that $d(\varphi(s, t, x), \varphi(s, t, y)) < \varepsilon$ for every $t \in [0, T]$. If we replace $\Lambda_K(\varphi, s)$ by $\Lambda_K^+(\varphi, s)$ in the above definition, then we say that E is *a positive $(s, T, \varepsilon, K, \varphi)$-spanning set (with respect to the set K)*

We denote by $S_\varphi^+(s, T, \varepsilon, K)$ (resp. $S_\varphi(s, T, \varepsilon, K)$) the minimal cardinal among all possible positive $(s, T, \varepsilon, K, \varphi)$-spanning sets with respect to K (resp. all possible $(s, T, \varepsilon, K, \varphi)$-spanning sets). In the particular case $s = 0$ we simply write $S_\varphi^+(T, \varepsilon, K)$ and $S_\varphi(T, \varepsilon, K)$; if additionally $K = X$, then we write $S_\varphi^+(T, \varepsilon)$ and $S_\varphi(T, \varepsilon)$. Note that if $\Lambda_K^+(s) = \emptyset$ or $\Lambda_K(s) = \emptyset$, then $S_\varphi^+(s, T, \varepsilon, K) = 0$ or $S_\varphi(s, T, \varepsilon, K) = 0$ respectively, because empty set fulfills the definition of spanning set in that case.

Lemma 4 *Let φ be a local process on a metric space (X,d), K be a closed and bouded subset of X and $s \in \mathbb{R}$. The following limits always exist*

$$a^{+}(s,K,\varphi) = \lim_{\varepsilon\to 0^{+}} \limsup_{T\to\infty} \frac{\log S^{+}_{\varphi}(s,T,\varepsilon,K)}{T}, \tag{3.6}$$

$$a(s,K,\varphi) = \lim_{\varepsilon\to 0^{+}} \limsup_{T\to\infty} \frac{\log S_{\varphi}(s,T,\varepsilon,K)}{T}. \tag{3.7}$$

Moreover, for every $s,t \in \mathbb{R}$, $s \le t$ the following conditions hold:

$$a(s,K,\varphi) = a(t,K,\varphi), \tag{3.8}$$

$$a(s,K,\varphi) \le a^{+}(s,K,\varphi), \tag{3.9}$$

$$a^{+}(s,K,\varphi) \le a^{+}(t,K,\varphi). \tag{3.10}$$

Proof The proof is standard. We present it for completeness.

Let us fix $T > 0$ and $0 < \varepsilon_1 < \varepsilon_2$. Then $S^{+}_{\varphi}(s,T,\varepsilon_1,K) \ge S^{+}_{\varphi}(s,T,\varepsilon_2,K)$ holds. Thus the map

$$\alpha_{(s,K,\varphi)} : (0,\infty) \ni \varepsilon \mapsto \limsup_{T\to\infty} \frac{\log S^{+}_{\varphi}(s,T,\varepsilon,K)}{T} \in [0,\infty]$$

is weakly decreasing, so $\lim_{\varepsilon\to 0^{+}} \alpha_{(s,K,\varphi)}(\varepsilon)$ exists.

Analogously, the limit (3.7) exists.

The inequality (3.9) follows by that fact that $\Lambda_K(\varphi,s) \subset \Lambda^{+}_K(\varphi,s)$ and so $S_{\varphi}(s,T,\varepsilon,K) \le S^{+}_{\varphi}(s,T,\varepsilon,K)$ holds.

To prove (3.10), let us fix $\varepsilon > 0$, $T > 0$, $s,t \in \mathbb{R}$, $s \le t$ and observe that the inclusion

$$\varphi_{(s,t-s)}\left(\Lambda^{+}_K(\varphi,s)\right) \subset \Lambda^{+}_K(\varphi,t)$$

may be strict. Nonetheless, is easy to see that

$$S^{+}_{\varphi}(s,T+t-s,\varepsilon,K) \le S^{+}_{\varphi}(s,T,\varepsilon,K) + S^{+}_{\varphi}(s,t-s,\varepsilon,K)$$

holds. Thus the inequality

$$\begin{aligned}
\limsup_{T+t-s\to\infty} \frac{\log S^{+}_{\varphi}(s,T+t-s,\varepsilon,K)}{T+t-s} &= \limsup_{T\to\infty} \frac{\log S^{+}_{\varphi}(s,T+t-s,\varepsilon,K)}{T} \\
&\le \limsup_{T\to\infty} \frac{\log S^{+}_{\varphi}(t,T,\varepsilon,K)}{T} \\
&\quad + \limsup_{T\to\infty} \frac{\log S^{+}_{\varphi}(s,t-s,\varepsilon,K)}{T} \\
&= \limsup_{T\to\infty} \frac{\log S^{+}_{\varphi}(t,T,\varepsilon,K)}{T}
\end{aligned}$$

is satisfied, so (3.10) follows.

To prove (3.8), let us fix $\varepsilon > 0$, $T > 0$, $s, t \in \mathbb{R}$, $s \leq t$ and observe that the equality

$$\varphi_{(s,t-s)}\left(\Lambda_K(\varphi, s)\right) = \Lambda_K(\varphi, t) \tag{3.11}$$

holds. As previously, the inequality

$$S_\varphi(s, T + t - s, \varepsilon, K) \leq S_\varphi(s, T, \varepsilon, K) + S_\varphi(s, t - s, \varepsilon, K)$$

is satisfied, so $a(s, K, \varphi) \leq a(t, K, \varphi)$ holds. Now, by (3.11), we have

$$S_\varphi(s, T + t - s, \varepsilon, K) \geq S_\varphi(s, T, \varepsilon, K),$$

so $a(s, K, \varphi) \geq a(t, K, \varphi)$ holds.

The above fact allows us to define the topological entropy.

Definition 5 *Let φ be a local process on a metric space (X, d), K be a closed and bounded subset of X and $s \in [-\infty, \infty]$.* The topological entropy of the process φ at a time section s (with respect to the set K) *or simply* entropy of φ at s *is the number $h_K(\varphi, s) \in [0, +\infty]$ defined by*

$$h_K(\varphi, s) = \begin{cases} a(0, K, \varphi), & \textit{for } s = -\infty, \\ a^+(s, K, \varphi), & \textit{for } s \in \mathbb{R}, \\ \lim_{t\to\infty} a^+(t, K, \varphi), & \textit{for } s = \infty. \end{cases}$$

The topological entropy of the process φ at s *is the number $h(\varphi, s) \in [0, +\infty]$ defined by*

$$h(\varphi, s) = \sup\left\{h_K(\varphi, s) \ : \ K \textit{ is a closed and bounded subset of } X\right\}. \tag{3.12}$$

Proposition 6 *Let φ be a local process on a metric space (X, d) and let $s, t \in \mathbb{R}$, $s \leq t$. In that case the following inequalities hold*

$$h(\varphi, -\infty) \leq \lim_{p\to-\infty} h(\varphi, p) \leq h(\varphi, s) \leq h(\varphi, t) \leq h(\varphi, \infty) \tag{3.13}$$

and furthermore

$$\lim_{p\to\infty} h(\varphi, p) = h(\varphi, \infty). \tag{3.14}$$

Proof By (3.10), $h(\varphi, p) \leq h(\varphi, q)$ for every $p, q \in \mathbb{R}$, $p \leq q$, so $\lim_{p\to-\infty} h(\varphi, p)$ exists.

By (3.8) and (3.9), $a(0, K, \varphi) \leq a^+(q, K, \varphi)$ for every $q \in \mathbb{R}$ and every closed and bouded $K \subset X$. Thus also $h_K(\varphi, -\infty) \leq h_K(\varphi, q)$ holds. It follows that $h(\varphi,-\infty) \leq h(\varphi,q)$ is satisfied for every $q \in \mathbb{R}$. Finally, the formula

$$h(\varphi,-\infty) \leq \inf_{q\in\mathbb{R}} h(\varphi,q) = \lim_{p\to-\infty} h(\varphi,p) \leq h(\varphi,s)$$

holds.

Since, by (3.10), $h_K(\varphi,q) \leq h_K(\varphi,\infty)$ holds for every $q \in \mathbb{R}$ and every closed and bounded $K \subset X$, we have

$$h(\varphi,t) \leq h(\varphi,\infty). \tag{3.15}$$

To prove (3.14), let us observe that, by (3.15), we have

$$\lim_{p\to\infty} h(\varphi,p) \leq h(\varphi,\infty).$$

Let us fix $p \in \mathbb{R}$ and K a closed and bounded subset of X. Then the following inequalities hold

$$\begin{aligned} h(\varphi,p) \geq& h_K(\varphi,p), \\ \lim_{p\to\infty} h(\varphi,p) \geq& h(\varphi,p) \geq h_K(\varphi,p), \\ \lim_{p\to\infty} h(\varphi,p) \geq& \lim_{p\to\infty} h_K(\varphi,p) = h_K(\varphi,\infty). \end{aligned}$$

Finally,

$$\lim_{p\to\infty} h(\varphi,p) \geq h(\varphi,\infty)$$

holds, so (3.14) is proved.

3.2 Entropy for PNDDS

We start with a definition of entropy which is obtained by simple rewriting of the definition introduced for processes.

Let $i \in \mathbb{Z}$, (X_∞, f_∞) be a proper nonautonomous discrete dynamical system with respect to Y and let K be a closed and bounded subset of Y. We define sets

$$\begin{aligned} \Lambda_K^+(f_\infty,i) &= \{x \in X_i \ : \ \text{the trajectory of } x \text{ is a subset of } K\}, \\ \Lambda_K(f_\infty,i) &= \{x \in X_i \ : \ \text{the full trajectory of } x \text{ exists and is contained in } K\}. \end{aligned}$$

Proposition 7 *Let $i \in \mathbb{Z}$, let (X_∞, f_∞) be a proper nonautonomous discrete dynamical system with respect to a metric space (Y,d) and let K be a closed and bounded subset of Y. Sets $\Lambda_K^+(f_\infty,i)$ and $\Lambda_K(f_\infty,i)$ are closed and bounded.*

Proof The proof is analogous to the proof of Proposition 3.

Fix $\varepsilon > 0$, $i \in \mathbb{Z}$, $N \in \mathbb{N}$ and a closed and bounded set $K \subset Y$. We say that a subset $E \subset K$ is a $(i, N, \varepsilon, K, f_\infty)$-*spanning set (with respect to K)* if for every $y \in \Lambda_K(f_\infty, i)$ there is $x \in E$ such that $d(f_i^n(x), f_i^n(x)) < \varepsilon$ for every integer $n \in [0, N]$. Again, if we replace $\Lambda_K(f_\infty, i)$ by $\Lambda_K^+(f_\infty, i)$ then we obtain the definition of a *positive* $(i, N, \varepsilon, K, f_\infty)$-*spanning set*.

We denote by $S_{f_\infty}^+(i, N, \varepsilon, K)$ and $S_{f_\infty}(i, N, \varepsilon, K)$ the minimal cardinal among all possible positive $(i, N, \varepsilon, K, f_\infty)$-spanning sets and $(i, N, \varepsilon, K, f_\infty)$-spanning sets with respect to K respectively. In the particular case $i = 0$, we simply write $S_{f_\infty}^+(N, \varepsilon, K)$ and $S_{f_\infty}(N, \varepsilon, K)$. Additionally, if $K = Y$ then we write $S_{f_\infty}^+(N, \varepsilon)$ or $S_{f_\infty}(N, \varepsilon)$ respectively.

Lemma 8 *Let $i \in \mathbb{Z}$, let (X_∞, f_∞) be a proper nonautonomous discrete dynamical system with respect to a metric space (Y, d) and let K be a closed and bounded subset of Y. The following limits always exist*

$$b^+(i, K, f_\infty) = \lim_{\varepsilon \to 0^+} \limsup_{N \to \infty} \frac{\log S_{f_\infty}^+(i, N, \varepsilon, K)}{N}, \tag{3.16}$$

$$b(i, K, f_\infty) = \lim_{\varepsilon \to 0^+} \limsup_{N \to \infty} \frac{\log S_{f_\infty}(i, N, \varepsilon, K)}{N}. \tag{3.17}$$

Moreover, for every $i, j \in \mathbb{Z}$, $i \leq j$ the following holds:

$$b(i, K, f_\infty) = b(j, K, f_\infty), \tag{3.18}$$

$$b(i, K, f_\infty) \leq b^+(i, K, f_\infty), \tag{3.19}$$

$$b^+(i, K, f_\infty) \leq b^+(j, K, f_\infty). \tag{3.20}$$

Proof The proof is analogous to the proof of Lemma 4.

The same way as usual, we define the topological entropy in this setting.

Definition 9 *Let (X_∞, f_∞) be a proper nonautonomous discrete dynamical system with respect to a metric space (Y, d), let K be a closed and bounded subset of Y and let $i \in \mathbb{Z} \cup \{-\infty, +\infty\}$.* The topological entropy of a PNDDS (X_∞, f_∞) at i (with respect to the set K) *is the number $h_K(f_\infty, i) \in [0, +\infty]$ defined by*

$$h_K(f_\infty, i) = \begin{cases} b(0, K, f_\infty), & \text{for } i = -\infty, \\ b^+(i, K, f_\infty), & \text{for } i \in \mathbb{Z}, \\ \lim_{n\to\infty} b^+(n, K, f_\infty), & \text{for } i = \infty. \end{cases}$$

The topological entropy of (X_∞, f_∞) at i *is the number* $h(f_\infty, i) \in [0, +\infty]$ *defined by*

$$h(f_\infty, i) = \sup \{h_K(f_\infty, i, \Upsilon) \ : \ K \text{ is a closed and bounded subset of } Y\}. \tag{3.21}$$

Proposition 10 *Let* (X_∞, f_∞) *be a PNDDS and let* $i, j \in \mathbb{Z}$, $i \leq j$. *Then*

$$\lim_{i\to\infty} h(f_\infty, i) = h(f_\infty, \infty) \tag{3.22}$$

and the following inequalities hold

$$h(f_\infty, -\infty) \leq \lim_{k\to-\infty} h(f_\infty, k) \leq h(f_\infty, i) \leq h(f_\infty, j) \leq h(f_\infty, \infty). \tag{3.23}$$

Proof The proof is analogous to the proof of Proposition 6.

Let $\varphi_\Upsilon = (X_\infty, f_\infty)$ be the Υ-discretisation of a local process φ. To simplify the notation we identify φ_Υ with the map f_∞ and write $h(\varphi_\Upsilon, i)$, $h_K(\varphi_\Upsilon, i)$, $S_{\varphi_\Upsilon}(i, T, \varepsilon, K)$ etc.

4 Relations between entropy of a process and its discretisation

Definition 11 *We call a local process* φ *on a metric space* (X, d) discretisable *if there exists a sequence of closed and bounded subsets* $\{M_i\}_{i\in\mathbb{N}}$ *of* X *such that*

$$M_i \subset M_{i+1}, \text{ for every } i \in \mathbb{N}, \tag{4.24}$$

$$X = \bigcup_{i=0}^{\infty} M_i, \tag{4.25}$$

$$\text{for every closed and bounded } K \subset X \text{ there exists } i \in \mathbb{N} \text{ such that } K \subset M_i \tag{4.26}$$

and for every $s \in \mathbb{R}, i \in N, x \in X \setminus M_i$ *it follows*

$$t^+_{(s,x)} \quad < \quad \infty, \tag{4.27}$$

provided there exists $t < 0$ *such that* $\varphi(s, t, x) \in M_i$.

Lemma 12 *Let* K *be a closed and bounded subset of a metric space* (X, d), φ *be a discretisable local process on* X *and* $\Upsilon = \{t_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ *be a strictly increasing sequence satisfying* (2.3). *Then for every* $i \in \mathbb{Z}$ *there exists a closed and bounded set* $M \subset X$ *satisfying*

$$\Lambda^+_K(\varphi_\Upsilon, i) \subset \Lambda^+_M(\varphi, t_i), \tag{4.28}$$

$$\Lambda_K(\varphi_\Upsilon, i) \subset \Lambda_M(\varphi, t_i). \tag{4.29}$$

Proof Let us fix $i \in \mathbb{Z}$. Then there exists $j \in \mathbb{N}$ such that $M = M_j \supset K$ where $\{M_i\}_{i\in\mathbb{N}}$ is as in Definition 11.

Let $x \in \Lambda_K^+(\varphi_\Upsilon, i)$. Thus, in particular, $t^+_{(t_i,x)} = \infty$. If there exists $t > 0$ such that $\varphi(t_i, t, x) \notin M$, then, by (4.27), $t^+_{(t_i,x)} < \infty$ which is a contradiction. Thus $\varphi(t_i, t, x) \in M$ for every $t > 0$, so $x \in \Lambda^+_M(\varphi, t_i)$ and (4.28) follows.

Let now $x \in \Lambda_K(\varphi_\Upsilon, i)$. As in the previous case, $\varphi(t_i, t, x) \in M$ for every $t > 0$. Suppose that $t < 0$ is such that $\varphi(t_i, t, x) \notin M$. By (2.3), there exists $j \in \mathbb{Z}$, $j < i$ such that $t_j < t_i + t$. Since $x \in \Lambda_K(\varphi_\Upsilon, i)$, we have $\varphi(t_i + t, t_j - t_i - t, \varphi(t_i, t, x)) = \varphi(t_i, t_j - t_i, x) = (\varphi_\Upsilon)_j^{j-i} \in K \subset M$. But, since $t_j - t_i - t < 0$, by (4.27), we have $t^+_{(t_i+t,\varphi(t_i,t,x))} < \infty$ which is equivalent to $t^+_{(t_i,x)} < \infty$ which is a desired contradiction.

Theorem 13 *Let φ be a dicretisable local process on a metric space (X, d) and $\Upsilon = (t_i)_{i\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly increasing sequence such that the condition*

$$\limsup_{n\to\infty} \frac{t_n}{n} \le \alpha \tag{4.30}$$

holds for some $\alpha \in [0, \infty]$. Then the inequality

$$h(\varphi_\Upsilon, i) \le \alpha\, h(\varphi, t_i) \tag{4.31}$$

is satisfied for every $i \in \mathbb{Z} \cup \{-\infty, \infty\}$ (here we define $c \cdot \infty = \infty \cdot c = \infty$ for every $c \in [0, +\infty]$).

Proof If $\alpha = +\infty$ or $h(\varphi, i) = +\infty$, then (4.31) is satisfied. So let us assume that $\alpha < \infty$ and $h(\varphi, t_i) < \infty$.

Fix any $i \in \mathbb{Z}$, $R \in \mathbb{N}$, $\varepsilon > 0$ and a closed and bounded subset $K \subset Y$.

Since, in general, the following inclusion

$$\Lambda^+_K(\varphi, t_i) \subset \Lambda^+_K(\varphi_\Upsilon, i)$$

can not be replaced by an equality, by Lemma 12, we use the following one

$$\Lambda^+_K(\varphi_\Upsilon, i) \subset \Lambda^+_M(\varphi, t_i),$$

where M is some closed and bounded subset of X.

Thus $S_{\varphi_\Upsilon}(i, R, \varepsilon, K) \le S_\varphi(t_i, t_{i+R} - t_i, \varepsilon, M)$ and, by (4.30),

$$\begin{aligned}
\limsup_{R\to\infty} \frac{\log S^+_{\varphi_\Upsilon}(i, R, \varepsilon, K)}{R} &\le \limsup_{R\to\infty} \frac{\log S^+_\varphi(t_i, t_{i+R} - t_i, \varepsilon, M)}{R} \\
&\le \limsup_{R\to\infty} \frac{\log S^+_\varphi(t_i, t_{i+R} - t_i, \varepsilon, M)}{t_{i+R} - t_i} \limsup_{R\to\infty} \frac{t_{i+R} - t_i}{R} \\
&\le \alpha \limsup_{T\to\infty} \frac{\log S^+_\varphi(t_i, T, \varepsilon, M)}{T}
\end{aligned}$$

hold. Thus $h_K(\varphi_\Upsilon, i) \leq \alpha h_M(\varphi, t_i)$ and immediately (4.31) is satisfied.

Now, by (3.14) and (3.22), (4.31) holds for $i = \infty$.

Let us observe that the inclusion

$$\Lambda_K(\varphi_\Upsilon, i) \subset \Lambda_M(\varphi, t_i),$$

is satisfied. So (4.31) holds for $i = -\infty$. The calculations are quite similar to the above ones.

Definition 14 *We call a local process φ on a metric space (X, d)* locally equicontinuous *if for every closed and bounded set $K \subset X$ exists nondecreasing continuous function $\psi_K : [0, \infty) \longrightarrow [0, \infty)$ such that the inequality*

$$d(\varphi(\sigma, t, x), \varphi(\sigma, t, y)) \leq d(x, y)\psi_K(t)$$

holds for every $\sigma \in \mathbb{R}$, $x, y \in K$, $t \in [0, \infty)$ which satisfy $\varphi(\sigma, [0, t], x), \varphi(\sigma, [0, t], y) \subset K$.

Theorem 15 *Let φ be a locally equicontinuous local process on a metric space (X, d) and $\Upsilon = (t_i)_{i\in\mathbb{Z}}$ be a forward syndetic strictly increasing sequence such that*

$$\liminf_{n\to\infty} \frac{t_n}{n} \geq \alpha \tag{4.32}$$

holds for some $\alpha \in [0, \infty]$. Then the inequality

$$h(\varphi_\Upsilon, i) \geq \alpha h(\varphi, t_i) \tag{4.33}$$

is satisfied for every $i \in \mathbb{Z} \cup \{-\infty, \infty\}$ (here we define $c \cdot 0 = 0 \cdot c = 0$ for every $c \in [0, +\infty]$).

Proof Let $\alpha > 0$ and $h(\varphi, t_i) > 0$ hold.

Let us fix $i \in \mathbb{Z}$, $R \in \mathbb{N}$, $\varepsilon > 0$ and K a closed and bounded subset of Y. Then $t_i \in \Upsilon$ and there exists $N > 0$ such that $t_{m+1} - t_m < N$ for every $m \geq i$.

By the inclusion

$$\Lambda_K^+(\varphi, t_i) \subset \Lambda_K^+(\varphi_\Upsilon, i)$$

and equicontinuity of φ, one gets $S^+_{\varphi_\Upsilon}(i, R, \varepsilon, K) \geq S^+_\varphi(t_i, t_{i+R} - t_i, 2\varepsilon\psi_K(N), K)$

and, by (4.32),

$$\begin{aligned}\limsup_{R\to\infty}\frac{\log S^+_{\varphi_\Upsilon}(i,R,\varepsilon,K)}{R} &\geq \limsup_{R\to\infty}\frac{\log S^+_{\varphi}(t_i,t_{i+R}-t_i,2\varepsilon\psi_K(N),K)}{R}\\ &\geq \limsup_{R\to\infty}\frac{\log S^+_{\varphi}(t_i,t_{i+R}-t_i,2\varepsilon\psi_K(N),K)}{t_{i+R}-t_i}\\ &\quad\cdot\liminf_{R\to\infty}\frac{t_{i+R}-t_i}{R}\\ &\geq \alpha\limsup_{T\to\infty}\frac{\log S^+_{\varphi}(t_i,T,2\varepsilon\psi_K(N),K)}{T}\end{aligned}$$

hold. Thus $h_K(\varphi_\Upsilon,i)\geq \alpha h_K(\varphi,t_i)$ and immediately (4.33) is satisfied.

Now, the case $i\in\{-\infty,\infty\}$ follows easily.

Example 16 *Let $\Upsilon=\left\{\frac{\pi m}{\kappa}\right\}_{m\in\mathbb{Z}}$ and φ be a locall process on $\mathbb{C}$ generated by the equation*

$$\dot z=\left(1+e^{i\kappa t}|z|^2\right)\overline{z}, \tag{4.34}$$

where $\kappa=0.5$. By the change of variables $w=e^{2t\sin(\kappa t)}z$ we get the equation

$$\dot w=\left(1+e^{-4t\sin(\kappa t)+i\kappa t}|w|^2\right)\overline{w}+w\left[2\sin(\kappa t)+2\kappa t\cos(\kappa t)\right]$$

which generates the process ψ. Let us observe that $\varphi_\Upsilon=\psi_\Upsilon$. Since, by [2, 4, 5], $h(\psi_\Upsilon,0)=h(\varphi_\Upsilon,0)\geq\log 3$ and $h(\psi,0)=0$ ($\Lambda^+_K(\psi,0)\subset\{0\}$ for every closed and bounded $K\subset\mathbb{C}$), the discretisability condition in Theorem 13 is essential.

Example 17 *Let φ be the locall process on $\mathbb{C}$ generated by the equation*

$$\dot z=\left[1+\left(\cos\left(t^2\right)+2\right)e^{i\kappa t}|z|^2\right]\overline{z} \tag{4.35}$$

where $\kappa\in(0,0.796]$ is a parameter. Let $\Upsilon=\{t_j\}_{j\in\mathbb{Z}}$ where $t_j=\frac{2\pi}{\kappa}j$. By [3, 2], $h(\varphi_\Upsilon,j)\geq\log 3$ for every $j\in\mathbb{Z}$.

Let $M_j=\{z\in\mathbb{C}:|z|\geq j+5\}$. By direct calculations, it is easy to see, that the sequence $\{M_j\}_{j\in\mathbb{N}}$ is such in Definition 11, i.e. every solution which leaves M_j blows up. Thus φ is discretisable. Finally, by Theorems 13 and 15, $h(\varphi,t_i)=\frac{\kappa}{2\pi}h(\varphi_\Upsilon,0)\geq\frac{\kappa}{2\pi}\log 3$ for every $j\in\mathbb{Z}\cup\{-\infty,+\infty\}$.

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