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ON THE BOUUNDEDNESS OF FRACTIONAL B-MAXIMAL OPERATORS IN THE LORENTZ SPACES $L_{p,q,\gamma}(\mathbb{R}^n_+)$ *

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Abstract

In this study, sharp rearrangement inequalities for the fractional B-maximal function $M_{\alpha,\gamma}f$ are obtained in the Lorentz spaces $L_{p,q,\gamma}$ and by using these inequalities the boundedness conditions of the operator $M_{\alpha,\gamma}$ are found. Then, the conditions for the boundedness of the B-maximal operator M_γ are obtained in $L_{p,q,\gamma}$.

1 Introduction

Let $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n, x_n > 0\}$. Denote by $L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}^n_+)$, $\gamma > 0$, the set of all classes of measurable functions with finite norm

$$\|f\|_{L_{p,\gamma}} \equiv \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\gamma dx \right)^{1/p} < \infty,$$

If $p = \infty$, we assume

$$L_{\infty,\gamma}(\mathbb{R}^n_+) = L_\infty(\mathbb{R}^n_+) = \{f : \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_+} |f(x)| < \infty\}.$$

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The generalized translation operator T^y, $y \in \mathbb{R}^n_+$, is defined (see [8, 9, 10]) for smooth functions on $\mathbb{R}^n_+$ by

$$T^y f(x) = C_\gamma \int_0^\pi f(x' - y', \sqrt{x_n^2 + y_n^2 - 2x_n y_n cos\,\theta}\,) sin^{\gamma-1}\theta d\theta,$$

where $x', y' \in \mathbb{R}^{n-1}$ and $C_\gamma = \Gamma((\gamma+1)/2)[\sqrt{\pi}\,\Gamma(\gamma/2)]^{-1}$.

It is well-known that the generalized shift operator T^y is closely related to the Laplace-Bessel differential operator

$$\Delta_B = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}.$$

Further, if f belongs to $L_{p,\gamma}$, $1 \le p \le \infty$, then for all $y \in \mathbb{R}^n_+$, the function $T^y f$ belongs to $L_{p,\gamma}$, and

$$\|T^y f\|_{L_{p,\gamma}} \le \|f\|_{L_{p,\gamma}}.$$

For $0 \le \alpha < n + \gamma$, the fractional maximal function associated with Δ_B (fractional B-maximal function) is defined at $f \in L^{loc}_{1,\gamma}(\mathbb{R}^n_+)$ by

$$(M_{\alpha,\gamma} f)(x) = \sup_{r>0} |B(0,r)|_\gamma^{\frac{\alpha}{n+\gamma}-1} \int_{B(0,r)} T^y\,|f(x)|\, y_n^\gamma dy,\; x \in \mathbb{R}^n_+,$$

where $B(0,r) = \{y \in \mathbb{R}^n_+ : |y| < r\}$. For $\alpha = 0$ we get the maximal function $M_\gamma f$ associated with the Laplace-Bessel differential operator (B-maximal function, see [6]).

The aim of this paper is to obtain sharp rearrangement estimates for the fractional B-maximal function $M_{\alpha,\gamma} f$. We give the necessary and sufficient conditions for the boundedness of the operator $M_{\alpha,\gamma}$ in the Lorentz spaces $L_{p,q,\gamma}(\mathbb{R}^n_+)$ by using the obtained sharp rearrangement estimates. As consequence of these results we find the boundedness conditions for the B-maximal operator M_γ in the Lorentz spaces $L_{p,q,\gamma}(\mathbb{R}^n_+)$.

It is well known that for the classical Hardy-Littlewood maximal operator the rearrangement inequality

$$cf^{**}(t) \le (Mf)^*(t) \le Cf^{**}(t), \quad t \in (0,\infty)$$

holds, where $f^*(t)$ is the nonincreasing rearrangement of f and $f^{**}(t) = \frac{1}{t}\int_0^t f^*(t)dt$.

For $f \in L_1^{loc}(\mathbb{R}^n)$, similar sharp rearrangement estimates are obtained for the maximal function of f in [1] and for the fractional maximal function of

f in [3], and for the fractional maximal function of $f \in L_{p,\gamma}(\mathbb{R}^n_+)$, $\gamma > 0$, associated with the Laplace-Bessel differential operator in [7]. These estimates are of great importance in the study of operators on rearrangement-invariant spaces as well as in interpolation theory.

Throughout the paper, we denote $M^+(0,\infty)$ the set of all non-negative measurable functions on $(0,\infty)$ with respect to the measure $x_n^\gamma dx$, and $M^+(0,\infty;\downarrow)$ the set of all non-increasing functions from $M^+(0,\infty)$. We use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence.

2 Preliminaries

Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a measurable function and for any measurable set E, $|E|_\gamma = \int_E x_n^\gamma dx$. We define γ-rearrangement of f in decreasing order by

$$f^*_\gamma(t) = \inf \{s > 0 \ : \ f_{*,\gamma}(s) \le t\}, \quad \forall t \in (0,\infty),$$

where$f_{*,\gamma}(s)$ denotes the γ-distribution function of f given by

$$f_{*,\gamma}(s) = \left|\{x \in (\mathbb{R}^n_+) : |f(x)| > s\}\right|_\gamma .$$

Some properties of γ-rearrangement of functions are given as follows (see [2, 4, 12]):

1) if $0 < p < \infty$, then

$$\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\gamma dx = \int_0^\infty \left(f^*_\gamma(t)\right)^p dt;$$

2) for any $t > 0$,

$$\sup_{|E|_\gamma = t} \int_E |f(x)|\, x_n^\gamma dx = \int_0^t f^*_\gamma(s) ds; \tag{1}$$

3)

$$\int_{\mathbb{R}^n_+} |f(x)g(x)|\, x_n^\gamma dx \le \int_0^\infty f^*_\gamma(t) g^*_\gamma(t) dt;$$

It is well known that

4)

$$(f+g)^*_\gamma(t) \le f^*_\gamma(\frac{t}{2}) + g^*_\gamma(\frac{t}{2}) \tag{2}$$

holds.

We denote by $WL_{p,\gamma}(0,\infty)$ the weak $L_{p,\gamma}$ space of all measurable functions f with finite norm

$$\|f\|_{WL_{p,\gamma}} = \sup_{t>0} t^{1/p} f_\gamma^*(t), \quad 1 \le p < \infty.$$

The function $f_\gamma^{**} : (0,\infty) \to [0,\infty]$ is defined as

$$f_\gamma^{**}(t) = \frac{1}{t}\int_0^t f_\gamma^*(s)ds.$$

Definition 1. *The Lorentz space $L_{p,q,\gamma}(\mathbb{R}_+^n)$ is the collection of all measurable functions f on $\mathbb{R}_+^n$ such that the quantity*

$$\|f\|_{p,q,\gamma} = \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} f_\gamma^*(t)\right)^q \frac{dt}{t}\right)^{1/q}, & 0<p<\infty, 0<q<\infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f_\gamma^*(t), & 0<p\le\infty, q=\infty \end{cases}$$

is finite.

If $0<p\le\infty, q=\infty$, then $L_{p,\infty,\gamma}(\mathbb{R}_+^n) = WL_{p,\gamma}(\mathbb{R}_+^n)$.

If $1 \le q \le p$ or $p=q=\infty$, then the functional $\|f\|_{p,q,\gamma}$ is a norm. (see [2], [5], [12]).

If $p=q=\infty$, then the space $L_{\infty,\infty,\gamma}(\mathbb{R}_+^n)$ is denoted by $L_{\infty,\gamma}(\mathbb{R}_+^n)$.

In the case $1<p,q<\infty$, we give a functional $\|.\|_{p,q,\gamma}^*$ by

$$\|f\|_{p,q,\gamma}^* = \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} f_\gamma^{**}(t)\right)^q \frac{dt}{t}\right)^{1/q}, & 0<p<\infty, 0<q<\infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f_\gamma^{**}(t), & 0<p\le\infty, q=\infty \end{cases}$$

(with the usual modification if $0<p\le\infty$, $q=\infty$) which is a norm on $L_{p,q,\gamma}(\mathbb{R}_+^n)$ for $1<p<\infty$, $1\le q\le\infty$ or $p=q=\infty$.

If $1<p\le\infty$, $1\le q\le\infty$, then

$$\|f\|_{p,q,\gamma} \le \|f\|_{p,q,\gamma}^* \le \frac{p}{p-1}\|f\|_{p,q,\gamma},$$

that is, the quasi-norms $\|f\|_{p,q,\gamma}$ and $\|f\|_{p,q,\gamma}^*$ are equivalent.

Lemma 1. *For any measurable set* $\mathcal{A} = (\mathcal{A}', \mathcal{A}_n) \subset \mathbb{R}^n_+$, $\mathcal{A}_n \subset (0, \infty)$, $\mathcal{A}' = \mathcal{A}_1 \times ... \times \mathcal{A}_{n-1} \subset \mathbb{R}^{n-1}$, *and* $y \in \mathbb{R}^n_+$, *then the following equality holds*

$$\int_{\mathcal{A}} T^y g(x) y_n^\gamma dy = C_\gamma \int_{(x,0)+\widetilde{\mathcal{A}}} g\left(z', \sqrt{z_n^2 + z_{n+1}^2}\right) d\mu\left(z, z_{n+1}\right),$$

where $\widetilde{\mathcal{A}} = \mathcal{A}' \times (-m, m) \times [0, m)$, $m = \sup \mathcal{A}_n$ *and* $d\mu\left(z, z_{n+1}\right) = z_{n+1}^{\gamma-1} dz_1 dz_2 ... dz_n dz_{n+1}$.

The proof of Lemma 1 is straightforward, after applying the following substitutions

$$z' = y' - x', z_n = x_n - y_n \cos\alpha, z_{n+1} = y_n \sin\alpha. \tag{3}$$

In the following lemma we give a relation between the generalized shift operator T^y and the γ-rearrangement of the function f.

Lemma 2. *For any measurable set* $\mathcal{A} \subset \mathbb{R}^n_+$ *and for any* $y \in \mathbb{R}^n_+$, *the following equality holds*

$$\sup_{|\mathcal{A}|_\gamma = t} \int_{\mathcal{A}} T^y |f(x)| y_n^\gamma dy = C_\gamma \int_0^t f_\gamma^*(s) ds. \tag{4}$$

Proof. From Lemma 1 we have

$$\int_{\mathcal{A}} T^y |f(x)| y_n^\gamma dy = C_\gamma \int_{(x,0)+\tilde{\mathcal{A}}} f\left(\sqrt{z_n^2 + z_{n+1}^2}\right) d\mu\left(z, z_{n+1}\right), \tag{5}$$

where $\widetilde{\mathcal{A}} = \mathcal{A}' \times (-m, m) \times [0, m)$, $m = \sup \mathcal{A}_n$ and $d\mu\left(z, z_{n+1}\right) = z_{n+1}^{\gamma-1} dz_1 dz_2 ... dz_n dz_{n+1}$. For the function $\tilde{f}(z, z_1) = f(\sqrt{z^2 + z_1^2})$, the analogue of the equality (1) is valid

$$\sup_{\mu(\tilde{\mathcal{A}})=t} \int_{(x,0)+\tilde{\mathcal{A}}} |\tilde{f}(z, z_{n+1})| d\mu(z, z_{n+1}) = \int_0^t (\tilde{f})_\mu^*(s) ds, \tag{6}$$

where $(\tilde{f})_\mu^*(s) = \inf\{t > 0 \ : \ \mu(\{(z, z_{n+1}) \ : \ |\tilde{f}(z, z_{n+1})| > t\}) \le s\}$.

Note that $\mu((x, 0) + \widetilde{\mathcal{A}})) = |\mathcal{A}|_\gamma$ and $(\tilde{f})_\mu^*(s) = f_\gamma^*(s)$.

Indeed, taking into account (3), we have

$$\mu(\{(z, z_{n+1}) \in R^n_+ : |\tilde{f}(z, z_{n+1})| > t\}) = \int_{\{x \in R_+ : |f(x)| > t\}} x_n^\gamma dx = f_{*,\gamma}(t).$$

Consequently,

$$(\tilde{f})_\mu^*(s) = \inf\{t > 0 \ : \ f_{*,\gamma}(t) \le s\} = f_\gamma^*(s).$$

From (5) and (6) we have

$$\sup_{|\mathcal{A}|_\gamma=t}\int_{\mathcal{A}} T^y|f(x)|y_n^\gamma dy = C_\gamma \sup_{\mu(\tilde{A})=t}\int_{(x,0)+\tilde{A}} |\tilde{f}(z,z_{n+1})|d\mu(z,z_{n+1})$$
$$= C_\gamma \int_0^t (\tilde{f})^*_\mu(s)ds = C_\gamma \int_0^t f^*_\gamma(s)ds.$$

Thus Lemma 2 is proved. □

3 Main results

We need the following lemma which is used in the proof of Theorem 1.

Lemma 3. *Let $0 \le \alpha < n+\gamma$. Then there exists a positive constant C, depending on α, n and γ, such that*

$$\sup_{t>0} t^{1-\frac{\alpha}{n+\gamma}}(M_{\alpha,\gamma}f)^*_\gamma(t) \le C \int\limits_{\mathbb{R}^n_+} |f(x)|x_n^\gamma dx \tag{7}$$

and

$$\sup_{t>0}(M_{\alpha,\gamma}f)^*_\gamma(t) \le C \sup_{t>0} t^{\frac{\alpha}{n+\gamma}} f^*_\gamma(t). \tag{8}$$

Proof. The estimate (7) follows from the Corollary 4 in [7].
For the estimate (8), by using (4), for every $B(0,r) \subset \mathbb{R}^n_+$, we get

$$|B(0,r)|_\gamma^{\frac{\alpha}{n+\gamma}-1}\int_{B(0,r)} T^y|f(x)|y_n^\gamma dy$$
$$\le C_\gamma |B(0,r)|_\gamma^{\frac{\alpha}{n+\gamma}-1}\int_0^{|B(0,r)|_\gamma} t^{\frac{\alpha}{n+\gamma}} f^*_\gamma(t) t^{-\frac{\alpha}{n+\gamma}} dt$$
$$\le C \sup_{t>0} t^{\frac{\alpha}{n+\gamma}} f^*_\gamma(t),$$

where $C = C_\gamma\left(\frac{n+\gamma}{n+\gamma-\alpha}\right)$. □

Theorem 1. *Let $0 \le \alpha < n+\gamma$. Then there is a positive constant C such that*

$$(M_{\alpha,\gamma}f)^*_\gamma(t) \le C \sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}} f^{**}_\gamma(\tau), t>0 \tag{9}$$

holds for all $f \in L_1^{loc}(\mathbb{R}^n_+)$, where C depends on α, n and γ. The inequality (9) is sharp for all $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$ and there exists a function f on $\mathbb{R}^n_+$ such that $f^*_\alpha = \varphi$ a.e. on $(0,\infty)$ and

$$(M_{\alpha,\gamma}f)^*_\gamma(t) \ge C_\gamma \sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}} f^{**}_\gamma(\tau), \quad t>0. \tag{10}$$

Proof. To prove (9), we may assume that

$$\sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}} f^{**}_\gamma(\tau) < \infty \tag{11}$$

otherwise there is nothing to prove. Then

$$\int_E |f(x)| x_n^\gamma dx \le \int\limits_0^t f^*_\gamma(s) ds$$

holds for for all $E \subset \mathbb{R}^n_+$ with $|E|_\gamma \le t$. In particular, if we put $E = \{x \in \mathbb{R}^n_+ : |f(x)| > f^*_\gamma(t)\}$ then $|E|_\gamma \le t$ and so $f \in L_{1,\gamma}(E)$.

Then the function

$$g_t(x) = \max\{|f(x)| - f^*_\gamma(t), 0\} sgn f(x),\ x \in \mathbb{R}^n_+$$

belongs to $L_{1,\gamma}(\mathbb{R}^n_+)$. Also the function

$$h_t(x) = \min\{|f(x)|, f^*_\gamma(t)\} sgn f(x),\ x \in \mathbb{R}^n_+$$

satisfies

$$h^*_{t_\gamma}(\tau) = \min\{f^*_\gamma(\tau), f^*_\gamma(t)\},\ \tau \in (0,\infty).$$

Hence,

$$\sup_{\tau>0} \tau^{\frac{\alpha}{n+\gamma}} (h_t)^*_\gamma(\tau) = \max\{\sup_{0<\tau<t} \tau^{\frac{\alpha}{n+\gamma}} f^*_\gamma(t), \sup_{t\le\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}} f^*_\gamma(\tau)\}$$

$$= \sup_{t\le\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}} f^*_\gamma(\tau) \le \sup_{t\le\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}} f^{**}_\gamma(t) \tag{12}$$

which, together with (11), implies that $h_t \in WL_{\frac{n+\gamma}{\alpha},\gamma}(\mathbb{R}^n_+)$.

Furthermore, since $f = g_t + h_t$ and

$$g^*_t(\tau) = \chi_{[0,t)}(\tau)(f^*_\gamma(\tau) - f^*_\gamma(t)),\ \tau \in (0,\infty). \tag{13}$$

by using $(2), (7), (8), (12)$ and (13), we get

$$(M_{\alpha,\gamma} f)^*_\gamma(t) \le (M_{\alpha,\gamma} g_t)^*_\gamma \left(\frac{t}{2}\right) + (M_{\alpha,\gamma} h_t)^*_\gamma \left(\frac{t}{2}\right)$$

$$\le C\left(\left(\frac{t}{2}\right)^{\frac{\alpha}{n+\gamma}-1} \int_{R^n_+} g_t(y) y_n^\gamma dy + \sup_{\tau>0} \tau^{\frac{\alpha}{n+\gamma}} (h_t)^*_\alpha(\tau)\right)$$

$$\leq C\left(t^{\frac{\alpha}{n+\gamma}-1}\int_0^t\left(\left(f_\gamma^*(\tau)-f_\gamma^*(t)\right)d\tau+\sup_{t<\tau<\infty}\tau^{\frac{\alpha}{n+\gamma}}f_\gamma^{**}(\tau)\right)\right)$$
$$\leq C\sup_{t<\tau<\infty}\tau^{\frac{\alpha}{n+\gamma}}f_\gamma^{**}(\tau)$$

and (9) follows.

Furthermore, (10) holds for all $t\in(0,\infty)$. Let $\varphi\in\mathcal{M}^+(0,\infty;\downarrow)$. Putting $f(x)=\varphi(\omega_n|x|^n)$, ω_n is the volume of the unit ball in $\mathbb{R}_+^n$, $\omega_n=|B(0,1)|_\gamma$; and $x\in\mathbb{R}_+^n$, we have $f_\gamma^*=\varphi(0,\infty)$. Moreover, given $y\in\mathbb{R}_+^n$, denote by $B(|y|)$ the ball with its center at the origin and having radius $|y|$. Then, for $x,y\in\mathbb{R}_+^n$ such that $|y|>|x|$,

$$\begin{aligned}(M_{\alpha,\gamma}f)(x)&=\sup_{r>0}|B(0,r)|_\gamma^{\frac{\alpha}{n+\gamma}-1}\int_{B(0,r)}T^y|f(x)|y_n^\gamma dy\\&\geq|B(0,|y|)|_\gamma^{\frac{\alpha}{n+\gamma}-1}\int_{B(0,|y|)}T^y|f(x)|y_n^\gamma dy\\&=C_\gamma|(B(0,|y|)|_\gamma^{\frac{\alpha}{n+\gamma}-1}\int_0^{\omega_n|y|^n}f_\gamma^*(\tau)d\tau\\&=C_\gamma H\left(\omega_n|y|^n\right),\end{aligned}$$

where $H\left(t\right)=t^{\frac{\alpha}{n+\gamma}-1}\int_0^t\varphi(\tau)d\tau, t\in(0,\infty)$.
Consequently,

$$(M_{\alpha,\gamma}f)(x)\geq C_\gamma\sup_{\tau>\omega_n|x|^n}H(\tau)$$

whence (10) follows on taking rearrangements.

From these estimates we find that $M_{\alpha,\gamma}f$ is bounded from $L_{1,\gamma}(\mathbb{R}_+^n)$ to $WL_{\frac{n+\gamma}{n+\gamma-\alpha},\gamma}(\mathbb{R}_+^n)$ and from $WL_{\frac{n+\gamma}{n+\gamma-\alpha},\gamma}(\mathbb{R}_+^n)$ to $L_{\infty,\gamma}(\mathbb{R}_+^n)$. □

In Theorem 1 if we take the limit as $\alpha\to0$, then we get

$$\lim_{\alpha\to0}(M_{\alpha,\gamma}f)_\gamma^*(t)=(M_\gamma f)_\gamma^*(t)$$

and

$$\lim_{\alpha\to0}\sup_{t<\tau<\infty}\tau^{\frac{\alpha}{n+\gamma}}f_\gamma^{**}(\tau)=f_\gamma^{**}(t),$$

and therefore we have the following:

Corollary 1. *There exists a positive constant C, depending on α, n and γ, such that*

$$(M_\gamma f)_\gamma^*(t)\leq Cf_\gamma^{**}(t),\ t>0,\tag{14}$$

for every $f \in L_1^{loc}(0,\infty)$. *Inequality* (14) *is sharp in the sense that for every* $\varphi \in M^+(0,\infty;\downarrow)$ *there exists a function* f *on* $(0,\infty)$ *such that* $f_\alpha^* = \varphi$ *a.e. on* $(0,\infty)$ *and*

$$(M_\gamma f)_\gamma^*(t) \geq C_\gamma f_\gamma^{**}(t), t > 0. \tag{15}$$

Theorem 2. *For* $0 \leq \alpha < n+\gamma$ *the following estimates are equivalent:*

(i) For $1 < p \leq q < \infty, 1 \leq r \leq s \leq \infty$, *fractional B-maximal operator* $M_{\alpha,\gamma} : L_{p,r,\gamma}(\mathbb{R}_+^n) \to L_{q,s,\gamma}(\mathbb{R}_+^n)$ *is bounded.*

(ii) For all $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$,

$$\left[\int_0^\infty \left(\sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}-1} \int_0^\tau \varphi(\sigma)d\sigma\right)^s t^{\frac{s}{q}-1}dt\right]^{1/s} \leq C\left(\int_0^\infty \varphi^p(t)t^{\frac{r}{p}-1}dt\right)^{1/r}.$$

(iii) $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+\gamma}$.

Proof. The equivalence of (ii) and (iii) follows from [11].

It suffices to prove $(i) \Leftrightarrow (ii)$.
Assume $M_{\alpha,\gamma} : L_{p,r,\gamma}(\mathbb{R}_+^n) \to L_{q,s,\gamma}(\mathbb{R}_+^n)$ is bounded. Then

$$\|M_{\alpha,\gamma} f\|_{L_{q,s,\gamma}(\mathbb{R}_+^n)} \leq C\|f\|_{L_{p,r,\gamma}(\mathbb{R}_+^n)}$$

holds. From Theorem 1 and by the definition of quasi-norm in Lorentz spaces, for every $\varphi = f_\gamma^*(t) \in \mathcal{M}^+(0,\infty;\downarrow)$, $f_\gamma^* = \varphi$ a.e. on $(0,\infty)$,

$$\begin{aligned}
&\left[\int_0^\infty \left(C_\gamma \sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}-1} \int_0^\tau f_\gamma^*(\sigma)d\sigma\right)^s t^{\frac{s}{q}-1}dt\right]^{1/s} \\
&= \left(\int_0^\infty \left(C_\gamma \sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}} f_\gamma^{**}(\tau)\right)^s t^{\frac{s}{q}-1}dt\right)^{1/s} \\
&\leq \left(\int_0^\infty ((M_{\alpha,\gamma} f)_\gamma^*(t))^s t^{\frac{s}{q}-1}dt\right)^{1/s} \\
&\leq C\left(\int_0^\infty f_\gamma^*(t)^p t^{\frac{r}{p}-1}dt\right)^{1/r}.
\end{aligned}$$

Conversely, for every $\varphi = f_\gamma^*(t) \in \mathcal{M}^+(0,\infty;\downarrow)$, $f_\gamma^* = \varphi$ a.e. on $(0,\infty)$,

$$\begin{aligned}
&\left(\int_0^\infty \left(\left(M_{\alpha,\gamma}f\right)_\gamma^*(t)\right)^s t^{\frac{s}{q}-1}dt\right)^{1/s} \\
&\le \left(\int_0^\infty \left(C \sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}} f_\gamma^{**}(\tau)\right)^s t^{\frac{s}{q}-1}dt\right)^{1/s} \\
&= \left(\int_0^\infty \left(C \sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n+\gamma}-1} \int_0^\tau f_\gamma^*(\sigma)d\sigma\right)^s t^{\frac{s}{q}-1}dt\right)^{1/s} \\
&\le C \left(\int_0^\infty f_\gamma^*(t)^p t^{\frac{r}{p}-1}dt\right)^{1/r}
\end{aligned}$$

and

$$\|M_{\alpha,\gamma}f\|_{L_{q,s,\gamma}(\mathbb{R}_+^n)} \le C\|f\|_{L_{p,r,\gamma}(\mathbb{R}_+^n)}$$

holds. □

From Theorem 2 and Corollary 1, we can now give the boundedness conditions of maximal operator

$$(M_\gamma f)(x) = \sup_{r>0} |B(0,r)|_\gamma^{-1} \int_{B(0,r)} T^y\,|f(x)|\, y_n^\gamma dy,\ x \in \mathbb{R}_+^n,$$

in $L_{p,q,\gamma}(\mathbb{R}_+^n)$ spaces.

Theorem 3. *For $f \in L_{1,\gamma}(\mathbb{R}_+^n)$ and C being a positive constant independent of f,*

$$\|M_\gamma f\|_{1,\infty,\gamma} \le C\|f\|_{1,\gamma}$$

holds.

Proof. For $f \in L_{1,\gamma}(\mathbb{R}_+^n)$ we have

$$\begin{aligned}
\|M_\gamma f\|_{1,\infty,\gamma} &= \sup_{t>0} t(M_\gamma f)_\gamma^*(t) \\
&\le C \sup_{t>0} t f_\gamma^{**}(t) \\
&= C \sup_{t>0} \int_0^t f_\gamma^*(s)ds \\
&\le C\|f\|_{L_{1,\gamma}}.
\end{aligned}$$

□

Theorem 4. *If $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = \infty$, then there is a positive constant C independent of f and for all $f \in L_{p,q,\gamma}$ such that*

$$\|M_\gamma f\|_{p,q,\gamma} \leq C\|f\|_{p,q,\gamma}.$$

Proof. In the case $1 < p < \infty$ and $1 \leq q < \infty$, we have

$$\begin{aligned}
\|M_\gamma f\|_{p,q,\gamma} &= \left(\int_0^\infty \left(t^{\frac{1}{p}} (M_\gamma f)_\gamma^*(t)\right)^q \frac{dt}{t}\right)^{1/q} \\
&\leq C_1 \left(\int_0^\infty \left(t^{\frac{1}{p}} f_\gamma^{**}(t)\right)^q \frac{dt}{t}\right)^{1/q} \\
&= C_1 \|f\|_{p,q,\gamma}^* \\
&\leq C_1 \frac{p}{p-1} \|f\|_{p,q,\gamma}, \\
&= C\|f\|_{p,q,\gamma}.
\end{aligned}$$

In the case $1 < p < \infty$ and $q = \infty$,

$$\begin{aligned}
\|M_\gamma f\|_{p,q,\gamma} &= \sup_{t>0} t^{\frac{1}{p}} (M_\gamma f)_\gamma^*(t) \\
&\leq C_2 \sup_{t>0} t^{\frac{1}{p}} f_\gamma^{**}(t) = C_2 \|f\|_{p,q,\gamma}^* \\
&\leq C_2 \frac{p}{p-1} \|f\|_{p,q,\gamma} = C\|f\|_{p,q,\gamma}.
\end{aligned}$$

If $p = q = \infty$, then the following inequalities hold

$$\begin{aligned}
\sup_{t>0} (M_\gamma f)_\gamma^*(t) &\leq \sup_{t>0} C f_\gamma^{**}(t) \\
&= C\|f\|_{p,q,\gamma}^* \leq C\|f\|_{p,q,\gamma}.
\end{aligned}$$

This completes the proof. □

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