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ON THE QUALITATIVE BEHAVIOR OF THE SOLUTIONS FOR A KIND OF NONLINEAR THIRD ORDER DIFFERENTIAL EQUATIONS WITH A RETARTED ARGUMENT

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Abstract

In this paper, by defining a Lyapunov functional, we discuss the stability and the boundedness of the solutions for nonlinear third order delay differential equations of the type:

$$\begin{aligned}
&x'''(t) + h(t, x(t), x'(t), x''(t), x(t-r(t)), x'(t-r(t)), x''(t-r(t)))x''(t)\\
&+g(x(t-r(t)), x'(t-r(t))) + d(t)\psi(x'(t))x'(t) + f(x(t-r(t)))\\
&= p(t, x(t), x'(t), x''(t), x(t-r(t)), x'(t-r(t)), x''(t-r(t)))
\end{aligned}$$

Our results include and improve some well-known results in the literature. An example is also given to illustrate the importance of the topic and the results obtained.

1 Introduction

It is well known that the systems with aftereffect, with time lag or with delay are of great theoretical interest and form an important class as regards their applications. This class of systems is described by functional differential equations, which are also called differential equations with deviating arguments. Among functional differential equations one may distinguish some special classes of equations, retarded functional differential equations, advanced

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functional differential equations and neutral functional differential equations. In particular, retarded functional differential equations describe those systems or processes whose rate of change of state is determined by their past and present states. These equations are frequently encountered as mathematical models of most dynamical process in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory, etc. Especially, since 1960's many good books, most of them are in Russian literature, have been published on the delay differential equations (see for example the books of Burton ([7], [8]), Èl'sgol'ts [10], Èl'sgol'ts and Norkin [11], Gopalsamy [12], Hale [13], Hale and Verduyn Lunel [14], Kolmanovskii and Myshkis [15], Kolmanovskii and Nosov [16], Krasovskii [17], Mohammed [20], Yoshizawa [53] and the references listed in these books).

However, with respect our observation from the literature; it is founded only a few papers on the stability and boundedness of solutions of nonlinear differential equations of third order with delay (see, for example, the papers of Afuwape and Omeike [3], Bereketoğlu and Karakoç [6], Omeike [25], Sadek ([29], [30]), Sinha [31], Tejumola and Tchegnani [32], Tunç([38-41], [43-45], [47-50]), Yao and Meng [52], Zhu [54]) and the references thereof).

It is worth mentioning that the use of the Lyapunov direct method [18] for equations with delays encountered some principal difficulties. In 1963, Krasovskii [17] suggested the use of functional defined on retarded equations' trajectories instead of Lyapunov function and proved general stability theorems based on the use of functionals. In this case, a positive functional with negative definite (or negative semi-definite) derivative is constructed. In fact, this functional is a tool to prove the stability and boundedness of the solutions of delay differential equation under consideration. It should be noted that finding appropriate Lyapunov functionals for higher order nonlinear delay differential equations is a more difficult task. That is to say that the construction of Lyapunov functionals remains as a problem in the literature. However, throughout all the paper listed above Lyapunov functionals are used to verify the results established there. At the same time, one can recognize that so far many significant theoretical results dealt with the stability and boundedness of solutions of nonlinear differential equations of third order without delay:

$$x'''(t) + b_1 x''(t) + b_2 x'(t) + b_3 x(t) = p(t, x(t), x'(t), x''(t)),$$

in which b_1, b_2 and b_3 are not necessarily constants. In particular, one can refer to the book of Reissig et al. [28] as a survey and the papers of Ademola et al. [2], Afuwape [4], Afuwape et al. [5], Mehri and Shadman [19], Ogundare [21], Ogundare and Okecha [22], Omeike ([23], [24]), Palusinski et al. [26], Ponzo [27], Tunç([33-37], [42], [46]), Tunç and Ateş [51] and the references cited in these sources for some publications performed on the topic. Meanwhile, in a recent paper, Afuwape and Omeike [3] discussed the same problems, the problems of the stability and boundedness of solutions, for nonlinear third order delay differential equation:

$$x'''(t)+h(x'(t))x''(t)+g(x(t-r(t)),x'(t-r(t)))+f(x(t-r(t))) = p(t,x(t),x'(t),x''(t)),$$

in the cases $p(t,x(t),x'(t),x''(t)) \equiv 0$ and $p(t,x(t),x'(t),x''(t)) \neq 0$, respectively.

In this paper, we consider nonlinear delay differential equation of third order of the type:

$$\begin{aligned}
&x'''(t) + h(t,x(t),x'(t),x''(t),x(t-r(t)),x'(t-r(t)),x''(t-r(t)))x''(t)\\
&+g(x(t-r(t)),x'(t-r(t))) + d(t)\psi(x'(t))x'(t) + f(x(t-r(t)))\\
&= p(t,x(t),x'(t),x''(t),x(t-r(t)),x'(t-r(t)),x''(t-r(t)))
\end{aligned} \tag{1}$$

or its associated system

$$\begin{aligned}
&x'(t) = y(t),\\
&y'(t) = z(t),\\
&z'(t) = -h(t,x(t),y(t),z(t),x(t-r(t)),y(t-r(t)),z(t-r(t)))z(t)\\
&-d(t)\psi(y(t))y(t) - g(x(t),y(t)) - f(x(t)) + \int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds\\
&+ \int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds + \int_{t-r(t)}^{t} f'(x(s))y(s)ds\\
&+p(t,x(t),y(t),z(t),x(t-r(t)),y(t-r(t)),z(t-r(t))),
\end{aligned} \tag{2}$$

where $r(t)$ is a variable and bounded delay, $0 \leq r(t) \leq \gamma$, γ is a positive constant which will be determined later, andthe derivative $r'(t)$ exists and $r'(t) \leq \beta$, $0 < \beta < 1$; the functions h, g, d, ψ, f and p depend only on the arguments displayed explicitly and the primes in Eq. (1) denote differentiation with respect to t, $t \in [0,\infty)$. It is principally assumed that the functions h, g, d, ψ, f and p are continuous for all values their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^6$, $\mathbb{R}^2$, $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}^6$, respectively. This fact guarantees the existence of the solution of Eq. (1) (see Èl'sgol'ts [10, pp.14]). Besides, it is also supposed that $g(x,0) = f(0) = 0$, and the derivatives $d'(t)$, $g_x(x,y) \equiv \frac{\partial}{\partial x}g(x,y)$, $g_y(x,y) \equiv \frac{\partial}{\partial y}g(x,y)$ and $f'(x) \equiv \frac{df}{dx}$ exist and are continuous; throughout the paper $x(t)$, $y(t)$ and $z(t)$ are abbreviated as x, y and z, respectively. In addition, it is also assumed that all solutions of Eq. (1) are real valued and the functions $h(t,x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t)))$, $g(x,y)$, $\psi(y)$, $h(x)$ and $p(t,x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t)))$ satisfy a Lipschitz condition in x, y, z, $x(t-r(t))$, $y(t-r(t))$ and $z(t-r(t))$. Then the solution is unique (see Èl'sgol'ts [10, pp.15]).

The motivation for the present work has been inspired basically by the paper of Afuwape and Omeike [3] and the papers mentioned above. Our aim here is to extend and improve the results established by Afuwape and Omeike [3] to nonlinear delay differential Eq. (1) for the stability of the zero solution and boundedness of all solutions of this equation, when $p \equiv 0$ and $p \neq 0$ in (1), respectively. We also give an explanatory example on the stability and boundedness of solutions of a specific delay differential equation of third order.

2 Preliminaries

In order to reach the main results of this paper, we will give some important basic information for general non-autonomous delay differential system. Consider the general non-autonomous delay differential system:

$$\dot{x} = F(t, x_t), x_t = x(t + \theta), -r \leq \theta \leq 0, t \geq 0, \tag{3}$$

where $F : [0, \infty) \times C_H \to \mathbb{R}^n$ is a continuous mapping, $F(t, 0) = 0$, and we suppose that F takes closed bounded sets into bounded sets of $\mathbb{R}^n$. Here $(C, \ \|.\|)$ is the Banach space of continuous function $\phi : [-r, \ 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$; C_H is the open H -ball in C; $C_H := \{\phi \in (C[-r, 0], \ \Re^n) : \|\phi\| < H\}$.

Definition 1 *(Yoshizawa [53]) A function $x(t_0, \phi)$ is said to be a solution of the system (3) with the initial condition $\phi \in C_H$ at $t = t_0$, $t_0 \geq 0$, if there is a constant $A > 0$ such that $x(t_0, \phi)$ is a function from $[t_0 - h, \ t_0 + A]$ into $\mathbb{R}^n$ with the properties:*
(i) *$x_t(t_0, \phi) \in C_H$ for $t_0 \leq t < t_0 + A$,*
(ii) *$x_{t_0}(t_0, \phi) = \phi$,*
(iii) *$x(t_0, \phi)$ satisfies (3) for $t_0 \leq t < t_0 + A$.*

Standard existence theory, see Burton [7], shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0, \ t_0 + \alpha)$ satisfying (3) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.|$ will denote a convenient norm in $\mathbb{R}^n$ with $|x| = \max_{1 \leq i \leq n} |x_i|$. Now, let us assume that $C(t)$ $= \{\phi : [t - \alpha] \to \Re^n \mid \phi \text{ is continuous}\}$ and ϕ_t denotes the ϕ in the particular $C(t)$, and that $\|\phi_t\| = \max_{t - \alpha \leq s \leq t} |\phi(t)|$. Clearly, Eq. (1) is also a particular case of (3).

Definition 2 *(Burton [7]) Let $F(t, 0) = 0$. The zero solution of (3) is:*
(i) *stable if for each $\varepsilon > 0$ and $t_1 \geq t_0$ there exists $\delta > 0$ such that $[\phi \in C(t_1), \ \ \|\phi\| < \delta, \ \ t \geq t_1]$ implies that $|x(t, t_1, \phi)| < \varepsilon$.*

(ii) *asymptotically stable if it is stable and if for each* $t_1 \geq t_0$ *there is an* $\eta > 0$ *such that* $[\phi \in C(t_1), \ \ \|\phi\| < \delta]$ *implies that* $x(t, t_0, \phi) \to 0$ *as* $t \to \infty$. *(If this is true for every* $\eta > 0$, *then* $x = 0$ *is asymptotically stable in the large or globally asymptotically stable.)*

Definition 3 *(Burton [7]) A continuous function* $W : [0, \infty) \to [0, \infty)$ *with* $W(0) = 0$, $W(s) > 0$ *if* $s > 0$, *and* W *strictly increasing is a wedge. We denote wedges by* W *or* W_i, *where* i *is an integer.*

Definition 4 *(Burton [7]) Let* D *be an open set in* $\mathbb{R}^n$ *with* $0 \in D$. *A function* $V : [0, \infty) \times D \to [0, \infty)$ *is called positive definite if* $V(t, 0) = 0$ *and if there is a wedge* W_1 *with* $V(t, x) \geq W_1(|x|)$, *and is called decrescent if there is a wedge* W_2 *with* $V(t, x) \leq W_2(|x|)$.

Definition 5 *(Burton [7]) Let* $V(t, \phi)$ *be a continuous functional defined for* $t \geq 0$, $\phi \in C_H$. *The derivative of* V *along solutions of (3) will be denoted by* $\dot{V}$ *and is defined by the following relation*

$$\dot{V}(t, \phi) = \limsup_{h \to 0} \frac{V(t + h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))}{h},$$

where $x(t_0, \phi)$ *is the solution of (3) with* $x_{t_0}(t_0, \phi) = \phi$.

Theorem 1 *(Burton and Hering [9]) Suppose that there exists a Lyapunov functional* $V(t, \phi)$ *for (3) such that the following conditions are satisfied:*
(i) $W_1(|\phi(0)|) \leq V(t, \phi)$, *where* $W_1(r)$ *is a wedge,* $V(t, 0) = 0$,
(ii) $\dot{V}(t, x_t) \leq 0$.
Then, the zero solution of (3) is stable.

3 Main results

In this section, we state and prove two theorems, which are our main results.

First, for the case $p(t, x, y, z, x(t - r(t)), y(t - r(t)), z(t - r(t))) \equiv 0$, the following result is introduced:

Theorem 2 *In addition to the basic assumptions imposed on the functions* h, g, d, ψ *and* f *appearing in Eq. (1), we assume there exist positive constants* a , b, b_0, c, γ, ε, ρ, K, L *and* M *such that that the following conditions hold:*
(i) $ab - c > 0$, $d(t) \geq 1$, $d'(t) \leq 0$ *for all* $t \in \mathbb{R}^+$.
(ii) $f(x) sgn x > 0$ *for all* $x \neq 0$, *sup* $\{f'(x)\} = c$, $|f'(x)| \leq L$ *for all* x.
(iii) $\psi(y) \geq b_0$, $\frac{g(x,y)}{y} \geq b + \varepsilon$, $(y \neq 0)$, $|g_x(x, y)| \leq K$, $|g_y(x, y)| \leq M$ *for all* x *and* y.
(iv) $h(t, x, y, z, x(t - r(t)), y(t - r(t)), z(t - r(t))) \geq a + \rho$,

$$\frac{\mu\{h(t,x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t)))-a\}^2}{4}\le\varepsilon\rho$$

for all t, x, y, z, $x(t-r(t))$, $y(t-r(t))$ *and* $z(t-r(t))$.

Then the zero solution of Eq. (1) is stable, provided that

$$\gamma<\min\left\{\frac{2(\mu b-c)}{\mu(K+L+M)+2\lambda},\ \frac{2(a-\mu)}{K+L+M+2\delta}\right\}$$

with $\mu=\frac{ab+c}{2b}$.

Proof. Define the Lyapunov functional $V_1=V_1(t,x_t,y_t,z_t):$

$$\begin{aligned}V_1=\ &\mu\int_0^x f(\xi)d\xi+yf(x)+\tfrac{1}{2}\mu ay^2+\int_0^y g(x,\eta)d\eta+\mu yz+d(t)\int_0^y\psi(\eta)\eta d\eta\\&+\tfrac{1}{2}z^2+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds\end{aligned}$$

so that

$$\begin{aligned}V_1\ &\ge\mu\int_0^x f(\xi)d\xi+f(x)y+\tfrac{\mu a}{2}y^2+\tfrac{b}{2}y^2+\tfrac{\varepsilon}{2}y^2+\tfrac{b_0}{2}y^2+\mu yz+\tfrac{1}{2}z^2\\&+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds\\&\ge\tfrac{1}{2b}[by+f(x)]^2+\mu\int_0^x f(\xi)d\xi+\tfrac{\mu a}{2}y^2+\tfrac{\varepsilon}{2}y^2+\tfrac{b_0}{2}y^2-\tfrac{1}{2b}f^2(x)\\&+\mu yz+\tfrac{1}{2}z^2+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds\\&=\tfrac{1}{2by^2}\left[4\int_0^x f(\xi)\left\{\int_0^y(\mu b-f'(\xi))\eta d\eta\right\}d\xi\right]+\tfrac{b_0}{2}y^2\\&+\tfrac{\varepsilon}{2}y^2+\tfrac{1}{2}(\mu y+z)^2+\tfrac{1}{2}\mu(a-\mu)y^2+\tfrac{1}{2b}\left[by+f(x)\right]^2\\&+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds\end{aligned}\tag{4}$$

by the assumptions $g(x,0)=f(0)=0$, $d(t)\ge1$, $\psi(y)\ge b_0$, $\frac{g(x,y)}{y}\ge b+\varepsilon$, $(y\ne0)$, $f(x)sgnx>0$, $(x\ne0)$, and $|f'(x)|\le L$, where λ and δ are positive constants which will be determined later in the proof. In view of the facts $a-\mu=\frac{ab-c}{2b}>0$ and $\mu b-f'(x)\ge\frac{ab-c}{2}>0$, from (4), it is clear that there exist sufficiently small positive constants D_i , $(i=1,\ 2,\ 3,)$, such that

$$\begin{aligned}V_1(t,x_t,y_t,z_t)\ &\ge D_1x^2+D_2y^2+D_3z^2\\&+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds\\&\ge D_4(x^2+y^2+z^2),\end{aligned}\tag{5}$$

where $D_4 = \min\{D_1,\ D_2,\ D_3\}$. Now, it can be easily verified the existence of a continuous function $W_1(|\phi(0)|)$ with $W_1(|\phi(0)|) \geq 0$ such that $W_1(|\phi(0)|) \leq V(t,\phi)$.

By a straightforward calculation, we obtain the time derivative of functional $V_1 = V_1(x_t, y_t, z_t)$ along the solutions of the system (2) as the following:

$$\begin{aligned}\frac{dV_1}{dt} &= f'(x)y^2 - \mu d(t)\psi(y)y^2 + \mu z^2 - \mu y g(x,y) + y\int_0^y g_x(x,\eta)d\eta \\ &-\mu\{h(t,x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t))) - a\}yz \\ &-h(t,x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t)),z)z^2 + d'(t)\int_0^y \psi(\eta)\eta d\eta \\ &+(\mu y + z)\int_{t-r(t)}^{t} f'(x(s))y(s)ds + (\mu y + z)\int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds \\ &+(\mu y + z)\int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds + \lambda y^2 r(t) + \delta z^2 r(t) \\ &-\lambda(1-r'(t))\int_{t-r(t)}^{t} y^2(s)ds - \delta(1-r'(t))\int_{t-r(t)}^{t} z^2(s)ds.\end{aligned} \tag{6}$$

Now, by help of the assumptions of Theorem 2 and the inequality $2|uv| \leq u^2 + v^2$, it results immediately the existence of the following:

$$-h(t,x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t)))z^2 \leq -(a+\rho)z^2,$$

$$-\mu\psi(y)y^2 \leq -(\mu b_0)y^2,$$

$$-\left(\mu\frac{g(x,y)}{y} - f'(x)\right)y^2 \leq -(\mu b + \mu\varepsilon - c)y^2,$$

$$d'(t)\int_0^y \psi(\eta)\eta d\eta \leq 0,$$

$$\begin{aligned}\mu y\int_{t-r}^{t} f'(x(s))y(s)ds &\leq \frac{\mu L r(t)}{2}y^2 + \frac{\mu L}{2}\int_{t-r(t)}^{t} y^2(s)ds \\ &\leq \frac{\mu L\gamma}{2}y^2 + \frac{\mu L}{2}\int_{t-r(t)}^{t} y^2(s)ds,\end{aligned}$$

$$\begin{aligned}z\int_{t-r(t)}^{t} f'(x(s))y(s)ds &\leq \frac{L r(t)}{2}z^2 + \frac{L}{2}\int_{t-r(t)}^{t} y^2(s)ds \\ &\leq \frac{L\gamma}{2}z^2 + \frac{L}{2}\int_{t-r(t)}^{t} y^2(s)ds,\end{aligned}$$

$$\begin{aligned}
\mu y \int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds &\le \tfrac{\mu K r(t)}{2} y^2 + \tfrac{\mu K}{2} \int_{t-r(t)}^{t} y^2(s)ds \\
&\le \tfrac{\mu K\gamma}{2} y^2 + \tfrac{\mu K}{2} \int_{t-r(t)}^{t} y^2(s)ds, \\
z \int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds &\le \tfrac{K r(t)}{2} z^2 + \tfrac{K}{2} \int_{t-r(t)}^{t} y^2(s)ds \\
&\le \tfrac{K\gamma}{2} z^2 + \tfrac{K}{2} \int_{t-r(t)}^{t} y^2(s)ds, \\
\mu y \int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds &\le \tfrac{\mu M r(t)}{2} y^2 + \tfrac{\mu M}{2} \int_{t-r(t)}^{t} z^2(s)ds \\
&\le \tfrac{\mu M\gamma}{2} y^2 + \tfrac{\mu M}{2} \int_{t-r(t)}^{t} z^2(s)ds, \\
z \int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds &\le \tfrac{M r(t)}{2} z^2 + \tfrac{M}{2} \int_{t-r(t)}^{t} z^2(s)ds \\
&\le \tfrac{M\gamma}{2} z^2 + \tfrac{M}{2} \int_{t-r(t)}^{t} z^2(s)ds,
\end{aligned}$$

$$\lambda y^2 r(t) \le \lambda\gamma y^2,$$
$$\delta z^2 r(t) \le \delta\gamma z^2.$$

Combining aforementioned inequalities into (6), we have

$$\begin{aligned}
\tfrac{dV_1}{dt} \quad &\le -\left(\mu b - c - \tfrac{\mu K}{2}\gamma - \tfrac{\mu L}{2}\gamma - \tfrac{\mu M}{2}\gamma - \lambda\gamma\right) y^2 \\
&- \left(a - \mu - \tfrac{K}{2}\gamma - \tfrac{L}{2}\gamma - \tfrac{M}{2}\gamma - \delta\gamma\right) z^2 - (\mu b_0) y^2 \\
&-(\mu\varepsilon) y^2 - \mu\{h(t,x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t))) - a\} yz \\
&-\rho z^2 + \left(\tfrac{K}{2} + \tfrac{L}{2} + \tfrac{\mu K}{2} + \tfrac{\mu L}{2} - (1-\beta)\lambda\right) \int_{t-r(t)}^{t} y^2(s)ds \\
&+ \left(\tfrac{M}{2} + \tfrac{\mu M}{2} - (1-\beta)\delta\right) \int_{t-r(t)}^{t} z^2(s)ds.
\end{aligned} \tag{7}$$

We now consider the terms

$$W =: (\mu\varepsilon) y^2 + \mu\{h(t,x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t))) - a\} yz + \rho z^2,$$

which are contained in (7). Clearly, W represents a quadratic form. These terms can be rearranged as the following:

$$[y \;\; z] \begin{bmatrix} \mu\varepsilon & \frac{\mu(h-a)}{2} \\ \frac{\mu(h-a)}{2} & \rho \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}.$$

By noting the basic information on the positive semi-definiteness of the above quadratic form, we can conclude that $W \geq 0$, provided that

$$\frac{\mu(h-a)^2}{4} \leq \varepsilon\rho.$$

Hence, by virtue of (7) it follows that

$$\begin{aligned} \frac{dV_1}{dt} &\leq -\left(\mu b - c - \tfrac{\mu K}{2}\gamma - \tfrac{\mu L}{2}\gamma - \tfrac{\mu M}{2}\gamma - \lambda\gamma\right) y^2 \\ &- \left(a - \mu - \tfrac{K}{2}\gamma - \tfrac{L}{2}\gamma - \tfrac{M}{2}\gamma - \delta\gamma\right) z^2 \\ &+ \left(\tfrac{K}{2} + \tfrac{L}{2} + \tfrac{\mu K}{2} + \tfrac{\mu L}{2} - (1-\beta)\lambda\right) \int_{t-r(t)}^{t} y^2(s)ds \\ &+ \left(\tfrac{M}{2} + \tfrac{\mu M}{2} - (1-\beta)\delta\right) \int_{t-r(t)}^{t} z^2(s)ds. \end{aligned} \tag{8}$$

Let $\lambda = \frac{1}{2(1-\beta)}(K+L)(1+\mu)$ and $\delta = \frac{1}{2(1-\beta)}M(1+\mu)$. Now, because of these choices, we get from (8) that

$$\begin{aligned} \frac{d}{dt}V_1(t, x_t, y_t, z_t) &\leq -\left(\mu b - c - \tfrac{\mu K}{2}\gamma - \tfrac{\mu L}{2}\gamma - \tfrac{\mu M}{2}\gamma - \lambda\gamma\right) y^2 \\ &- \left(a - \mu - \tfrac{K}{2}\gamma - \tfrac{L}{2}\gamma - \tfrac{M}{2}\gamma - \delta\gamma\right) z^2. \end{aligned} \tag{9}$$

Then, from the inequality (9) for some positive constants k_1 and k_2, it follows that

$$\frac{d}{dt}V_1(t, x_t, y_t, z_t) \leq -k_1 y^2 - k_2 z^2 \leq 0 \tag{10}$$

provided that

$$\gamma < \min\left\{\frac{2(\mu b - c)}{\mu(K+L+M) + 2\lambda}, \ \frac{2(a-\mu)}{K+L+M+2\delta}\right\}.$$

The proof Theorem 2 is now complete.

In the case $p(t, x, y, z, x(t-r(t)), y(t-r(t)), z(t-r(t))) \neq 0$, we establish the following result

Theorem 3 *Let us assume that the assumptions (i)-(iv) of Theorem 2 hold. In addition, we suppose that*

$$|p(t, x, y, z, x(t-r(t)), y(t-r(t)), z(t-r(t)))| \leq q(t)$$

for all t, x, y, z, $x(t-r(t))$, $y(t-r(t))$ and $z(t-r(t))$, where $q \in L^1(0,\infty)$, $L^1(0,\infty)$ is space of Lebesgue integrable functions. Then, there exists a finite positive constant K_1 such that the solution $x(t)$ of Eq. (1) defined by the initial functions

$$x(t) = \phi(t), x'(t) = \phi'(t), x''(t) = \phi''(t)$$

satisfies the inequalities

$$|x(t)| \le \sqrt{K_1}, |x'(t)| \le \sqrt{K_1}, |x''(t)| \le \sqrt{K_1}$$

for all $t \ge t_0$, *where* $\phi \in C^2([t_0 - r, t_0], \Re)$, *provided that*

$$\gamma < \min\left\{\frac{2(\mu b - c)}{\mu(K + L + M) + 2\lambda}, \ \frac{2(a - \mu)}{K + L + M + 2\delta}\right\}$$

with $\mu = \frac{ab+c}{2b}$.

Proof. Taking into account the assumptions of the Theorem 3 and the result of the Theorem 2, that is, the inequality (10), a straightforward calculation leads to

$$\begin{aligned} \tfrac{d}{dt}V_1(t, x_t, y_t, z_t) \ &\le -k_1 y^2 - k_2 z^2 \\ &+(\mu y + z)p(t, x, y, z, x(t - r(t)), y(t - r(t)), z(t - r(t))). \end{aligned}$$

Hence,

$$\begin{aligned} \tfrac{d}{dt}V_1(t, x_t, y_t, z_t) \ &\le (\mu\,|y| + |z|)\,|p(t, x, y, z, x(t - r(t)), y(t - r(t)), z(t - r(t)))| \\ &\le (\mu\,|y| + |z|)q(t) \\ &\le D_5(|y| + |z|)q(t), \end{aligned}$$

where $D_5 = \max\{1, \ \mu\}$. By virtue of the inequalities $|y| < 1 + y^2$ and $|z| < 1 + z^2$, we have

$$\frac{d}{dt}V_1(t, x_t, y_t, z_t) \le D_5(2 + y^2 + z^2)q(t).$$

Obviously, the inequality (5) implies that

$$y^2 + z^2 \le D_4^{-1}\, V_1(t, x_t, y_t, z_t).$$

Hence, it follows that

$$\begin{aligned} \tfrac{d}{dt}V_1(t, x_t, y_t, z_t) \ &\le D_5(2 + D_4^{-1}V_1(t, x_t, y_t, z_t))q(t) \\ &= 2D_5 q(t) + D_5 D_4^{-1}V_1(t, x_t, y_t, z_t)q(t). \end{aligned} \qquad (11)$$

Now, integrating (11) from 0 to t, using the assumption $q \in L^1(0, \infty)$ and Gronwall-Reid-Bellman inequality (see Ahmad and Rama Mohana Rao [1]), we obtain

$$\begin{aligned} V_1(t, x_t, y_t, z_t) \ &\le V_1(0, x_0, y_0, z_0) + 2D_5 A + D_5 D_4^{-1}\int_0^t \{V_1(s, x_s, y_s, z_s)\}q(s)ds \\ &\le \{V_1(0, x_0, y_0, z_0) + 2D_5 A\}\exp\left(D_5 D_4^{-1}\int_0^t q(s)ds\right) \\ &= \{V_1(0, x_0, y_0, z_0) + 2D_5 A\}\exp(D_5 D_4^{-1} A) = K_2 < \infty, \end{aligned} \qquad (12)$$

where $K_2 > 0$ is a constant, $K_2 = \{V_1(0, x_0, y_0, z_0) + 2D_5 A\} \exp(D_5 D_4^{-1} A)$ and $A = \int\limits_0^\infty q(s)ds$. In view of (5) and (12), it follows that

$$x^2 + y^2 + z^2 \le D_4^{-1} V_1(t, x_t, y_t, z_t) \le K_1,$$

where $K_1 = K_2 D_4^{-1}$. Hence, we deduce that

$$|x(t)| \le \sqrt{K_1}, |y(t)| \le \sqrt{K_1}, |z(t)| \le \sqrt{K_1}$$

for all $t \ge t_0$. That is,

$$|x(t)| \le \sqrt{K_1}, |x'(t)| \le \sqrt{K_1}, |x''(t)| \le \sqrt{K_1}$$

for all $t \ge t_0$. The proof of the Theorem 3 is now complete.

Example 1 *We consider the following nonlinear delay differential equation of third order:*

$$\begin{aligned} &x'''(t) + \left(4 + \tfrac{1}{1+t^2+x^2(t)+x'^2(t)+x''^2(t)+x^2(t-r(t))+x'^2(t-r(t))+x''^2(t-r(t))^2}\right) x''(t) \\ &+4x'(t-r(t)) + \sin x'(t-r(t)) + 4(1+e^{-t})x'(t) + x(t-r(t)) \\ &= \tfrac{1}{1+t^2+x^2(t)+x'^2(t)+x''^2(t)+x^2(t-r(t))+x'^2(t-r(t))+x''^2(t-r(t))}. \end{aligned} \tag{13}$$

Eq. (13) is a special case of Eq. (1), and it can be stated as the following system:

$$\begin{aligned} &x' = y, \\ &y' = z, \\ &z' = -\left(4 + \tfrac{1}{1+t^2+x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))}\right) z \\ &-(4y + \sin y) + \int\limits_{t-r(t)}^{t} (4 + \cos y(s))z(s)ds - 4(1+e^{-t})y - x + \int\limits_{t-r(t)}^{t} y(s)ds \\ &+\tfrac{1}{1+t^2+x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))}. \end{aligned}$$

We now observe the following relations:

$$\begin{aligned} &h(t, x, y, z, x(t-r(t)), y(t-r(t)), z(t-r(t))) = \\ &4 + \tfrac{1}{1+t^2+x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))}, \end{aligned}$$

$$\begin{aligned} &4 + \tfrac{1}{1+t^2+x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))} \\ &\ge 4 = a + \rho, \end{aligned}$$

$a = 2,\ \rho = 2,$

$$\left[2 + \tfrac{1}{1+t^2+x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))}\right]^2 \le 9 = \tfrac{4\varepsilon\rho}{\mu},$$

$$9\mu = 8\varepsilon,$$

$$g(y) = 4y + \sin y, g(0) = 0,$$

$$\frac{g(y)}{y} = 4 + \frac{\sin y}{y}, (y \ne 0, |y| < \pi),$$

$$4 + \frac{\sin y}{y} \ge 3 = b + \varepsilon,$$

$$g'(y) = 4 + \cos y,$$

$$|g'(y)| \le 5 = M,$$

$$d(t)\psi(y) = 4(1 + e^{-t}),$$

$$d(t) = 1 + e^{-t} \ge 1,$$

$$\psi(y) = 4 = b_0,$$

$$f(x) = x, f(0) = 0,$$

$$f'(x) = 1, c = L = 1.$$

$$p(t, x, y, z, x(t - r(t)), y(t - r(t)), z(t - r(t))) \\ = \tfrac{1}{1+t^2+x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))},$$

$$\frac{1}{1 + t^2 + x^2 + y^2 + z^2 + x^2(t - r(t)) + y^2(t - r(t)) + z^2(t - r(t))} \le \frac{1}{1 + t^2},$$

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1 + s^2}ds = \frac{\pi}{2} < \infty, \text{ that is, } q \in L^1(0, \infty).$$

It should be noted that the constants b, ε and γ can also be specified such that all the assumptions of the Theorems 2 and 3 hold.

This shows that the zero solution of Eq. (13) is stable and all solutions of the same equation are bounded, when $p(t, x, y, z, x(t - r(t)), y(t - r(t)), z(t - r(t))) = 0$ and $\ne 0$, respectively.

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