An. Şt. Univ. Ovidius Constanţa Vol. 17(2), 2009, 131–142

ON MANNHEIM PARTNER CURVE IN DUAL SPACE

Siddika Özkaldi, Kazim İlarslan and Yusuf Yayli

Abstract

In this paper, we define Mannheim partner curves in three dimensional dual space $\mathbb{D}^3$ and we obtain the necessary and sufficient conditions for the Mannheim partner curves in dual space $\mathbb{D}^3$.

1 Introduction

In the differential geometry of a regular curve in the Euclidean 3-space $\mathbb{E}^3$, it is well-known that one of the important problem is the characterization of a regular curve. The curvature functions k_1 (curvature $\varkappa$) and k_2 (torsion τ) of a regular curve play an important role to determine the shape and size of the curve ([2, 6]). For example: If $k_1 = k_2 = 0$, then the curve is a geodesic. If $k_1 \neq 0$ (constant) and $k_2 = 0$, then the curve is a circle with radius $1/k_1$. If $k_1 \neq 0$ (constant) and $k_2 \neq 0$ (constant), then the curve is a helix in the space, etc.

Another way to classification and characterization of curves is the relationship between the Frenet vectors of the curves. For example (in 1845) Saint Venant proposed the question whether upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal the principal normal of the given curve. This question was answered by Bertrand in 1850; he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients exists between the first and second curvatures of

Key Words: Mannheim partner curve, dual space, dual space curve
Mathematics Subject Classification: 53A04, 53A25, 53A40
Received: March, 2009
Accepted: September, 2009

the given original curve. The pairs of curves of this kind have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves ([2, 6, 8]) . There are many works related with Bertrand curves in the Euclidean space and Minkowski space. Another kind of associated curves are called Mannheim curve and Mannheim partner curve. If there exists a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of β, then α is called a Mannheim curve, and β Mannheim partner curve of α. Mannheim partner curves was studied by Liu and Wang ([7]) in Euclidean $3-$ space and in the Minkowski $3-$space.

Dual numbers had been introduced by W.K. Clifford (1849 - 1879) as a tool for his geometrical investigations. After him E. Study used dual numbers and dual vectors in his research on line geometry and kinematics. He devoted special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping: The set of oriented lines in an Euclidean three-dimension space $\mathbb{E}^3$ is one to one correspondence with the points of a dual space $\mathbb{D}^3$ of triples of dual numbers ([3]).

In this paper we study Mannheim partner curves in dual space $\mathbb{D}^3$.

2 Preliminary

By a dual number $\widehat{x}$, we mean an ordered pair of the form (x, x^*) for all $x, x^* \in \mathbb{R}$. Let the set $\mathbb{R} \times \mathbb{R}$ be denoted as $\mathbb{D}$. Two inner operations and an equality on $\mathbb{D} = \{\ (x, x) |\ x, x^* \in \mathbb{R}\}$ are defined as follows:

(i) $\oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ for $\widehat{x} = (x, x^*)$, $\widehat{y} = (y, y^*)$ defined as

$$\widehat{x} \oplus \widehat{y} = (x, x^*) \oplus (y, y^*) = (x + y,\ x^* + y^*)$$

is called the addition in $\mathbb{D}$.

(ii) $\odot : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ for $\widehat{x} = (x, x^*)$, $\widehat{y} = (y, y^*)$ defined as

$$\widehat{x} \odot \widehat{y} = \widehat{x}\widehat{y} = (x, x^*) \odot (y, y^*) = (xy,\ xy^* + x^*y)$$

is called the multiplication in $\mathbb{D}$.

(iii) If $x = y$, $x^* = y^*$ for $\widehat{x} = (x, x^*)$, $\widehat{y} = (y, y^*) \in \mathbb{D}$, $\widehat{x}$ and $\widehat{y}$ are equal, and it is indicated as $\widehat{x} = \widehat{y}$.

If the operations of addition, multiplication and equality on $\mathbb{D} = \mathbb{R} \times \mathbb{R}$ with set of real numbers $\mathbb{R}$ are defined as above, the set $\mathbb{D}$ is called the dual numbers system and the element (x, x^*) of $\mathbb{D}$ is called a dual number. In a dual number $\widehat{x} = (x, x^*) \in \mathbb{D}$, the real number x is called the real part of $\widehat{x}$ and the real number x is called the dual part of $\widehat{x}$. The dual number $(1, 0) = 1$ is called unit element of multiplication operation in $\mathbb{D}$ or real unit in $\mathbb{D}$. The dual number $(0,1)$ is to be denoted with ε in short, and the $(0,1)=\varepsilon$ is to be called dual unit. In accordance with the definition of the operation of multiplication, it can easily be seen that $\varepsilon^2=0$. Also, the dual number $\widehat{x}=(x,x^*)\in\mathbb{D}$ can be written as $\widehat{x}=x+\varepsilon x^*$ (see[9, 5]).

The set of $\mathbb{D}=\{\widehat{x}=x+\varepsilon x^* |\ x,x^*\in\mathbb{R}\}$ of dual numbers is a commutative ring according to the operations

$$(i)(x+\varepsilon x^*)+(y+\varepsilon y^*)=(x+y)+\varepsilon(x^*+y^*),$$

$$(ii)(x+\varepsilon x^*)(y+\varepsilon y^*)=xy+\varepsilon(xy^*+y^*x).$$

The dual number $\widehat{x}=x+\varepsilon x^*$ divided by the dual number $\widehat{y}=y+\varepsilon y^*$ provided $y\neq 0$ can be defined as

$$\frac{\widehat{x}}{\widehat{y}}=\frac{x+\varepsilon x^*}{y+\varepsilon y^*}=\frac{x}{y}+\varepsilon\frac{x^*y-xy^*}{y^2}.$$

The set of

$$\begin{aligned}\mathbb{D}^3 &= \mathbb{D}\times\mathbb{D}\times\mathbb{D}\\ &= \{\overrightarrow{\widehat{x}}\,|\quad \overrightarrow{\widehat{x}}=(x_1+\varepsilon x_1^*,x_2+\varepsilon x_2^*,x_3+\varepsilon x_3^*)\}\\ &= \{\overrightarrow{\widehat{x}}\,|\quad \overrightarrow{\widehat{x}}=(x_1,x_2,x_3)+\varepsilon\,(x_1^*,x_2^*,x_3^*)\}\\ &= \{\overrightarrow{\widehat{x}}\,|\quad \overrightarrow{\widehat{x}}=\overrightarrow{x}+\varepsilon\overrightarrow{x^*},\ \overrightarrow{x},\overrightarrow{x^*}\ \in\mathbb{R}^3\}\end{aligned}$$

is a module on the ring $\mathbb{D}$. For any $\overrightarrow{\widehat{x}}=\overrightarrow{x}+\varepsilon\overrightarrow{x^*}$, $\overrightarrow{\widehat{y}}=\overrightarrow{y}+\varepsilon\overrightarrow{y^*}\in\mathbb{D}^3$, the scalar or inner product and the vector product of $\overrightarrow{\widehat{x}}$ and $\overrightarrow{\widehat{y}}$ are defined by, respectively,

$$\langle\ \overrightarrow{\widehat{x}},\overrightarrow{\widehat{y}}\rangle=\langle\ \overrightarrow{x},\overrightarrow{y}\rangle+\varepsilon\left(\langle\ \overrightarrow{x},\overrightarrow{y^*}\rangle+\langle\ \overrightarrow{x^*},\overrightarrow{y}\rangle\right),$$

$$\overrightarrow{\widehat{x}}\,\Lambda\,\overrightarrow{\widehat{y}}=(\widehat{x}_2\ \widehat{y}_3-\widehat{x}_3\ \widehat{y}_2,\ \widehat{x}_3\ \widehat{y}_1-\widehat{x}_1\ \widehat{y}_3,\ \widehat{x}_1\ \widehat{y}_2-\widehat{x}_2\ \widehat{y}_1),$$

where $\widehat{x}_i=x_i+\varepsilon x_i^*$, $\widehat{y}_i=y_i+\varepsilon y_i^*\in\mathbb{D}$, $1\leq i\leq 3$. If $x\neq 0$, the norm $\left\|\overrightarrow{\widehat{x}}\right\|$ of $\overrightarrow{\widehat{x}}=\overrightarrow{x}+\varepsilon\overrightarrow{x^*}$ is defined by

$$\left\|\overrightarrow{\widehat{x}}\right\|=\sqrt{\langle\ \overrightarrow{\widehat{x}},\overrightarrow{\widehat{x}}\rangle}=\|\overrightarrow{x}\|+\varepsilon\frac{\langle\ \overrightarrow{x},\overrightarrow{x^*}\rangle}{\|\overrightarrow{x}\|}.$$

A dual vector $\overrightarrow{\widehat{x}}$ with norm 1 is called a dual unit vector. Let $\overrightarrow{\widehat{x}}=\overrightarrow{x}+\varepsilon\overrightarrow{x^*}$ $\in\mathbb{D}^3$. The set

$$\mathbb{S}^2 = \{ \ \overrightarrow{\widehat{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x^*} \mid \left\| \overrightarrow{\widehat{x}} \right\| = (1,0); \ \overrightarrow{x}, \overrightarrow{x^*} \ \in \mathbb{R}^3 \}$$

is called the dual unit sphere with the center $\widehat{O}$ in $\mathbb{D}^3$.

If every $x_i(t)$ and $x_i^*(t)$, $1 \leq i \leq 3$ real valued functions, are differentiable, the dual space curve

$$\begin{aligned} \widehat{x} \quad &: \quad I \subset \mathbb{R} \to \mathbb{D}^3 \\ t \quad &\to \quad \overrightarrow{\widehat{x}}(t) = (x_1(t) + \varepsilon x_1^*(t), \ x_2(t) + \varepsilon x_2^*(t), \ x_3(t) + \varepsilon x_3^*(t)) \\ &= \quad \overrightarrow{x}(t) + \varepsilon \overrightarrow{x^*}(t) \end{aligned}$$

in $\mathbb{D}^3$ is differentiable. We call the real part $\overrightarrow{x}(t)$ the indicatrix of $\overrightarrow{\widehat{x}}(t)$. The dual arc length of the curve $\overrightarrow{\widehat{x}}(t)$ from t_1 to t is defined as

$$\widehat{s} = \int_{t_1}^{t} \left\| \overrightarrow{\widehat{x}}(t)' \right\| dt = \int_{t_1}^{t} \| \overrightarrow{x}(t)' \| \, dt + \varepsilon \int_{t_1}^{t} \left\langle \overrightarrow{t}, \left(\overrightarrow{x^*} \right)' \right\rangle = s + \varepsilon s^*, \tag{2.1}$$

where $\widehat{t}$ is a unit tangent vector of $\overrightarrow{x(t)}$. From now on we will take the arc length s of $\overrightarrow{x(t)}$ as the parameter instead of t.

Now we will obtain equations relatively to the derivatives of dual Frenet vectors throughout the curve in $\mathbb{D}^3$. Let

$$\begin{aligned} \widehat{x} \quad &: \quad I \to \mathbb{D}^3 \\ s \quad &\to \quad \overrightarrow{\widehat{x}(s)} = \overrightarrow{x(s)} + \varepsilon \overrightarrow{x^*(s)} \end{aligned}$$

be a dual curve with the arc length parameter s of the indicatrix. Then,

$$\frac{d \overrightarrow{\widehat{x}}}{d \widehat{s}} = \frac{d \overrightarrow{\widehat{x}}}{ds} \frac{ds}{d \widehat{s}} = \overrightarrow{\widehat{t}}$$

is called the dual unit tangent vector of $\overrightarrow{\widehat{x}(s)}$. With the aid of equation(2.1), we have

$$\widehat{s} = s + \varepsilon \int_{s_1}^{s} \left\langle \overrightarrow{t}, \left(\overrightarrow{x^*} \right)' \right\rangle ds$$

and from this $\frac{d\widehat{s}}{ds} = 1 + \varepsilon \Delta$, where the prime denotes differentiation with respect to the arc length s of indicatrix and $\Delta = \left\langle \overrightarrow{t}, \left(\overrightarrow{x^*} \right)' \right\rangle$. Since $\overrightarrow{\widehat{t}}$ has constant length 1, its differentiation with respect to $\widehat{s}$, which is given by

$$\frac{d \overrightarrow{\widehat{t}}}{d \widehat{s}} = \frac{d \overrightarrow{\widehat{t}}}{ds} \frac{ds}{d \widehat{s}} = \frac{d^2 \overrightarrow{\widehat{x}}}{d \widehat{s}^2} = \widehat{\kappa} \overrightarrow{\widehat{n}},$$

measures the way the curve is turning in $\mathbb{D}^3$. The norm of the vector $\frac{d\overrightarrow{\widehat{t}}}{d\widehat{s}}$ is called curvature function of $\overrightarrow{\widehat{x}(s)}$. We impose the restriction that the function $\widehat{\kappa}: I \to \mathbb{D}$ is never pure dual. Then, the dual unit vector $\overrightarrow{\widehat{n}} = \frac{1}{\widehat{\kappa}}\frac{d\overrightarrow{\widehat{t}}}{d\widehat{s}}$ is called the principal normal of $\overrightarrow{\widehat{x}(s)}$. The dual vector $\overrightarrow{\widehat{b}}$ is called the binormal of $\overrightarrow{\widehat{x}(s)}$. The dual vectors $\overrightarrow{\widehat{t}}, \overrightarrow{\widehat{n}}, \overrightarrow{\widehat{b}}$ are called the dual Frenet trihedron of $\overrightarrow{\widehat{x}(s)}$ at the point $\widehat{x}(s)$. The equalities relative to derivatives of dual Frenet vectors $\overrightarrow{\widehat{t}}, \overrightarrow{\widehat{n}}, \overrightarrow{\widehat{b}}$ throughout the dual space curve are written in the matrix form

$$\frac{d}{d\widehat{s}}\begin{bmatrix}\overrightarrow{\widehat{t}}\\ \overrightarrow{\widehat{n}}\\ \overrightarrow{\widehat{b}}\end{bmatrix} = \begin{bmatrix}0 & \widehat{\kappa} & 0\\ -\widehat{\kappa} & 0 & \widehat{\tau}\\ 0 & -\widehat{\tau} & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{\widehat{t}}\\ \overrightarrow{\widehat{n}}\\ \overrightarrow{\widehat{b}}\end{bmatrix}, \tag{2.2}$$

where $\widehat{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature and $\widehat{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion. The formulae (2.2) are called the Frenet formulae of dual curve in $\mathbb{D}^3$(see [5]).

3 Mannheim partner curves in $\mathbb{D}^3$

In this section, we define Mannheim partner curves in dual space $\mathbb{D}^3$ and we give some characterization for Mannheim partner curves in the same space.

Definition 1. *Let $\mathbb{D}^3$ be the dual space with the standard inner product $\langle,\rangle$. If there exists a corresponding relationship between the dual space curves $\widehat{\alpha}$ and $\widehat{\beta}$ such that, at the corresponding points of the dual curves, the principal normal lines of $\widehat{\alpha}$ coincides with the binormal lines of $\widehat{\beta}$, then $\widehat{\alpha}$ is called a dual Mannheim curve, and $\widehat{\beta}$ a dual Mannheim partner curve of $\widehat{\alpha}$. The pair $\{\widehat{\alpha}, \widehat{\beta}\}$ is said to be a dual Mannheim pair.*

Let $\widehat{\alpha}: \widehat{x}(\widehat{s})$ be a dual Mannheim curve in $\mathbb{D}^3$ parameterized by its arc length $\widehat{s}$ and $\widehat{\beta}: \widehat{x}_1(\widehat{s}_1)$ the dual Mannheim partner curve of with an arc length parameter $\widehat{s}_1$. Denote by $\left\{\overrightarrow{\widehat{t}}(\widehat{s}), \overrightarrow{\widehat{n}}(\widehat{s}), \overrightarrow{\widehat{b}}(\widehat{s})\right\}$ the Frenet frame field along $\widehat{\alpha}: \widehat{x}(\widehat{s})$.

In the following theorems, we give a necessary and sufficient condition for a dual space curve to be a Mannheim curve.

Theorem 1. *A dual space curve in $\mathbb{D}^3$ is a dual Mannheim curve if and only if its curvature $\widehat{\kappa}$ and torsion $\widehat{\tau}$ satisfy the formula $\widehat{\kappa} = \widehat{\lambda}(\widehat{\kappa}^2 + \widehat{\tau}^2)$, where $\widehat{\lambda}$ is never pure dual constant.*

Proof. Let $\widehat{\alpha} : \widehat{x}(\widehat{s})$ be a dual Mannheim curve in $\mathbb{D}^3$ with the arc length parameter $\widehat{s}$ and $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ the dual Mannheim partner curve of with an arc length parameter $\widehat{s}_1$. Then by the definition we can assume that

$$\widehat{x}_1(\widehat{s}) = \widehat{x}(\widehat{s}) + \widehat{\lambda}(\widehat{s})\overrightarrow{\widehat{n}}(\widehat{s}) \tag{3.1}$$

for some never pure dual constant $\widehat{\lambda}(\widehat{s})$. By taking the derivative of (3.1) with respect to $\widehat{s}$ and applying the Frenet formulas we have

$$\frac{d\widehat{x}_1(\widehat{s})}{d\widehat{s}} = \left(1 - \widehat{\lambda}\widehat{\kappa}\right)\overrightarrow{\widehat{t}} + \frac{d\widehat{\lambda}}{d\widehat{s}}\overrightarrow{\widehat{n}} + \widehat{\lambda}\widehat{\tau}\overrightarrow{\widehat{b}}.$$

Since $\overrightarrow{\widehat{t_1}}$ is coincident with $\overrightarrow{\widehat{b_1}}$ in direction, we get

$$\frac{d\widehat{\lambda}(\widehat{s})}{d\widehat{s}} = 0.$$

This means that $\widehat{\lambda}$ is a never pure dual constant. Thus we have

$$\frac{d\widehat{x}_1(\widehat{s})}{d\widehat{s}} = \left(1 - \widehat{\lambda}\widehat{\kappa}\right)\overrightarrow{\widehat{t}} + \widehat{\lambda}\widehat{\tau}\overrightarrow{\widehat{b}}.$$

On the other hand, we have

$$\overrightarrow{\widehat{t_1}} = \frac{d\widehat{x}_1}{d\widehat{s}}\frac{d\widehat{s}}{d\widehat{s}_1} = \left(\left(1 - \widehat{\lambda}\widehat{\kappa}\right)\overrightarrow{\widehat{t}} + \widehat{\lambda}\widehat{\tau}\overrightarrow{\widehat{b}}\right)\frac{d\widehat{s}}{d\widehat{s}_1}.$$

By taking the derivative of this equation with respect to $\widehat{s}_1$ and applying the Frenet formulas we obtain

$$\begin{aligned} \frac{d\overrightarrow{\widehat{t_1}}}{d\widehat{s}}\frac{d\widehat{s}}{d\widehat{s}_1} &= \left(-\widehat{\lambda}\frac{d\widehat{\kappa}}{d\widehat{s}}\overrightarrow{\widehat{t}} + \left(\widehat{\kappa} - \widehat{\lambda}\widehat{\kappa}^2 - \widehat{\lambda}\widehat{\tau}^2\right)\overrightarrow{\widehat{n}} + \widehat{\lambda}\frac{d\widehat{\tau}}{d\widehat{s}}\overrightarrow{\widehat{b}}\right)\frac{d\widehat{s}}{d\widehat{s}_1} \\ &\quad + \left(\left(1 - \widehat{\lambda}\widehat{\kappa}\right)\overrightarrow{\widehat{t}} + \widehat{\lambda}\widehat{\tau}\overrightarrow{\widehat{b}}\right)\frac{d^2\widehat{s}}{d\widehat{s}_1^2}. \end{aligned}$$

From this equation we get

$$\left(\widehat{\kappa} - \widehat{\lambda}\widehat{\kappa}^2 - \widehat{\lambda}\widehat{\tau}^2\right)\frac{d\widehat{s}}{d\widehat{s}_1} = 0,$$

$$\widehat{\kappa} = \widehat{\lambda}(\widehat{\kappa}^2 + \widehat{\tau}^2).$$

This completes the proof. □

Theorem 2. *Let $\widehat{\alpha} : \widehat{x}(\widehat{s})$ be a dual Mannheim curve in $\mathbb{D}^3$ with the arc length parameter $\widehat{s}$. Then $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ is the dual Mannheim partner curve of if and only if the curvature $\widehat{\kappa}_1$ and the torsion $\widehat{\tau}_1$ of $\widehat{\beta}$ satisfy the following equation*

$$\frac{d\widehat{\tau}_1}{d\widehat{s}_1} = \frac{\widehat{\kappa}_1}{\widehat{\mu}}\left(1 + \widehat{\mu}^2\widehat{\tau}_1^2\right)$$

for some never pure dual constant $\widehat{\mu}$.

Proof. Suppose that $\widehat{\alpha} : \widehat{x}(\widehat{s})$ is a dual Mannheim curve. Then by the definition we can assume that

$$\widehat{x}(\widehat{s}_1) = \widehat{x}_1(\widehat{s}_1) + \widehat{\mu}(\widehat{s}_1)\overrightarrow{\widehat{b}_1}(\widehat{s}_1), \tag{3.2}$$

for some function $\widehat{\mu}(\widehat{s}_1)$. By taking the derivative of (3.2) with respect to $\widehat{s}_1$ and applying the Frenet formulas, we have

$$\overrightarrow{\widehat{t}}\frac{d\widehat{s}}{d\widehat{s}_1} = \overrightarrow{\widehat{t}}_1 + \overrightarrow{\widehat{b}} - \widehat{\mu}\widehat{\tau}_1\overrightarrow{\widehat{n}}_1. \tag{3.3}$$

Since $\overrightarrow{\widehat{b}}$ is coincident with $\overrightarrow{\widehat{n}}$ in direction, we get

$$\frac{d\widehat{\mu}}{d\widehat{s}_1} = 0.$$

This means that $\widehat{\mu}$ is never a pure dual constant. Thus we have

$$\overrightarrow{\widehat{t}}\frac{d\widehat{s}}{d\widehat{s}_1} = \overrightarrow{\widehat{t}}_1 - \widehat{\mu}\widehat{\tau}_1\overrightarrow{\widehat{n}}_1. \tag{3.4}$$

On the other hand, we have

$$\overrightarrow{\widehat{t}} = \overrightarrow{\widehat{t}}_1\cos\widehat{\theta} + \overrightarrow{\widehat{n}}_1\sin\widehat{\theta}, \tag{3.5}$$

where $\widehat{\theta}$ is the dual angle between $\overrightarrow{\widehat{t}}$ and $\overrightarrow{\widehat{t}}_1$ at the corresponding points of $\widehat{\alpha}$ and $\widehat{\beta}$. By taking the derivative of this equation with respect to $\widehat{s}_1$, we obtain

$$\widehat{\kappa}\overrightarrow{\widehat{n}}\frac{d\widehat{s}}{d\widehat{s}_1} = -\left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1}\right)\sin\widehat{\theta}\,\overrightarrow{\widehat{t}}_1 + \left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1}\right)\cos\widehat{\theta}\,\overrightarrow{\widehat{n}}_1 + \widehat{\tau}_1\sin\widehat{\theta}\,\overrightarrow{\widehat{b}}_1.$$

From this equation and the fact that the direction of $\overrightarrow{\widehat{n}}$ is coincident with $\overrightarrow{\widehat{b}}_1$, we get

$$\begin{cases} \left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1}\right)\sin\widehat{\theta} = 0 \\ \left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1}\right)\cos\widehat{\theta} = 0. \end{cases}$$

Therefore we have

$$\frac{d\widehat{\theta}}{d\widehat{s}_1} = -\widehat{\kappa}_1. \tag{3.6}$$

From (3.4), (3.5) and notice that $\overrightarrow{\widehat{t}}_1$ is orthogonal to $\overrightarrow{\widehat{b}}_1$, we find that

$$\frac{d\widehat{s}}{d\widehat{s}_1} = \frac{1}{\cos\widehat{\theta}} = -\frac{\widehat{\mu}\widehat{\tau}_1}{\sin\widehat{\theta}}.$$

Then we have

$$\widehat{\mu}\widehat{\tau}_1 = -\tan\widehat{\theta}.$$

By taking the derivative of this equation and applying (3.6), we get

$$\widehat{\mu}\frac{d\widehat{\tau}_1}{d\widehat{s}_1} = \widehat{\kappa}_1\left(1+\widehat{\mu}^2\widehat{\tau}_1^2\right),$$

that is

$$\frac{d\widehat{\tau}_1}{d\widehat{s}_1} = \frac{\widehat{\kappa}_1}{\widehat{\mu}}\left(1+\widehat{\mu}^2\widehat{\tau}_1^2\right).$$

Conversely, if the curvature $\widehat{\kappa}_1$and torsion $\widehat{\tau}_1$of the dual curve $\widehat{\beta}$ satisfy

$$\frac{d\widehat{\tau}_1}{d\widehat{s}_1} = \frac{\widehat{\kappa}_1}{\widehat{\mu}}\left(1+\widehat{\mu}^2\widehat{\tau}_1^2\right)$$

for some never pure dual constant $\widehat{\mu}(\widehat{s})$, then we define a dual curve by

$$\widehat{x}(\widehat{s}_1) = \widehat{x}_1(\widehat{s}_1) + \widehat{\mu}\overrightarrow{\widehat{b}_1}(\widehat{s}_1) \tag{3.7}$$

and we will prove that $\widehat{\alpha}$ is a Mannheim curve and $\widehat{\beta}$ is the partner curve of $\widehat{\alpha}$. By taking the derivative of (3.7) with respect to $\widehat{s}_1$ twice, we get

$$\overrightarrow{\widehat{t}}\frac{d\widehat{s}}{d\widehat{s}_1} = \overrightarrow{\widehat{t}}_1 - \widehat{\mu}\widehat{\tau}_1\overrightarrow{\widehat{n}}_1, \tag{3.8}$$

$$\widehat{\kappa}\overrightarrow{\widehat{n}}\left(\frac{d\widehat{s}}{d\widehat{s}_1}\right)^2 + \overrightarrow{\widehat{t}}\frac{d^2\widehat{s}}{d\widehat{s}_1^2} = \widehat{\mu}\widehat{\kappa}_1\widehat{\tau}_1\overrightarrow{\widehat{t}}_1 + \left(\widehat{\kappa}_1 - \widehat{\mu}\frac{d\widehat{\tau}_1}{d\widehat{s}_1}\right)\overrightarrow{\widehat{n}}_1 - \widehat{\mu}\widehat{\tau}_1^2\overrightarrow{\widehat{b}}_1, \tag{3.9}$$

respectively. Taking the cross product of (3.8) with (3.9) and noticing that

$$\widehat{\kappa}_1 - \widehat{\mu}\frac{d\widehat{\tau}_1}{d\widehat{s}_1} + \widehat{\mu}^2\widehat{\kappa}_1\widehat{\tau}_1^2 = 0,$$

we have

$$\widehat{\kappa}\overrightarrow{\widehat{b}}\left(\frac{d\widehat{s}}{d\widehat{s}_1}\right)^3 = \widehat{\mu}^2\widehat{\tau}_1^3\overrightarrow{\widehat{t}}_1 + \widehat{\mu}\widehat{\tau}_1^2\overrightarrow{\widehat{n}}_1. \tag{3.10}$$

By taking the cross product of (3.10) with (3.8), we obtain also

$$\widehat{\kappa}\overrightarrow{\widehat{n}}\left(\frac{d\widehat{s}}{d\widehat{s}_1}\right)^4 = -\widehat{\mu}\widehat{\tau}_1^2\left(1+\widehat{\mu}^2\widehat{\tau}_1^2\right)\overrightarrow{\widehat{b}}_1.$$

This means that the principal normal direction $\overrightarrow{\widehat{n}}$ of $\widehat{\alpha} : \widehat{x}(\widehat{s})$ coincides with the binormal direction $\overrightarrow{\widehat{b}}_1$ of $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$. Hence $\widehat{\alpha} : \widehat{x}(\widehat{s})$ is a dual Mannheim curve and $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ is its dual Mannheim partner curve. □

Definition 2. *A dual helix is a dual curve for which the tangent makes a constant dual angle with a dual fixed line.*

Proposition 1. *Let $\widehat{\alpha} : \widehat{x}(\widehat{s})$ be a dual Mannheim curve in $\mathbb{D}^3$ with the arc length parameter $\widehat{s}$ and $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ the dual Mannheim partner curve of with an arc length parameter $\widehat{s}_1$. If $\widehat{\alpha} : \widehat{x}(\widehat{s})$ is a generalized dual helix, then $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ is a dual straight line.*

Proof. Let $\overrightarrow{\widehat{t}}, \overrightarrow{\widehat{n}}, \overrightarrow{\widehat{b}}$ be the tangent, principal normal and binormal vector field of the curve $\widehat{\alpha} : \widehat{x}(\widehat{s})$, respectively. From the properties of generalized dual helices and the definition of dual Mannheim curves, we have

$$\overrightarrow{\widehat{b}}_1.\widehat{p} = \overrightarrow{\widehat{n}}.\widehat{p} = 0$$

for some constant dual vector $\widehat{p}$. Then it is easy to obtain that $\widehat{\tau}_1 = \widehat{\kappa}_1 \equiv 0$. □

Proposition 2. *If a generalized dual helix is the Mannheim partner curve of some curve $\widehat{\alpha} : \widehat{x}(\widehat{s})$ in $\mathbb{D}^3$, then the ratio of torsion and curvature of the curve $\widehat{\alpha} : \widehat{x}(\widehat{s})$ is*

$$\frac{\widehat{\tau}}{\widehat{\kappa}} = \frac{\widehat{c}_2}{2}e^{\widehat{c}_1\widehat{s}} - \frac{1}{2\widehat{c}_2}e^{-\widehat{c}_1\widehat{s}},$$

for some nonzero constant $\widehat{c}_1$ and for some never pure dual constant $\widehat{c}_2$, and $\widehat{s}$ is the arc length parameter of $\widehat{\alpha}$. In particular, if we put $\widehat{c}_1 = \widehat{c}_2 = 1$, we have

$$\frac{\widehat{\tau}}{\widehat{\kappa}} = \frac{e^{\widehat{s}} - e^{-\widehat{s}}}{2} = \sinh\widehat{s}.$$

Proof. Let $\overrightarrow{\widehat{t}}, \overrightarrow{\widehat{n}}, \overrightarrow{\widehat{b}}$ be the tangent, principal normal and binormal vector field of the curve $\widehat{\alpha} : \widehat{x}(\widehat{s})$, respectively. From the properties of generalized dual helices and the definition of dual Mannheim curves, we have

$$\overrightarrow{\widehat{b}}_1.\widehat{p} = \sin\widehat{\theta}_0,$$

for some constant dual vector $\widehat{p}$ and some constant dual angle $\widehat{\theta}_0$. From the last equation we know that $\sin\widehat{\theta}_0 \neq 0$ and $\frac{\widehat{\tau}}{\widehat{\kappa}} \neq$ dual constant. By taking the derivative of this equation with respect to $\widehat{s}$ twice, we get

$$-\widehat{\kappa}\,\overrightarrow{\widehat{t}}.\widehat{p} + \widehat{\tau}\,\overrightarrow{\widehat{n}}.\widehat{p} = 0,$$

$$-\frac{d\widehat{\kappa}}{d\widehat{s}}\overrightarrow{\widehat{t}}.\widehat{p} + \frac{d\widehat{\tau}}{d\widehat{s}}\overrightarrow{\widehat{n}}.\widehat{p} = \left(\widehat{\kappa}^2 + \widehat{\tau}^2\right)\sin\widehat{\theta}_0.$$

By a direct calculation and using $\widehat{\kappa} = \widehat{\lambda}(\widehat{\kappa}^2 + \widehat{\tau}^2)$, we obtain

$$\overrightarrow{\widehat{t}}.\widehat{p} = \frac{\widehat{\tau}}{\widehat{\lambda}\widehat{\kappa}\frac{d\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right)}{d\widehat{s}}}\sin\widehat{\theta}_0,$$

$$\overrightarrow{\widehat{n}}.\widehat{p} = \frac{1}{\widehat{\lambda}\frac{d\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right)}{d\widehat{s}}}\sin\widehat{\theta}_0.$$

Taking the derivative, we have

$$\widehat{\kappa} = \frac{1}{\widehat{\lambda}}\left(1 - \frac{\widehat{\tau}\frac{d^2\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right)}{d\widehat{s}^2}}{\widehat{\kappa}\left(\frac{d\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right)}{d\widehat{s}}\right)^2}\right),$$

$$\widehat{\tau} = \frac{\frac{d^2\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right)}{d\widehat{s}^2}}{\widehat{\lambda}\left(\frac{d\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right)}{d\widehat{s}}\right)^2},$$

respectively. From these equations, we find that

$$\frac{\widehat{\tau}}{\widehat{\kappa}} = \frac{\frac{d^2\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right)}{d\widehat{s}^2}}{\left(\frac{d\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right)}{d\widehat{s}}\right)^2 - \frac{\widehat{\tau}}{\widehat{\kappa}}\frac{d^2\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right)}{d\widehat{s}^2}}.$$

Let $\frac{\widehat{\tau}}{\widehat{\kappa}} = \widehat{y}(\widehat{s})$, then we get the following differential equation

$$\left(1 + \widehat{y}^2\right)\frac{d^2\widehat{y}}{d\widehat{s}^2} - \widehat{y}\left(\frac{d\widehat{y}}{d\widehat{s}}\right)^2 = 0.$$

Solving this equation, we obtain that

$$\widehat{y}(\widehat{s}) = \widehat{c}_0$$

or

$$\widehat{y}(\widehat{s}) = \frac{\widehat{c}_2}{2} e^{\widehat{c}_1 \widehat{s}} - \frac{1}{2\widehat{c}_2} e^{-\widehat{c}_1 \widehat{s}},$$

for some nonzero constant $\widehat{c}_0$,$\widehat{c}_1$ and for some never pure dual constant $\widehat{c}_2$. Thus, the proposition is proved. □

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University of Ankara
Faculty of Science
Department of Mathematics
Tandoğan, Ankara, Turkey
Email: sozkaldi@kku.edu.tr

University of Kırıkkale
Faculty of Science and Arts
Department of Mathematics
Kırıkkale, Turkey
Email: kilarslan@yahoo.com

University of Ankara
Faculty of Science
Department of Mathematics
Tandoğan, Ankara, Turkey
Email: yayli@science.ankara.edu.tr