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Viscosity approximation method for m-accretive mapping and variational inequality in Banach space

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Abstract

This paper introduces a composite viscosity iterative scheme to approximate a zero of $m-$accretive operator A defined on Banach spaces E with uniformly Gâteaux differentiable norm. It is also shown that the zero is a solution of some variation inequalities. The results in this paper improve and extend the corresponding that of [3] and some others.

1 Introduction and preliminaries

Let E be a real Banach space and E^* its dual space. Let J denote the normalized duality mapping from E into 2^{E^*} defined by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, where $\langle \cdot, \cdot\rangle$ denote the generalized duality pairing between E and E^*. It is well-known that if E^* is strictly convex then J is sing-valued. In the sequel, we shall denote the single-valued normalized duality mapping by j.

Let K be a nonempty subset of E. We first recall some definitions and conclusions:

Definition 1.1 *$T : K \to K$ is said to be a $L-$Lipschitz mapping, if $\forall$ $x, y \in K$, $\|Tx-Ty\| \le L\|x-y\|$. Especially, if $L = 1$, i.e. $\|Tx-Ty\| \le \|x-y\|$, then T is said to non-expansive; if $0 < L < 1$, then T is said to contraction*

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mapping.

Definition 1.2 *An operator A (possibly multivalued) with domain $D(A)$ and range $R(A)$ in E is called accretive mapping, if $\forall\ x_i \in D(A)$ and $y_i \in Ax_i$(i=1,2), there exists $j(x_2-x_1) \in J(x_2-x_1)$ such that $\langle y_2-y_1, j(x_2-x_1)\rangle \geq 0$. Especially, an accretive operator A is called m-accretive if $R(I+rA)=E$ for all $r>0$.*

Note that if A is *accretive*, then $J_A := (I+A)^{-1}$ is a nonexpansive single-valued mapping from $R(I+A)$ to $D(A)$ and $F(J_A)=N(A)$, where $N(A)=\{x \in D(A) : Ax=0\}$.

Definition 1.3. *$T : K \to K$ is called pseudocontractive mapping, if there exists $j(x-y) \in J(x-y)$ such that $\langle Tx-Ty, j(x-y)\rangle \leq \|x-y\|^2, \forall\ x,y \in K$.*

Remark. If T is pseudocontractive, then $I-T$ is accretive, where I is an identity operator.

Let $S=\{x \in E : \|x\|=1\}$ denote the unit sphere of the real Banach space E. E is said to have a *Gâteaux differentiable norm* if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x,y \in S$; and E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S$, the limit is attained uniformly for $x \in S$. Furthermore, it is well known that if E has a *uniformly Gâteaux differentiable norm*, then the dual space E^* is *uniformly convex* and so the duality map j is *single valued* and *uniformly continuous* on bounded subsets of E. Let E be a normed space with dim $E \geq 2$, the modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\{\frac{\|x+y\|+\|x-y\|}{2}-1 : \|x\|=1; \|y\|=\tau\}.$$

The space E is called uniformly smooth if and only if $\lim_{\tau\to 0^+}\rho_E\tau/\tau=0$.

In 2006, H.K. Xu considered the following algorithm,

$$x_{n+1}=\alpha_n u+(1-\alpha_n)J_{r_n}x_n, n\geq 0, \tag{1}$$

where $u \in K$ is arbitrary (but fixed), $J_{r_n}=(I+r_nA)^{-1}$, $\{\alpha_n\}$ is a sequence in (0,1), and $\{r_n\}$ is a sequence of positive numbers. Xu proved that if E is a uniformly smooth Banach space, then the sequence $\{x_n\}$ given by (1.1) converges strongly to a point in $N(A)$ provided the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy certain conditions.

Inspired by (1.1), R. Chen and Z. Zhu [3] studied the following two iterative schemes:

$$x_{t,n_t} = tf(x_{t,n_t}) + (1-t)J_{r_{n_t}}x_{t,n_t}, \quad t \in (0,1) \tag{2}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{r_n}x_n, \quad n \geq 0. \tag{3}$$

where $J_{r_n} = (I + r_nA)^{-1}$, $\sigma \in (0,1)$ is arbitrary (but fixed). I denotes identity operator.

Under appropriate conditions, R. Chen and Z. Zhu [3] proved that if E is a uniformly smooth Banach space, then the sequence $\{x_{t,n_t}\}$ and $\{x_n\}$ given by (1.2) and (1.3) converge strongly to a zero point of $m-$accretive operator A, respectively.

Motivated by Chen and Zhu's work, in this paper, we study two new iterative schemes in reflexive Banach spaces E with uniformly Gâteaux differentiable norm as follows:

$$x_t = tf(x_t) + (1-t)S_rx_t, \quad t \in (0,1) \tag{4}$$

and

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \\ y_n = \beta_n x_n + (1-\beta_n)S_{r_n}x_n, \quad n \geq 0, \end{cases} \tag{5}$$

where $S_r = (1-\sigma)I + \sigma J_r$, $J_r = (I + rA)^{-1}$, $S_{r_n} = (1-\sigma)I + \sigma J_{r_n}$, $J_{r_n} = (I + r_nA)^{-1}$, $\sigma \in (0,1)$ is arbitrary(but fixed), I denotes identity operator. Especially, if $\beta_n = 0$, then (1.5) reduces to following iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)S_{r_n}x_n, \quad n \geq 0, \tag{6}$$

Obviously, the iterative scheme (1.4) and (1.6) are still different from that of (1.2) and (1.3), respectively.

Under appropriate conditions, this paper proves that $\{x_t\}$ defined by (1.4) converge strongly to a $p \in N(A)$ which is a solution of some variational inequalities in the framework of reflexive Banach spaces E with uniformly Gâteaux differentiable norm. At the same time, we also prove that $\{x_n\}$ converges strongly to a $p \in N(A)$. The results obtained in this paper improve and extend that of Chen and Zhu [3] and some others.

In what follows, we shall make use of the following Lemmas.

Lemma 1.1([2]). *Let E be a real normed linear space and J the normalized duality mapping on E, then for each $x, y \in E$ and $j(x+y) \in J(x+y)$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

Lemma 1.2(Suzuki, [6]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \geq 0$*

and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0$, *then,* $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 1.3([9]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1}\le(1-\alpha_n)a_n+\alpha_n\sigma_n+\gamma_n, n\ge 0,$$

if (i) $\alpha_n\in[0,1]$, $\sum\alpha_n=\infty$*; (ii)* $\limsup\sigma_n\le 0$*; (iii)* $\gamma_n\ge 0$, $\sum\gamma_n<\infty$*, then* $a_n\to 0$*, as* $n\to\infty$*.*

Theorem I(see,e.g.,[4,10]). *Let* A *be a continuous and accretive operator on the real Banach space* E *with* $D(A)=E$*. Then* A *is* $m-$*accretive.*

Let μ be a continuous linear functional on l^∞ satisfying $\|\mu\|=1=\mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n;n\in N\}\le\mu(a)\le\sup\{a_n;n\in N\}$$

for every $a=(a_1,a_2,...)\in l^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a Banach limit if $\mu_n(a_n)=\mu_n(a_{n+1})$ for every $a=(a_1,a_2,...)\in l^\infty$. Furthermore, we know the following result [8, Lemma 1] and[7, Lemma4.5.4].

Lemma1.4([8], Lemma 1). *Let* K *be a nonempty closed convex subset of a Banach space* E *with a uniformly Gâteaux differentiable norm. Let* $\{x_n\}$ *be a bounded sequence of* E *and let* μ *be a mean on* N*.Let* $z\in K$*. Then*

$$\mu_n\|x_n-z\|=\min_{y\in K}\mu_n\|x_n-y\|$$

if and only if

$$\mu_n\langle y-z,j(x_n-z)\rangle\le 0,\ \ \forall\, y\in K,$$

where j *is the duality mapping of* E*.*

Lemma 1.5([1, 5]). *For* $\lambda>0$ *and* $\mu>0$ *and* $x\in E$*,*

$$J_\lambda x=J_\mu\left(\frac{\mu}{\lambda}x+\left(1-\frac{\mu}{\lambda}\right)J_\lambda x\right).$$

2 Main results

Throughout this paper, suppose that
(a) E is a real reflexive Banach space E which has a uniformly Gâteaux differentiable norms;
(b) K is a nonempty closed convex subset of E;
(c) every nonempty closed bounded convex subset of E has the fixed point

property for nonexpansive mappings.

Theorem 2.1. *Let $A : K \to E$ be a $m-$accretive mapping with $N(A) \neq \emptyset$. Let $f : K \to K$ be a contraction with contraction constant$\alpha \in (0, 1)$, then there exists $x_t \in K$ such that*

$$x_t = tf(x_t) + (1-t)S_r x_t, \tag{1}$$

where $S_r = (1-\sigma)I + \sigma J_r$ with $J_r = (I + rA)^{-1}$ and $\sigma \in (0, 1)$, I denotes identity operator. Further, as $t \to 0^+$, x_t converges strongly a zero $p \in N(A)$ which solutes the following variational inequality:

$$\langle p - f(p), j(p-q)\rangle \leq 0, \ \ \forall\, q \in N(A). \tag{2}$$

Proof. Firstly, S_r is nonexpansive mapping and $F(S_r) = N(A) \neq \emptyset$. Secondly, let H_t^f denote a mapping defined by

$$H_t^f x = tf(x) + (1-t)S_r x, \ \ \forall\, t \in (0, 1), \ \forall\, x \in K.$$

Obviously, H_t^f is contraction, then by Banach contraction mapping principle there exists $x_t \in K$ such that

$$x_t = tf(x_t) + (1-t)S_r x_t.$$

Now, let $p \in N(A)$, then

$$\|x_t - p\| = \|t(f(x_t) - p) + (1-t)(S_r x_t - p)\| \leq t\alpha\|x_t - p\| + t\|f(p) - p\| + (1-t)\|x_t - p\|,$$

i.e.,

$$\|x_t - p\| \leq \frac{\|f(p) - p\|}{1-\alpha}.$$

Hence $\{x_t\}$ is bounded. Assume that $t_n \to 0^+$ as $n \to \infty$. Set $x_n := x_{t_n}$, define a function g on K by

$$g(x) = \mu_n \|x_n - x\|^2.$$

Let

$$C = \{x \in K; g(x) = \min_{y \in K} \mu_n \|x_n - y\|^2\}.$$

It is easy to see that C is a closed convex bounded subset of E. Since $\|x_n - S_r x_n\| \to 0 (n \to \infty)$, hence

$$g(S_r x) = \mu_n \|x_n - S_r x\|^2 = \mu_n \|S_r x_n - S_r x\|^2 \leq \mu_n \|x_n - x\|^2 = g(x),$$

it follows that $S_r(C) \subset C$, that is C is invariant under S_r. By assumption (c), non-expansive mapping S_r has fixed point $p \in C$. Using Lemma 1.4 we obtain

$$\mu_n \langle x - p, j(x_n - p)\rangle \leq 0.$$

Taking $x = f(p)$, then

$$\mu_n\langle f(p) - p, j(x_n - p)\rangle \leq 0. \tag{3}$$

Since

$$x_t - p = t(f(x_t) - p) + (1 - t)(S_r x_t - p),$$

then

$$\|x_t-p\|^2 = t\langle f(x_t)-p, j(x_t-p)\rangle + (1-t)\langle S_r x_t-p, j(x_t-p)\rangle \leq t\langle f(x_t)-p, j(x_t-p)\rangle + (1-t)\|x_t-p\|^2$$

Further,

$$\|x_t - p\|^2 \leq \langle f(x_t) - p, j(x_t - p)\rangle = \langle f(x_t) - f(p), j(x_t - p)\rangle + \langle f(p) - p, j(x_t - p)\rangle.$$

Thus,

$$\mu_n\|x_n - p\|^2 \quad \leq \quad \mu_n\alpha\|x_n - p\|^2 + \mu_n\langle f(p) - p, j(x_n - p)\rangle.$$

it follows from (2.3) that

$$\mu_n\|x_n - p\|^2 = 0.$$

Hence there exists a subsequence of $\{x_n\}$ which is still denoted by $\{x_n\}$ such that $\lim_{n\to\infty}\|x_n - p\| = 0$. Now assume that another subsequence $\{x_m\}$ of $\{x_n\}$ converge strongly to $\bar{p} \in N(A)$. Since j is *uniformly continuous* on bounded subsets of E, then for any $q \in N(A)$,we have

$$\begin{aligned}
&|\langle x_m - f(x_m), j(x_m - q)\rangle - \langle \bar{p} - f(\bar{p}), j(\bar{p} - q)\rangle| \\
&= |\langle x_m - f(x_m) - (\bar{p} - f(\bar{p})), j(x_m - q)\rangle + \langle (\bar{p} - f(\bar{p})), j(x_m - q)\rangle - \langle \bar{p} - f(\bar{p}), j(\bar{p} - q)\rangle| \\
&\leq \|(I - f)x_m - (I - f)\bar{p}\|\|x_m - q\| + |\langle \bar{p} - f(\bar{p}), j(x_m - q) - j(\bar{p} - q)\rangle| \to 0 \ (m \to \infty) \quad (4)
\end{aligned}$$

i.e.,

$$\langle \bar{p} - f(\bar{p}), j(\bar{p} - q)\rangle = \lim_{n\to\infty}\langle x_m - f(x_m), j(x_m - q)\rangle. \tag{5}$$

Since $x_m = tf(x_m) + (1 - t)S_r x_m$, we have

$$(I - f)x_m = -\frac{1 - t}{t}(I - S_r)x_m,$$

hence for any $q \in N(A)$,

$$\langle (I - f)x_m, j(x_m - q)\rangle = -\frac{1 - t}{t}\langle (I - S_r)x_m - (I - S_r)q, j(x_m - q)\rangle \leq 0, \tag{6}$$

it follows from (2.5) and (2.6) that

$$\langle \bar{p} - f(\bar{p}), j(\bar{p} - q)\rangle \leq 0. \tag{7}$$

Interchange p and q to obtain

$$\langle \bar{p} - f(\bar{p}), j(\bar{p} - p)\rangle \leq 0, \tag{8}$$

i.e.,

$$\langle \bar{p} - p + p - f(\bar{p}), j(\bar{p} - p)\rangle \leq 0, \tag{9}$$

hence

$$\|\bar{p} - p\|^2 \leq \langle f(\bar{p}) - p, j(\bar{p} - p)\rangle. \tag{10}$$

Interchange p and $\bar{p}$ to obtain

$$\|\bar{p} - p\|^2 \leq \langle f(p) - \bar{p}, j(p - \bar{p})\rangle. \tag{11}$$

Adding up (2.10) and (2.11) yields that

$$2\|\bar{p} - p\|^2 \leq (1 + \alpha)\|\bar{p} - p\|, \tag{12}$$

this implies that $p = \bar{p}$. Hence $x_t \to p$ as $t \to 0^+$ and p is a solution of the following variational inequality

$$\langle p - f(p), j(p - q)\rangle \leq 0, \quad \forall\, q \in \mathrm{N(A)}.$$

This completes the proof of Theorem 2.1.

It is well known that the duality mapping j is identity mapping on Hilbert space. Next we give an example for the variational inequality (2.2).

Example 1. *Let $Tx = \frac{1}{2}x^2 - \frac{1}{4(|a|+|b|)}x^3$, $\forall\, x \in [a, b]$, $a, b \in R^1, a < b$. By Weierstrass Theorem we know that there exists $x_0 \in [a, b]$ such that*

$$Tx_0 = \min_{a \leq x \leq b} Tx.$$

Moreover, there has following results:

(i) If $x_0 \in (a, b)$, then $T'x_0 = 0$;

(ii) If $x_0 = a$, then $T'x_0 \geq 0$;

(iii) If $x_0 = b$, then $T'x_0 \leq 0$.

By (i)-(iii), we have $T'x_0(x - x_0) \geq 0$, $\forall\, x \in [a, b]$. Thus the following variational inequality is obtained by inner product of R^1:

$$\langle T'x_0, x - x_0\rangle \geq 0, \ \ \forall\, x \in [a, b]. \quad (*)$$

Notice that

$$T'x = x - \frac{3}{4(|a| + |b|)}x^2.$$

Let $f(x) = \frac{3}{4(|a|+|b|)}x^2, \forall\, x \in [a, b]$, then it is obvious that f is a contraction. This shows that the variate inequality $(*)$ is a special case of the variational *inequality (2.2).*

Theorem 2.2. *Let $A : K \to E$ be $m-$accretive with $N(A) \neq \emptyset$ and $f : K \to K$ be contractive with constant $\alpha \in (0,1)$. For given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n \\ y_n = \beta_n x_n + (1-\beta_n)S_{r_n}x_n, \end{cases} \tag{13}$$

where $S_{r_n} := (1-\sigma)I + \sigma J_{r_n}$ with $J_{r_n} := (I + r_n A)^{-1}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. $\sigma \in (0,1)$ is arbitrary (but fixed). Suppose that $\{\alpha_n\}, \{r_n\}$ satisfy the following conditions:

(i) $0 \leq \alpha_n \leq 1$ for all $n \geq 0$, $\lim \alpha_{n\to\infty} = 0$, $\Sigma_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $r_n \geq \varepsilon > 0$ for all $n \geq 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

then $\{x_n\}$ converges strongly to a zero $p \in N(A)$, where $p = \lim_{t\to 0^+} x_t$ is a solution of variational inequality (2.2).

Proof. We is easy to know that $F(S_{r_n}) = F(J_{r_n}) = N(A) \neq \emptyset$ and S_{r_n} is nonexpansive. Since $p \in N(A)$, then $p \in F(S_{r_n})$. It follows from (2.13)

$$\|y_n - p\| \leq \|x_n - p\|, \;\; \|x_{n+1} - p\| \leq (1-(1-\alpha)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\|,$$

which yields that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\alpha}\}.$$

Hence, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{S_{r_n}x_n\}$.

Let M be a constant such that for all $n \geq 0$,

$$\max\{\|f(x_n\|, \|f(x_{n+1}\|, \|J_{r_{n+1}}x_{n+1} - x_{n+1}\|, \|J_{r_{n+1}}x_{n+1}\|\} \leq M.$$

Then from (2.13) and Lemma 1.5 we have

$$\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| \leq \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| M. \tag{14}$$

and

$$\|S_{r_{n+1}}x_{n+1} - S_{r_n}x_n\| \leq \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| M, \tag{15}$$

Now, we shall show $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. We shall split two cases to study it.

Case 1. If $\limsup_{n\to\infty} \beta_n = 1$, then it follows from (2.13) that

$$x_{n+1} - x_n = \alpha_n f(x_n) + (1-\alpha_n)(1-\beta_n)(S_{r_n}x_n - x_n),$$

which implies that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$.

Case 2. Let $\limsup_{n\to\infty}\beta_n \le a < 1$. Let $\gamma_n = \alpha_n + (1-\beta_n)(1-\alpha_n)\sigma$, $\overline{y}_n = \frac{x_{n+1}-x_n+\gamma_n x_n}{\gamma_n}$, i.e. $\overline{y}_n = \frac{\alpha_n f(x_n)+(1-\alpha_n)(1-\beta_n)\sigma J_{r_n}x_n}{\gamma_n}$, then

$$
\begin{aligned}
\overline{y}_{n+1}-\overline{y}_n &= \frac{\alpha_{n+1}}{\gamma_{n+1}}f(x_{n+1})-\frac{\alpha_n}{\gamma_n}f(x_n)+\frac{(1-\alpha_{n+1})(1-\beta_{n+1})\sigma J_{r_{n+1}}x_{n+1}}{\gamma_{n+1}}-\frac{(1-\alpha_n)(1-\beta_n)\sigma J_{r_n}x_n}{\gamma_n}\\
&= \frac{\alpha_{n+1}}{\gamma_{n+1}}f(x_{n+1})-\frac{\alpha_n}{\gamma_n}f(x_n)+\frac{(1-\alpha_n)(1-\beta_n)\sigma}{\gamma_n}(J_{r_{n+1}}x_{n+1}-J_{r_n}x_n)\\
&\quad+\left(\frac{(1-\alpha_{n+1})(1-\beta_{n+1})}{\gamma_{n+1}}-\frac{(1-\alpha_n)(1-\beta_n)}{\gamma_n}\right)\sigma J_{r_{n+1}}x_{n+1},
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|\overline{y}_{n+1}-\overline{y}_n\| \le{}& \frac{\alpha_{n+1}+\alpha_n}{\gamma_{n+1}\gamma_n}M+\frac{(1-\alpha_n)(1-\beta_n)\sigma}{\gamma_n}\|x_{n+1}-x_n\|+\frac{1}{\gamma_n}|1-\frac{r_n}{r_{n+1}}|M\\
&+\left|\frac{(1-\alpha_{n+1})(1-\beta_{n+1})}{\gamma_{n+1}}-\frac{(1-\alpha_n)(1-\beta_n)}{\gamma_n}\right|M. \qquad (16)
\end{aligned}
$$

Using the conditions (i-ii), from (2.16) we get that

$$\limsup_{n\to\infty}\{\|\overline{y}_{n+1}-\overline{y}_n\|-\|x_{n+1}-x_n\|\}\le 0. \qquad (17)$$

Based on Lemma 1.2 and (2.17), we obtain $\lim_{n\to\infty}\|\overline{y}_n - x_n\| = 0$, which implies

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

By case 1 and case 2 we know $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Since $\|x_{n+1}-y_n\| = \alpha_n\|f(x_n)-y_n\| \to 0$ as $n\to\infty$, then $\|x_n - y_n\|\to 0$ and

$$\|x_n - S_{r_n}x_n\| \le \frac{1}{1-a}\|x_n-y_n\| \to 0 \text{ as } n\to\infty. \qquad (18)$$

Since

$$\|x_n - J_{r_n}x_n\| = \frac{1}{\sigma}\|x_n - S_{r_n}x_n\|,$$

it follows from (2.18) that $\|x_n - J_{r_n}x_n\|\to 0$ as $n\to\infty$.

Let $r>0$ is a constant such that $\varepsilon > r > 0$, then

$$
\begin{aligned}
\|x_n - J_r x_n\| &\le \|x_n - J_{r_n}x_n\| + \|J_r x_n - J_{r_n}x_n\|\\
&= \|x_n - J_{r_n}x_n\| + \|J_r x_n - J_r(\frac{r}{r_n}x_n + (1-\frac{r}{r_n})J_{r_n}x_n)\|\\
&\le 2\|x_n - J_{r_n}x_n\| \to 0 (n\to\infty). \qquad (19)
\end{aligned}
$$

It follows from (2.19) that $\|x_n - S_r x_n\| \to 0$ as $n \to \infty$, where $S_r x_n = (1-\sigma)x_n + \sigma J_r x_n$. Let x_t be defined by (2.1), i.e.,

$$x_t = tf(x_t) + (1-t)S_r x_t, \quad \forall\ t \in (0,1).$$

Then, using Lemma 1.1, we have

$$\begin{aligned}
&\|x_t - x_n\|^2 = \|t(f(x_t) - x_n) + (1-t)(S_r x_t - x_n)\|^2 \\
&\leq (1-t)^2\|S_r x_t - x_n\|^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle \\
&\leq (1-t)^2(\|S_r x_t - S_r x_n\| + \|S_r x_n - x_n\|)^2 + 2t\langle f(x_t) - x_t + x_t - x_n, j(x_t - x_n)\rangle \\
&\leq (1+t^2)\|x_t - x_n\|^2 + \|S_r x_n - x_n\|(2\|x_t - x_n\| + \|S_r x_n - x_n\|) + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle,
\end{aligned}$$

hence,

$$\langle f(x_t) - x_t, j(x_n - x_t)\rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{\|S_r x_n - x_n\|}{2t}(2\|z_t - x_n\| + \|S_r x_n - x_n\|),$$

let $n \to \infty$ in the last inequality, then we obtain

$$\limsup_{n\to\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle \leq \frac{t}{2}M',$$

where $M' \geq 0$ is a constant such that$\|x_t - x_n\|^2 \leq M$ for all $t \in (0,1)$ and $n \geq 0$. Now letting $t \to 0^+$, then we have that

$$\limsup_{t\to 0^+}\limsup_{n\to\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle \leq 0.$$

Thus , for $\forall\, \varepsilon > 0$, there exists a positive number δ' such that for any $t \in (0, \delta')$,

$$\limsup_{n\to\infty}\langle f(x_t) - x_t, j(x_n - x_t)\rangle \leq \frac{\varepsilon}{2}.$$

On the other hand, By Theorem 2.1 we have $x_t \to p \in F(S_r) = N(A)$ as $t \to 0^+$. In addition, j is norm-to-weak* uniformly continuous on bounded subsets of E, so there exists $\delta'' > 0$ such that, for any $t \in (0, \delta'')$, we have

$$\begin{aligned}
&|\langle(f(p) - p, j(x_n - p)\rangle - \langle f(x_t) - x_t, j(x_n - x_t)\rangle| \\
&\leq |\langle f(p) - p, j(x_n - p)\rangle - \langle f(p) - p, j(x_n - x_t)\rangle| + |\langle f(p) - p, j(x_n - x_t)\rangle - \langle f(x_t) - x_t, j(x_n - x_t)\rangle| \\
&\leq \|f(p) - p\|\|j(x_n - p) - j(x_n - x_t)\| + (1+\alpha)\|\ x_t - p\|\|x_n - x_t\| \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Taking $\delta = \min\{\delta', \delta''\}$, for $t \in (0, \delta)$, we have that

$$\langle f(p) - p, j(x_n - p)\rangle \leq \langle f(x_t) - x_t, j(x_n - x_t)\rangle + \frac{\varepsilon}{2}.$$

Hence,

$$\limsup_{n\to\infty}\langle f(p) - p, j(x_n - p)\rangle \leq \varepsilon, \quad \text{where } \varepsilon > 0 \text{ is arbitrary,}$$

which yields that

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n-p)\rangle \le 0. \tag{20}$$

Now we prove that $\{x_n\}$ converges strongly to p. It follows from Lemma 1.1 and (2.13) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(y_n-p)\|^2\\ &\le (1-\alpha_n)^2\|y_n-p\|^2+2\alpha_n\langle f(x_n)-p, j(x_{n+1}-p)\rangle\\ &= (1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n\langle f(x_n)-f(p)+f(p)-p, j(x_{n+1}-p)\rangle\\ &\le (1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n\alpha\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\ &\le (1-\alpha_n)^2\|x_n-p\|^2+\alpha_n\alpha(\|x_n-p\|^2+\|x_{n+1}-p\|^2)+2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle \end{aligned}\tag{21}$$

which yields that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\|x_n-p\|^2+\frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-p\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(p)-p, j(x_{n+1}-p)\rangle\\ &= (1-\bar{\alpha}_n)\|x_n-p\|^2+\frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-p\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(p)-p, j(x_{n+1}-p)\rangle, \end{aligned}\tag{22}$$

where $\bar{\alpha}_n=\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$. By boundness of $\{x_n\}$ the condition (i) and Lemma 1.3, $\{x_n\}$ converges strongly to p. This completes the proof of Theorem 2.2.

Theorem 2.3. *Let E and α_n, β_n satisfy the conditions of Theorem 2.2. Let $A: E\to E$, be a continuous accretive mapping with $N(A)\neq\emptyset$. For given $x_0\in E$, let $\{x_n\}$ be generated by the algorithm (2.13), then $\{x_n\}$ converges strongly to a zero $p\in N(A)$ which solutes the variational inequality (2.2).*

Proof. It follows from Theorem I that A is $m-$accretive mapping. Then by Theorem 2.2 we know that Theorem 2.3 is true. This completes the proof of Theorem 2.3.

Theorem 2.4. *Let E and α_n, β_n satisfy the conditions of Theorem 2.2. Let $T: K\to E$, be a pseudocontractive mapping such that $(I-T)$ is $m-$accretive on K with $F(T)\neq\emptyset$. For given $x_0\in E$, let $\{x_n\}$ be generated by the algorithm*

$$\begin{cases} x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)y_n\\ y_n=\beta_n x_n+(1-\beta_n)S_{r_n}x_n, \end{cases}\tag{23}$$

where $S_{r_n}:=(1-\sigma)I+\sigma J_{r_n}$ with $J_{r_n}:=(I+r_n(I-T))^{-1}$ and $0<\sigma<1$. Then $\{x_n\}$ converges strongly to a a fixed point $p\in F(T)$ which solutes the variational inequality (2.2).

Proof. Let $A=(I-T)$, then A is $m-$accretive. Note that $N(A)=F(T)$, which yields that $N(A)=F(T)\neq\emptyset$. We complete the proof of Theorem 2.4 by Theorem 2.2.

If $\beta_n \equiv 0$, from Theorem 2.2-2.4 we have the following Corollary 2.5-2.7, respectively.

Corollary 2.5. *We choose K, E, A, S_{r_n}, r_n, α_n such that they satisfy the conditions of Theorem 2.2. For given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm (1.6), then $\{x_n\}$ converges strongly to $p \in N(A)$ which solutes the variational inequality (2.2).*

Corollary 2.6. *Let E and α_n satisfy the conditions of Theorem 2.2. Let $A : E \to E$, be a continuous accretive mapping with $N(A) \neq \emptyset$. For given $x_0 \in E$, let $\{x_n\}$ be generated by the algorithm (1.6) . Then $\{x_n\}$ converges strongly to a a zero $p \in N(A)$ which solutes the variational inequality (2.2).*

Corollary 2.7. *Let E and K and α_n satisfy the conditions of Theorem 2.2. Let $T : K \to E$, be a continuous pseudocontractive mapping with $F(T) \neq \emptyset$. For given $x_0 \in E$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_{r_n} x_n, \tag{24}$$

where $S_{r_n} := (1 - \sigma)I + \sigma J_{r_n}$ with $J_{r_n} := (I + r_n(I - T))^{-1}$ and $0 < \sigma < 1$. Then $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$ which solutes the variational inequality (2.2).

Remark 2.8. Since Corollary 2.5 is obtained under the coefficient α_n satisfying $\lim \alpha_n = 0$ and $\Sigma_{n=0}^{\infty} \alpha_n = \infty$, then it is an improvement of Theorem 3.2 of [3].

Example 2. *Let*

$$\alpha_n = \begin{cases} 0, & \text{if } n = 2k; \\ \frac{1}{n}, & \text{if } n = 2k - 1. \end{cases} \quad \text{and} \quad r_n = \begin{cases} \frac{1}{2}, & \text{if } n = 2k; \\ \frac{1}{2} - \frac{1}{n}, & \text{if } n = 2k - 1. \end{cases}$$

where k is some positive integer. Obviously, the coefficient α_n and r_n satisfy the condition of this paper. But because of $\Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| = \infty$, $\Sigma_{n=1}^{\infty} |r_{n+1} - r_n| = \infty$, hence the coefficient α_n and r_n do not satisfy the condition of Theorem 3.2 of [3].

Remark 2.9. If E is uniformly smooth then E is reflexive and has a uniformly Gâteaux differentiable norm with the property that every nonempty closed and bounded subset of E has the fixed point property for nonexpansive mappings(see, remark 3.5 of [10]). Thus, if E is a real uniformly smooth Banach space, then the results in this paper are true, too.

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