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A Weighted Hermite Hadamard Inequality for Steffensen–Popoviciu and Hermite–Hadamard Weights on Time Scales

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Abstract

In this paper, we present a weighted version of the Hermite–Hadamard inequality for convex functions on time scales, with weights that are allowed to take some negative values, these are the Steffensen–Popoviciu and the Hermite–Hadamard weights. We also present some applications of this inequality.

1 Introduction

In the past years, new developments in the theory and applications of dynamic derivatives on time scales emerged. This study can be considered an unification of the discrete theory with the continuous theory. Also, it is a tool of the utmost importance in many computational and numerical applications. A combined dynamic derivative, so called $\Diamond_\alpha$ (diamond-α) dynamic derivative, was introduced as a linear combination of the well-known Δ (delta) and ∇ (nabla) dynamic derivatives on time scales. Using the delta and nabla derivatives, the notions of delta and nabla integrals were defined. We assume, throughout this paper, that the basic notions of the time scales are well known and understood. For the basic rules of calculus on time scales, please refer to [1, 2, 4, 8, 12, 13].

The classical Hermite–Hadamard inequality gives an estimation, from below and from above, of the mean value of a convex function. The aim of

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this paper is to establish a full analogue of this inequality if we compute the mean value with the help of the delta, nabla and diamond-α integral and some classes of weights that are not necessarily positive.

The left hand side of the Hermite–Hadamard inequality is a special case of the Jensen inequality.

In [11], a generalized version of the diamond-α Jensen's inequality was proved, for positive weights:

Theorem 1 *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $g \in C([a,b]_{\mathbb{T}}, (c,d))$, $h \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |h(s)| \Diamond_\alpha s > 0$ and $f \in C((c,d), \mathbb{R})$ is convex, then*

$$f\left(\frac{\int_a^b |h(s)| g(s) \Diamond_\alpha s}{\int_a^b |h(s)| \Diamond_\alpha s}\right) \le \frac{\int_a^b |h(s)| f(g(s)) \Diamond_\alpha s}{\int_a^b |h(s)| \Diamond_\alpha s}.$$

In [4], *C. Dinu* proved several versions of the Hermite–Hadamard inequality for positive weights.

In section 2, we give our main results, regarding the weighted version of the Hermite–Hadamard inequality.

2 Hermite–Hadamard Inequality

Let $\mathbb{T}$ be a time scale and let $a, b \in \mathbb{T}$. The aim is to prove that the Hermite–Hadamard inequality is also true, even if there are used weights that are not positive on the entirely time scale interval. More exactly, the research focuses on the classes of weights that satisfy the following inequality:

$$\begin{aligned} f\left(\frac{\int_a^b g(t) w(t) \Diamond_\alpha t}{\int_a^b w(t) \Diamond_\alpha t}\right) &\le \frac{\int_a^b f(g(t)) w(t) \Diamond_\alpha t}{\int_a^b w(t) \Diamond_\alpha t} \\ &\le \frac{M - \frac{\int_a^b g(t) w(t) \Diamond_\alpha t}{\int_a^b w(t) \Diamond_\alpha t}}{M - m} f(m) + \frac{\frac{\int_a^b g(t) w(t) \Diamond_\alpha t}{\int_a^b w(t) \Diamond_\alpha t} - m}{M - m} f(M). \end{aligned} \tag{1}$$

An immediate version is given by nonnegative weights, but there are also some weights that are allowed to take negative values and they also extend that inequality; these are the α-Steffensen–Popoviciu weights for the left side of the inequality (1) and α-Hermite–Hadamard weights for the right side of (1).

2.1 The α-Steffensen–Popoviciu weights

Definition 1 *Let $\mathbb{T}$ be a time scale and $g : \mathbb{T} \to \mathbb{R}$ a continuous function. The continuous function $w : \mathbb{T} \to \mathbb{R}$ is an α-Steffensen–Popoviciu weight for*

g on $[a,b]_{\mathbb{T}}$ (abbreviated α-SP weight) if

$$\int_a^b w(t)\Diamond_\alpha t > 0 \quad \textit{and} \quad \int_a^b f(g(t))^+ w(t)\Diamond_\alpha t \geq 0, \tag{2}$$

for every $f : [m,M] \to \mathbb{R}$ continuous convex function, where $m = \inf_{t\in[a,b]_{\mathbb{T}}} g(t)$ and $M = \sup_{t\in[a,b]_{\mathbb{T}}} g(t)$.

Some conditions of end positivity for α-SP weights in the case that g is a nondecreasing function are necessary. Thus, the following Lemma is important; it improves Theorem 1.5.7 in [7] due to *T. Popoviciu* (see also [9]):

Lemma 1 *Let $\mathbb{T}$ be a time scale and $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ a nondecreasing continuous function and $f : [g(a), g(b)] \to \mathbb{R}$ be a piecewise linear convex function. Then f is the sum of an affine function of g and a linear combination, with positive coefficients, of translates of the absolute value function of g, that is,*

$$f(g(t)) = ug(t) + v + \sum_{i=1}^{N} c_i |g(t) - g(t_i)|,$$

where $u, v \in \mathbb{R}$ and $c_1, ..., c_N$ are suitable nonnegative coefficients.

Proof: The interval $[g(a), g(b)]$ is divided in N intervals $[x_0, x_1]$, $[x_1, x_2]$, ... , $[x_{N-1}, x_N]$, with $g(a) = x_0 < g(t_1) = x_1 ... < x_N = g(b)$, such that the restriction of f to each interval is affine. If $ux + v$ is the restriction of f to $[x_0, x_1]$ and $u_1 x + v_1$ is the restriction of f to $[x_1, x_2]$, then $u < u_1$ from the convexity of f and the same holds for each of the other intervals. Then $f(g(t)) - ug(t) - v$ is a nondecreasing convex function which is equal to 0 on $[t_0, t_1]$. Then we have a constant $c_1 \geq 0$ such that $f(g(t)) - ug(t) - v = c_1(g(t) - g(t_1))^+$ on $[t_0, t_2]$. Reiterating the argument, it yields:

$$f(g(t)) = ug(t) + v + \sum_{i=1}^{N-1} c_i (g(t) - g(t_i))^+,$$

where $c_1, ..., c_N$ are suitable nonnegative coefficients.

Replacing the positive part with modulus by means of the formula $y^+ = (|y| + y)/2$ ends the proof. ■

Using the same methods as *T. Popoviciu* in [10] and Lemma 1 we get the following lemma, which characterizes the α-SP weights for a nondecreasing function g on a time scale.

Lemma 2 *Let $\mathbb{T}$ be a time scale and w be a continuous function defined on $\mathbb{T}$ and $a, b \in \mathbb{T}$ such that $\int_a^b w(t)\Diamond_\alpha t > 0$. Then w is an α-Steffensen–Popoviciu* *weight for a nondecreasing continuous function* $g : [a, b]_{\mathbb{T}} \to \mathbb{R}$ *if and only if it verifies the following condition (called of end positivity):*

$$\int_a^x (g(x) - g(t))w(t)\Diamond_\alpha t \geq 0 \quad \text{and} \quad \int_x^b (g(t) - g(x))w(t)\Diamond_\alpha t \geq 0, \tag{3}$$

for every $x \in [a, b]_{\mathbb{T}}$.

If the following stronger (but more suitable) condition holds

$$0 \leq \int_a^x w(t)\Diamond_\alpha t \leq \int_a^b w(t)\Diamond_\alpha t \quad \text{for every } x \in [a, b]_{\mathbb{T}}, \tag{4}$$

then w *is also an* α*-Steffensen–Popoviciu weight for the nondecreasing continuous function* g.

Proof: It is obvious that (2) implies (3) since the positive part of any linear combination of t is a nonnegative continuous and convex function on $[a, b]_{\mathbb{T}}$.

The other implication is based on Lemma 1. If f is nonnegative piecewise linear convex function, then f is a finite combination with nonnegative coefficients of functions 1, $(g(t) - g(x))^+$ and $(g(x) - (t))^+$ and so, $\int_a^b f(g(t))w(t)\Diamond_\alpha t \geq 0$.

If f is just a continuous convex function, then it can be approximated by piecewise linear convex functions thus giving that (3) implies (2).

The implication (3) $\Rightarrow$ (4) is immediate using the fact that g is nondecreasing. ■

It is obvious that all the positive weights are α-SP weights, for any continuous function g and every $\alpha \in [0, 1]$. But there are some α-SP weights that are allowed to take negative values.

Corollary 1 *Let* $\mathbb{T} = \mathbb{R}$, $g : [a, b] \to \mathbb{R}$, $g(t) = t$ *and* $w : [a, b] \to \mathbb{R}$ *be a continuous function. Then* w *is a* SP *weight for* g *on* $[a, b]$ *if and only if*

$$\int_a^b w(t)\,\mathrm{d}t > 0, \quad \int_a^x (x - t)w(t)\,\mathrm{d}t \geq 0 \quad \text{and} \quad \int_x^b (t - x)w(t)\,\mathrm{d}t \geq 0,$$

for every $x \in [a, b]$. *If the following condition holds*

$$\int_a^b w(t)\,\mathrm{d}t > 0 \quad \text{and} \quad 0 \leq \int_a^x w(t)\,\mathrm{d}t \leq \int_a^b w(t)\,\mathrm{d}t \quad \text{for every } x \in [a, b],$$

then w *is also a* SP *weight for* g *on* $[a, b]$.

It follows that $w(t) = t^2 + a$ is an α-SP weight for $g(t) = t$ on $[-1, 1]$ if $a > -1/3$.

Corollary 2 *Let $\mathbb{T} = \mathbb{Z}$, $n \in \mathbb{N}$, $\alpha = 1$, $a = 1$, $b = n+1$, $g : \{1, ..., n+1\} \to \mathbb{R}$, $g(i) = x_i$, with $x_1 \leq x_2 \leq ... \leq x_n$ and $w : \{1, ..., n+1\} \to \mathbb{R}$ $w(i) = w_i$. Then w is an 1-SP weight for g on $\{1, ..., n+1\}$ (or, w_i are SP weights for x_i) if and only if*

$$\sum_{i=1}^{n} w_i > 0, \quad \sum_{i=1}^{m} w_i(x_m - x_i) \geq 0 \quad \text{and} \quad \sum_{i=m}^{n} w_i(x_i - x_m) \geq 0,$$

for all $m \in \{1, ..., n\}$. If the following condition holds

$$\sum_{i=1}^{n} w_i > 0 \quad \text{and} \quad 0 \leq \sum_{i=1}^{m} w_i \leq \sum_{i=1}^{n} w_i, \quad \text{for all} \;\; m \in \{1, ..., n\}.$$

then w_i are also SP weights for x_i.

We will denote $A([m, M])$ the set of all affine functions defined on $[m, M]$. For a continuous function $f : [m, M] \to \mathbb{R}$, we can attach its *lower envelope*

$$\underline{f}(x) = \sup\{h(x) | h \in A([m, M]), f \geq h\}$$

and its *upper envelope*

$$\overline{f}(x) = \inf\{h(x) | h \in A([m, M]), f \leq h\}.$$

Some of the properties of $\underline{f}$ can be seen also in [7, Lemma 4.2.2]. We will use here the convexity of $\underline{f}$ and the fact that $\underline{f} \leq f$ and also $f = \underline{f}$ for convex f.

If $g : \mathbb{T} \to [m, M]$ is a continuous function, we will denote

$$x_{g,w,\alpha} = \frac{\int_a^b g(t)w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}.$$

Obviously, $x_{g,w,\alpha} \in [m, M]$. If h is an affine function on a time scale $\mathbb{T}$, then there exist $u, v \in \mathbb{T}$ such that $h(t) = ut + v$. Then

$$\frac{\int_a^b h(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t} = \frac{\int_a^b (ug(t) + v)w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t} = h(x_{g,w,\alpha}).$$

The previous steps allow the following theorem:

Theorem 2 (A complete weighted Jensen inequality). *Let $a, b \in \mathbb{T}$ and $m, M \in \mathbb{R}$. If $g \in C([a,b]_{\mathbb{T}}, [m,M])$ and $w \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b w(t)\Diamond_\alpha t > 0$, then the following assertions are equivalent:*

(i) w is an α-SP weight for g on $[a,b]_{\mathbb{T}}$;

(ii) for every $f \in C([m,M], \mathbb{R})$ convex function, we have

$$f\left(\frac{\int_a^b g(t)w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right) \le \frac{\int_a^b f(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}. \tag{5}$$

Proof: The implication (i) $\Rightarrow$ (ii). Since f is convex, then $f = \underline{f}$ and, with the above notation, we have

$$\begin{aligned} f(x_{g,w,\alpha}) = \underline{f}(x_{g,w,\alpha}) &= \sup\{h(x_{g,w,\alpha}) | h \in A([m,M]), f \ge h\} \\ &= \sup\left\{\frac{\int_a^b h(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t} | h \in A([m,M]), f \ge h\right\} \\ &\le \frac{\int_a^b f(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}. \end{aligned}$$

For the last inequality, we have used $\int_a^b (f(g(t)) - h(g(t)))w(t)\Diamond_\alpha t \ge 0$ since w is an α-SP weight for g and $f - h \ge 0$ is convex.

The implication (ii) $\Rightarrow$ (i) is clear if we consider f^+ instead of f, in (5). ■

The inequality (5) gives a better version of Jensen's inequality for time scales, than Theorem 1 (see also [11]), since we allow w to take some negative values. Moreover, the weighted version of Jensen inequality is true if and only if that weight is an α-SP weight for g and so, we have a complete characterization of all the weights that make the Jensen's inequality true.

2.2 The α-Hermite–Hadamard weights

Definition 2 *Let $\mathbb{T}$ be a time scale and $g : \mathbb{T} \to \mathbb{R}$ a continuous function. The continuous function $w : \mathbb{T} \to \mathbb{R}$ is an α-Hermite–Hadamard weight for g on $[a,b]_{\mathbb{T}}$ (abbreviated α-HH weight) if*

$$\int_a^b w(t)\Diamond_\alpha t > 0 \quad \text{and} \quad \frac{\int_a^b f(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t} \le \frac{M - x_{g,w,\alpha}}{M-m} f(m) + \frac{x_{g,w,\alpha} - m}{M-m} f(M), \tag{6}$$

for every $f : [m, M] \to \mathbb{R}$ *continuous convex function, where* $m = \inf_{t\in[a,b]_{\mathbb{T}}} g(t)$ *and* $M = \sup_{t\in[a,b]_{\mathbb{T}}} g(t)$.

We will give some conditions of end positivity for α-HH weights in the case that g is a nondecreasing function, with the arguments used by *C. P. Niculescu* and *A. Florea* in [6].

Lemma 3 *Let* $\mathbb{T}$ *be a time scale and* w *be a continuous function defined on* $\mathbb{T}$ *and* $a, b \in \mathbb{T}$ *such that* $\int_a^b w(t)\Diamond_\alpha t > 0$. *Then* w *is an* α*-Hermite–Hadamard weight for a nondecreasing continuous function* $g : [a, b]_{\mathbb{T}} \to \mathbb{R}$ *on* $[a, b]_{\mathbb{T}}$ *if and only if it verifies the following condition :*

$$\frac{g(b)-g(s)}{g(b)-g(a)}\int_a^s (g(t)-g(a))w(t)\Diamond_\alpha t+\frac{g(s)-g(a)}{g(b)-g(a)}\int_s^b (g(b)-g(t))w(t)\Diamond_\alpha t \geq 0, \tag{7}$$

for every $s \in [a, b]_{\mathbb{T}}$.

Proof: Let $m = g(a)$ and $M = g(b)$. Using an easy approximation argument, it can be proven that the inequality (6) is true for all continuous convex functions $f \in C([m, M], \mathbb{R})$ if and only if it is true for all functions $f \in C^2([m, M], \mathbb{R})$.

In the same time, it is well known (see, for example, [7]) that every function from $C^2([m, M], \mathbb{R})$ can be represented in the form

$$f(x) = \frac{M-x}{M-m}f(m) + \frac{x-m}{M-m}f(M) + \int_m^M G(x, y)f''(y)\,\mathrm{d}y,$$

where

$$G(x, y) = -\begin{cases} \frac{(x-m)(M-y)}{M-m} & \textit{if } m \leq x \leq y \leq M, \\ \\ \frac{(y-m)(M-x)}{M-m} & \textit{if } m \leq y \leq x \leq M \end{cases}$$

is the Green function associated to the second order derivative operator $\mathrm{d}^2/\,\mathrm{d}t^2$ and the boundary condition $u(m) = u(M) = 0$. Substituting x by $g(t)$ and y by $g(s)$ and denoting $g'_+(t)$ the right derivative of g in t, we get

$$f(g(t)) = \frac{M-g(t)}{M-m}f(g(a))+\frac{g(t)-m}{M-m}f(g(b))+\int_a^b G(g(t), g(s))f''(g(s))g'_+(s)\,\mathrm{d}s,$$

where

$$G(g(t), g(s)) = -\begin{cases} \frac{(g(t)-m)(M-g(s))}{M-m} & \textit{if } a \leq t \leq s \leq b, \\ \\ \frac{(g(s)-m)(M-g(t))}{M-m} & \textit{if } a \leq s \leq t \leq b. \end{cases}$$

Multiplying the above equality by $w(t)$ and taking the diamond-α integral with respect to t, if $f \in C^2([m,M],\mathbb{R})$, we have

$$\begin{aligned}
&\frac{\int_a^b f(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t} - \frac{M - x_{g,w,\alpha}}{M-m} f(m) - \frac{x_{g,w,\alpha} - m}{M-m} f(M) \\
&\quad = \frac{1}{\int_a^b w(t)\Diamond_\alpha t} \int_a^b \left[f(g(t)) - \frac{M-g(t)}{M-m} f(m) - \frac{g(t)-m}{M-m} f(M) \right] w(t)\Diamond_\alpha t \\
&\quad = \frac{1}{\int_a^b w(t)\Diamond_\alpha t} \int_a^b \left[\int_a^b G(g(t),g(s)) f''(g(s)) g'_+(s)\,\mathrm{d}s \right] w(t)\Diamond_\alpha t \\
&\quad = \frac{1}{\int_a^b w(t)\Diamond_\alpha t} \int_a^b \left[f''(g(s)) g'_+(s) \int_a^b G(g(t),g(s)) w(t)\Diamond_\alpha t \right] \mathrm{d}s.
\end{aligned}$$

Since f is a convex function from $C^2([m,M])$, then $f'' \geq 0$ and since g is a nondecreasing function, then $g'_+ \geq 0$. So, the inequality (6) holds if and only if

$$\int_a^b G(g(t),g(s)) w(t)\Diamond_\alpha t \leq 0,$$

for all $s \in [a,b]_{\mathbb{T}}$, that is inequality (7). ■

Until now, we have characterized the weights that make the left part of the Hermite–Hadamard inequality true and also the ones for the right part. The question that arises naturally concerns the connection, if any, between this two classes of weights on time scales. The next theorem emphasizes the relation.

Theorem 3 *Let $\mathbb{T}$ be a time scale and $g : \mathbb{T} \to \mathbb{R}$ a continuous function. Every α-Steffensen–Popoviciu weight for g on $[a,b]_{\mathbb{T}}$ is an α-Hermite–Hadamard weight for g on $[a,b]_{\mathbb{T}}$, for all $\alpha \in [0,1]$.*

Proof: It suffices to prove that an α-Steffensen–Popoviciu weight for g on $[a,b]_{\mathbb{T}}$ makes true the right side of Hermite–Hadamard inequality. For that, we will define the functional

$$p : C([m,M]) \to \mathbb{R}, \quad p(f) = \frac{\int_a^b \overline{f}(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t},$$

where $\overline{f}$ is the upper envelope of f. As it was shown before (see, for example, [7]), p is sublinear. In the same time, we define the following linear functional

$$l : A([m,M]) \to \mathbb{R}, \quad l(h) = h(x_{g,w,\alpha}).$$

Obviously, $p = l$ on $A([m, M])$. According to the Hahn-Banach extension theorem, there exists a linear functional $L : C([m, M]) \to \mathbb{R}$ that extends l and

$$L(f) \leq p(f) \quad \text{for every} \quad f \in C([m, M]).$$

If f is a continuous function on $[m, M]$ with $f \leq 0$, then $L(f) \leq 0$. Indeed, if $f \leq 0$, then $\overline{f} \leq 0$ and $L(f) \leq p(f) \leq 0$, (since w is an α-SP weight for g). Moreover, if f is a continuous convex function on $[m, M]$, then $-f$ is concave and

$$L(-f) \leq p(-f) = \frac{\int_a^b \overline{-f}(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t} = \frac{\int_a^b -f(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t},$$

that is,

$$L(f) \geq \frac{\int_a^b f(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}, \tag{8}$$

for every continuous convex function f.

Now, we consider the affine function $h : [m, M] \to \mathbb{R}$ given by

$$h(x) = f(m) + \frac{f(M) - f(m)}{M - m}(x - m).$$

Obviously, we have $h(g(t)) \geq f(g(t))$ for all $t \in [a, b]_{\mathbb{T}}$. Then $L(f - h) \leq 0$ and, since L is linear, we have $L(f) \leq L(h) = l(h) = h(x_{g,w,\alpha})$, (since h is affine) which can be rewritten, using (8), as

$$\frac{1}{\int_a^b w(t)\Diamond_\alpha t} \int_a^b f(g(t))w(t)\Diamond_\alpha t \leq \frac{M - x_{g,w,\alpha}}{M - m} f(m) + \frac{x_{g,w,\alpha} - m}{M - m} f(M).$$

■

Corollary 3 (A weighted version of Hermite–Hadamard inequality) *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $g : \mathbb{T} \to [m, M]$ be a continuous function, $f : [m, M] \to \mathbb{R}$ be a continuous convex function and $w : \mathbb{T} \to \mathbb{R}$ be an α-Steffensen–Popoviciu weight for g on $[a, b]_{\mathbb{T}}$. Then*

$$f(x_{g,w,\alpha}) \leq \frac{1}{\int_a^b w(t)\Diamond_\alpha t} \int_a^b f(g(t))w(t)\Diamond_\alpha t \leq \frac{M - x_{g,w,\alpha}}{M - m} f(m) + \frac{x_{g,w,\alpha} - m}{M - m} f(M). \tag{9}$$

Remark 1 *If we consider concave functions instead of convex functions, the above Hermite–Hadamard inequality (9) is reversed.*

Corollary 4 (The continuous case). *Let $\mathbb{T} = \mathbb{R}$, $g : [a,b] \to [m,M]$ and $w : [a,b] \to \mathbb{R}$ be continuous functions such that w is a SP weight for g on $[a,b]$. Let $f : [m,M] \to \mathbb{R}$ be a continuous convex function. Then we have*

$$\begin{aligned} f\left(\frac{\int_a^b g(t)w(t)\,\mathrm{d}t}{\int_a^b w(t)\,\mathrm{d}t}\right) &\leq \frac{1}{\int_a^b w(t)\,\mathrm{d}t}\int_a^b f(g(t))w(t)\,\mathrm{d}t \\ &\leq \frac{M - \frac{\int_a^b g(t)w(t)\,\mathrm{d}t}{\int_a^b w(t)\,\mathrm{d}t}}{M-m} f(m) + \frac{\frac{\int_a^b g(t)w(t)\,\mathrm{d}t}{\int_a^b w(t)\,\mathrm{d}t} - m}{M-m} f(M). \end{aligned}$$

Corollary 5 (The discrete case). *Let $\mathbb{T} = \mathbb{Z}$, $n \in \mathbb{N}$, $\alpha = 1$, $a = 1$, $b = n+1$, $g : \{1,...,n+1\} \to \mathbb{R}$, $g(i) = x_i$, with $m = x_1 \leq x_2 \leq ... \leq x_{n+1} = M$ and $w : \{1,...,n+1\} \to \mathbb{R}$, $w(i) = w_i$, such that w_i are SP weights for x_i. Let $f : [m,M] \to \mathbb{R}$ be a convex function. Then we get*

$$\begin{aligned} f\left(\frac{\sum_{i=1}^n x_i w_i}{\sum_{i=1}^n w_i}\right) &\leq \frac{1}{\sum_{i=1}^n w_i}\sum_{i=1}^n f(x_i)w_i \\ &\leq \frac{M - \frac{\sum_{i=1}^n x_i w_i}{\sum_{i=1}^n w_i}}{M-m} f(m) + \frac{\frac{\sum_{i=1}^n x_i w_i}{\sum_{i=1}^n w_i} - m}{M-m} f(M). \end{aligned}$$

Corollary 6 (The quantum calculus case). *Let $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$, $a = q^l$, $b = q^n$ with $l < n$. Let $g : \{q^l, q^{l+1}, ..., q^n\} \to [m,M]$ and $w : \{q^l, q^{l+1}, ..., q^n\} \to \mathbb{R}$ an α-SP weight for g. For every convex function $f : [m,M] \to \mathbb{R}$, we have*

$$\begin{aligned} & f\left(\frac{\alpha\sum_{i=l}^{n-1} q^i g(q^i)w(q^i) + (1-\alpha)\sum_{i=l+1}^n q^i g(q^i)w(q^i)}{\alpha\sum_{i=l}^{n-1} q^i w(q^i) + (1-\alpha)\sum_{i=l+1}^n q^i w(q^i)}\right) \\ &\leq \frac{\alpha\sum_{i=l}^{n-1} q^i f(g(q^i))w(q^i) + (1-\alpha)\sum_{i=l+1}^n q^i f(g(q^i))w(q^i)}{\alpha\sum_{i=l}^{n-1} q^i w(q^i) + (1-\alpha)\sum_{i=l+1}^n q^i w(q^i)} \\ &\leq \frac{M - \frac{\alpha\sum_{i=l}^{n-1} q^i g(q^i)w(q^i) + (1-\alpha)\sum_{i=l+1}^n q^i g(q^i)w(q^i)}{\alpha\sum_{i=l}^{n-1} q^i w(q^i) + (1-\alpha)\sum_{i=l+1}^n q^i w(q^i)}}{M-m} f(m) \\ &\quad + \frac{\frac{\alpha\sum_{i=l}^{n-1} q^i g(q^i)w(q^i) + (1-\alpha)\sum_{i=l+1}^n q^i g(q^i)w(q^i)}{\alpha\sum_{i=l}^{n-1} q^i w(q^i) + (1-\alpha)\sum_{i=l+1}^n q^i w(q^i)} - m}{M-m} f(M). \end{aligned}$$

Example 1 *(i) Let $g : [a,b]_{\mathbb{T}} \to [m,M]$, with $m > 0$ and $f : (0,+\infty) \to (0,+\infty)$, $f(t) = t^\beta$. An easy calculus shows that f is convex on $(0,+\infty)$ for $\beta \in (-\infty,0] \cup [1,+\infty)$ and concave on $(0,+\infty)$ for $\beta \in [0,1]$. Let $w : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be an α-SP weight for g. Then*

$$\begin{aligned}\left(\frac{\int_a^b w(t)g(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right)^\beta &\le \frac{\int_a^b w(t)g^\beta(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\\ &\le \frac{M - \frac{\int_a^b w(t)g(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}}{M-m} m^\beta + \frac{\frac{\int_a^b w(t)g(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t} - m}{M-m} M^\beta,\end{aligned}$$

if $\beta \in (-\infty,0] \cup [1,+\infty)$ and

$$\begin{aligned}\left(\frac{\int_a^b w(t)g(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right)^\beta &\ge \frac{\int_a^b w(t)g^\beta(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\\ &\ge \frac{M - \frac{\int_a^b w(t)g(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}}{M-m} m^\beta + \frac{\frac{\int_a^b w(t)g(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t} - m}{M-m} M^\beta,\end{aligned}$$

if $\beta \in [0,1]$. In the cases $\beta = 0$ or $\beta = 1$, the above inequalities become equalities.

(ii) Let $g : [a,b]_{\mathbb{T}} \to [m,M]$, with $m > 0$ and $f : (0,+\infty) \to \mathbb{R}$, $f(t) = \log t$. Let $w : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be an α-SP weight for g. Since f is concave, we get

$$\begin{aligned}\log\left(\frac{\int_a^b w(t)g(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right) &\ge \frac{\int_a^b w(t)\log(g(t))\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\\ &\ge \frac{M - \frac{\int_a^b w(t)g(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}}{M-m} \log(m) + \frac{\frac{\int_a^b w(t)g(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t} - m}{M-m} \log(M).\end{aligned}$$

(iii) Let $\mathbb{T} = \mathbb{Z}$, $n \in \mathbb{N}$, $\alpha = 1$, $a = 1$, $b = n+1$, $g : \{1,...,n+1\} \to \mathbb{R}$, $g(i) = x_i$, with $m = x_1 \le x_2 \le ... \le x_{n+1} = M$ and $w : \{1,...,n+1\} \to \mathbb{R}$, $w(i) = w_i$, such that w_i are SP weights for x_i and we suppose that $\sum_{i=1}^n w_i = 1$, (otherwise, we divide each w_i by $\sum_{i=1}^{n+1} w_i$). Since $\log : [m,M] \to \mathbb{R}$ is a concave function, we have

$$\log\left(\alpha\sum_{i=1}^{n}x_iw_i+(1-\alpha)\sum_{i=2}^{n+1}x_iw_i\right)\geq\alpha\sum_{i=1}^{n}w_i\log(x_i)+(1-\alpha)\sum_{i=2}^{n+1}w_i\log(x_i)$$
$$\geq\frac{x_{n+1}-\alpha\sum_{i=1}^{n}x_iw_i-(1-\alpha)\sum_{i=2}^{n+1}x_iw_i}{x_{n+1}-x_1}\log(x_1)$$
$$+\frac{\alpha\sum_{i=1}^{n}x_iw_i+(1-\alpha)\sum_{i=2}^{n+1}x_iw_i-x_1}{x_{n+1}-x_1}\log(x_{n+1}),$$

that is,

$$\log\left(\alpha x_1w_1+\sum_{i=2}^{n}x_iw_i+(1-\alpha)x_{n+1}w_{n+1}\right)$$
$$\geq\alpha w_1\log(x_1)+\sum_{i=2}^{n}w_i\log(x_i)+(1-\alpha)w_{n+1}\log(x_{n+1})$$
$$\geq\frac{x_{n+1}-\alpha x_1w_1-\sum_{i=2}^{n}x_iw_i-(1-\alpha)x_{n+1}w_{n+1}}{x_{n+1}-x_1}\log(x_1)$$
$$+\frac{\alpha x_1w_1+\sum_{i=2}^{n}x_iw_i+(1-\alpha)x_{n+1}w_{n+1}-x_1}{x_{n+1}-x_1}\log(x_{n+1}),$$

or,

$$\alpha\sum_{i=1}^{n}x_iw_i+(1-\alpha)\sum_{i=2}^{n+1}x_iw_i\geq\prod_{i=1}^{n}(x_i)^{\alpha w_i}\prod_{i=2}^{n+1}(x_i)^{(1-\alpha)w_i}$$
$$\geq x_1^{\left(x_{n+1}-\alpha\sum_{i=1}^{n}x_iw_i-(1-\alpha)\sum_{i=2}^{n+1}x_iw_i\right)/(x_{n+1}-x_1)}$$
$$\cdot x_{n+1}^{\left(\alpha\sum_{i=1}^{n}x_iw_i+(1-\alpha)\sum_{i=2}^{n+1}x_iw_i-x_1\right)/(x_{n+1}-x_1)}.$$

If $\alpha=1$, then we obtain

$$\log\left(\sum_{i=1}^{n}x_iw_i\right)\geq\sum_{i=1}^{n}w_i\log(x_i)$$
$$\geq\frac{M-\sum_{i=1}^{n}x_iw_i}{M-m}\log(m)+\frac{\sum_{i=1}^{n}x_iw_i-m}{M-m}\log(M).$$

The left side of the above inequality gives a weighted version of the well-known arithmetic-mean geometric-mean inequality for SP weights:

$$\sum_{i=1}^{n} x_i w_i \geq \prod_{i=1}^{n} (x_i)^{w_i}$$

while the right side gives us:

$$\prod_{i=1}^{n} (x_i)^{w_i} \geq x_1^{\left(x_{n+1}-\sum_{i=1}^{n} x_i w_i\right)/(x_{n+1}-x_1)} \cdot x_{n+1}^{\left(\sum_{i=1}^{n} x_i w_i - x_1\right)/(x_{n+1}-x_1)}.$$

(iv) Let $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$, $a = 1$, $b = q^n$ and $\alpha = 1$. Let $g : \{1, q, ..., q^n\} \to [m, M]$, $g(q^i) = x_i$ and $w : \{1, q, ..., q^n\} \to \mathbb{R}$, $w(q^i) = w_i$, such that w_i are SP weights for x_i. For every convex function $f : [m, M] \to \mathbb{R}$, we have

$$\begin{aligned} f\left(\frac{\sum_{i=0}^{n-1} q^i x_i w_i}{\sum_{i=0}^{n-1} q^i w_i}\right) &\leq \frac{\sum_{i=0}^{n-1} q^i f(x_i) w_i}{\sum_{i=0}^{n-1} q^i x_i w_i} \\ &\leq \frac{M - \frac{\sum_{i=0}^{n-1} q^i x_i w_i}{\sum_{i=0}^{n-1} q^i w_i}}{M-m} f(m) + \frac{\frac{\sum_{i=0}^{n-1} q^i x_i w_i}{\sum_{i=0}^{n-1} q^i w_i} - m}{M-m} f(M). \end{aligned}$$

Using Theorem 2 we can give improved versions of the results obtained in [11], such as Hölder's inequality.

Theorem 4 (Hölder's inequality). *Let $\mathbb{T}$ be a time scale, $a < b \in \mathbb{T}$, $f, g \in C([a, b]_{\mathbb{T}}), [0, +\infty)$ and $w \in C([a, b]_{\mathbb{T}}), \mathbb{R}$ such that wg^q is an α-SP weight for $fg^{-q/p}$, where p and q are Hölder conjugates (that is, $1/p + 1/q = 1$ and $p > 1$). Then we have*

$$\int_a^b h(t)f(t)g(t)\Diamond_\alpha t \leq \left(\int_a^b h(t)f^p(t)\Diamond_\alpha t\right)^{1/p} \left(\int_a^b h(t)g^q(t)\Diamond_\alpha t\right)^{1/q}.$$

The proof follows the algorithm of the proof of Theorem 4.1 in [11], after we notice that we can apply Theorem 2 to the convex function $x \mapsto x^p$ for $p > 1$, and we replace g by $fg^{-q/p}$ and w by wg^q.

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