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From the elliptic regulator to exotic relations

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Abstract

In this paper we prove an identity between the elliptic regulators of some 2-isogenous elliptic curves. This allow us to prove a new exotic relation for the elliptic curve $20A$ of Cremona's tables. Also we prove the (conjectured) exotic relation for the curve $20B$ given by Bloch and Grayson in [3].

1 Introduction

For some elliptic curves, the elliptic dilogarithm satisfies linear relations, called exotic by Bloch and Grayson [3].
In [3] a list of elliptic curves that satisfies exotic relations is given. Recently some of these relations are proved by Bertin [1], Touafek [7].

We note that whenever we can find a tempered model of the elliptic curve, the existence of exotic relations is related to elements in the second group of the K-theory $K_2(E)$ hence to elliptic regulators, so we can prove the exotic relations.

Bloch and Grayson conjectured the following fact.

Conjecture 1 *Suppose that $E(\mathbb{Q})_{tors}$ is cyclic and $d = \#E(\mathbb{Q})_{tors} > 2$. Write Σ for the number of fibres of type I_ν with $\nu \geq 3$ in the Néron model, and suppose $\left[\frac{d-1}{2}\right] - \Sigma > 1$. Then there should be at least $\left[\frac{d-1}{2}\right] - \Sigma - 1$ exotic relations*

$$\sum_{r=1}^{\left[\frac{d-1}{2}\right]} a_r D^E(rP) = 0,$$

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where P is a d-torsion point and $a_r \in \mathbb{Z}$.

In particular, these conditions are satisfied for the curve $20A$ of Cremona's tables [4]: $E(\mathbb{Q})_{tors}$ cyclic, $d = 6$ and $\Sigma = 0$.

In section 3 we use an identity between regulators and some equalities between the elliptic dilogarithm to prove a new exotic relation for the elliptic curve $20A$ and also we prove the (conjectured) exotic relation for the curve $20B$ of Cremona's tables given by Bloch and Grayson in [3].

2 Preliminaries

Let E be an elliptic curve defined over $\mathbb{Q}$.

Throughout this paper, the notation $E = [a_1, a_2, a_3, a_4, a_6]$ means that the elliptic curve E is in the Weierstrass form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

2.1 The elliptic regulator

Let F be a field. By Matsumoto's theorem, the second group of K-theory $K_2(F)$ can be described in terms of symbols $\{f, g\}$, for $f, g \in F^*$ and relations between them.

The relations are

$$\begin{cases} \{f_1f_2, g\} & = \{f_1, g\} + \{f_2, g\} \\ \{f, g_1g_2\} & = \{f, g_1\} + \{f, g_2\} \\ \{1 - f, f\} & = 0. \end{cases}$$

For example, if v is a discrete valuation on F with maximal ideal $\mathfrak{M}$ and residual field k, the Tate's tame symbol

$$(x, y)_\nu \equiv (-1)^{\nu(x)\nu(y)} \frac{x^{\nu(y)}}{y^{\nu(x)}} \mod \mathfrak{M}$$

defines a homomorphism

$$\lambda_v : K_2(F) \longrightarrow k^*.$$

Let $\mathbb{Q}(E)$ be the rational function field of the elliptic curve E. To any $P \in E(\overline{\mathbb{Q}})$ is associated a valuation on $\mathbb{Q}(E)$ that gives the homomorphism

$$\lambda_P : K_2(\mathbb{Q}(E)) \longrightarrow \mathbb{Q}(P)^*$$

and the exact sequence

$$0 \to K_2(E) \otimes \mathbb{Q} \to K_2(\mathbb{Q}(E)) \otimes \mathbb{Q} \to \coprod_{P \in E(\overline{\mathbb{Q}})} \mathbb{Q}(P)^* \otimes \mathbb{Q} \to ...$$

By definition $K_2(E)$ is defined modulo torsion by

$$K_2(E) \simeq \ker \lambda = \bigcap_P \ker \lambda_P \subset K_2(\mathbb{Q}(E))$$

where

$$\lambda : K_2(\mathbb{Q}(E)) \otimes \mathbb{Q} \longrightarrow \coprod_{P \in E(\overline{\mathbb{Q}})} (\mathbb{Q}(P)^* \otimes \mathbb{Q}).$$

Definition 1 *A polynomial in two variables is tempered if the polynomial of the faces of its Newton polygon has only roots of unity.*

When drawing the convex hull of points $(i, j) \in \mathbb{Z}^2$ corresponding to the monomials $a_{i,j}x^iy^j$, $a_{i,j} \neq 0$, you also draw points located on the faces. The polynomial of the face is a polynomial in one variable t which is a combination of the monomials 1, t, t^2,.... The coefficients of the combination are given when going along the face, that is $a_{i,j}$ if the lattice point of the face belongs to the convex hull and 0 otherwise.

In particular, the polynomials

$$P_1(X_1, Y_1) = Y_1^2 + 2X_1Y_1 - X_1^3 + X_1$$

and

$$P_2(X_2, Y_2) = Y_2^2 + 2X_2Y_2 + 2Y_2 - (X_2 - 1)^3$$

are tempered, so we get $\{X_1, Y_1\} \in K_2(E_1)$ and $\{X_2, Y_2\} \in K_2(E_2)$, see Rodriguez-Villegas [5]. Here E_1 is the elliptic curve defined by $P_1(X_1, Y_1) = 0$ and E_2 the elliptic curve defined by $P_2(X_2, Y_2) = 0$.

Let f and g be in $\mathbb{Q}(E)^*$. Let us define

$$\eta(f, g) = \log|f|d(\arg g) - \log|g|d(\arg f).$$

Definition 2 *The elliptic regulator r of E is given by*

$$\begin{array}{rrcc} r: & K_2(E) & \longrightarrow & \mathbb{R} \\ & \{f, g\} & \longmapsto & \frac{1}{2\pi}\int_\gamma \eta(f, g) \end{array}$$

for a suitable loop γ generating the subgroup $H_1(E, \mathbb{Z})^-$ of $H_1(E, \mathbb{Z})$, where the complex conjugation acts by -1.

2.2 The elliptic dilogarithm

We have two representations for $E(\mathbb{C})$

$$\begin{array}{ccccc} E(\mathbb{C}) & \overset{\sim}{\longrightarrow} & \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) & \overset{\sim}{\longrightarrow} & \mathbb{C}^*/q^{\mathbb{Z}} \\ (\wp(u),\acute{\wp}(u)) & \longrightarrow & u(\mod \Lambda) & \longrightarrow & z=e^{2\pi iu} \end{array}$$

where $\wp$ is the Weierstrass function, $\Lambda=\{1,\tau\}$ the lattice associated to the elliptic curve and $q=e^{2\pi i\tau}$.

Definition 3 *The elliptic dilogarithm D^E [2] is defined by*

$$D^E(P)=\sum_{n=-\infty}^{n=+\infty} D(q^n z),$$

where $P\in E(\mathbb{C})$ is the image of $z\in\mathbb{C}^*$, $z=e^{2\pi iu}$, $u=\xi\tau+\eta$ and D is the Bloch-Wigner dilogarithm

$$D(x):=\Im Li_2(x)+\log|x|arg(1-x).$$

Remark 1 1. *The Bloch-Wigner dilogarithm is a function univalued, real analytic in $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$ [9].*

2. *There is a second representation of the elliptic dilogarithm given by Bloch [2], [10] in terms of Eisenstein-Kronecker series*

$$D^E(P)=\frac{(\Im\tau)^2}{\pi}\Re\Big(\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{\exp(\ 2\pi i(n\xi-m\eta))}{(m\tau+n)^2(m\bar{\tau}+n)}\Big). \tag{1}$$

3. *The elliptic dilogarithm can be extended to divisors on $E(\mathbb{C})$*

$$D^E((f))=\sum_i n_i D^E(P_i),$$

where

$$(f)=\sum_i n_i\,[P_i]\,.$$

2.3 The diamond operation

Let $\mathbb{Z}[E(\mathbb{C})]^-$ be the subgroup of $\mathbb{Z}[E(\mathbb{C})]$ modulo the equivalence relation

$$cl([-P])=-cl([P])\,.$$

Definition 4 *The diamond operation is defined by*

$$\begin{array}{rccl} \Diamond : & \mathbb{Z}\left[E(\mathbb{C})\right] \times \mathbb{Z}\left[E(\mathbb{C})\right] & \longrightarrow & \mathbb{Z}\left[E(\mathbb{C})\right]^{-} \\ & ((f),(g)) & \longmapsto & \sum_{i,j} n_i m_j cl([P_i - P_j]) \end{array}$$

where

$$(f) = \sum_i n_i \left[P_i\right] \text{ and } (g) = \sum_j m_j \left[P_j\right].$$

The following theorem [2], establishes the relation between the elliptic regulator and the elliptic dilogarithm.

Theorem 1 *The elliptic dilogarithm D^E can be extended to a morphism*

$$\mathbb{Z}\left[E(\mathbb{C})\right]^{-} \longrightarrow \mathbb{R}.$$

If f, g are functions on E and $\{f,g\} \in K_2(E)$, then

$$\pi r(\{f,g\}) = D^E((f)\Diamond(g));$$

in particular

$$D^E((f)\Diamond(1-f)) = 0.$$

3 An identity between regulators

Let E_1 be the elliptic curve, isomorphic to the curve $20A$, with equation

$$Y_1^2 + 2X_1Y_1 = X_1^3 - X_1$$

and let E_2 be the elliptic curve, isomorphic to the curve $20B$, with equation

$$Y_2^2 + 2X_2Y_2 + 2Y_2 = (X_2 - 1)^3.$$

Proposition 1 *We have*

$$\begin{array}{rcl} \pi r(\{X_1, Y_1\}) & = & -4D^{E_1}(P_1) - 4D^{E_1}(2P_1) \\ \pi r(\{X_2, Y_2\}) & = & 6D^{E_2}(P_2) + 6D^{E_2}(2P_2) \end{array}$$

where $P_1 = (-1,2)$ is the 6-torsion point of the curve $E_1 = [2,0,0,-1,0]$ and $P_2 = (5,4)$ is the 6-torsion point of the curve $E_2 = [2,-3,2,3,-1]$.

Proof. We need to compute the following divisors

$$\begin{array}{rr} (X_1) = & 2\left[3P_1\right] - 2\left[O_1\right] \\ (Y_1) = & \left[3P_1\right] + \left[4P_1\right] + \left[5P_1\right] - 3\left[O_1\right] \\ (X_2) = & 2\left[3P_2\right] - 2\left[O_2\right] \\ (Y_2) = & 3\left[2P_2\right] - 3\left[O_2\right] \end{array}$$

hence

$$(X_1)\Diamond(Y_1) = -4cl([P_1]) - 4cl([2P_1])$$

and

$$(X_2)\Diamond(Y_2) = 6cl([P_2]) + 6cl([2P_2])$$

so, by theorem 1 we get

$$\pi r(\{X_1, Y_1\}) = -4D^{E_1}(P_1) - 4D^{E_1}(2P_1)$$

and

$$\pi r(\{X_2, Y_2\}) = 6D^{E_2}(P_2) + 6D^{E_2}(2P_2).$$

■

Remark 2 *Using the previous proposition and formula (1), we have find by the computer,*

$$r(\{X_2, Y_2\}) \stackrel{?}{=} r(\{X_1, Y_1\}),$$

where the notation $A \stackrel{?}{=} B$, means " A is conjectured to be equal to B ", that is A and B are numerically equal to at least 25 decimal places.

Theorem 2 *We have the following identity*

$$r(\{X_2, Y_2\}) = r(\{X_1, Y_1\}).$$

Proof. Let $\Xi^1 = 20A$ be the elliptic curve with equation

$$S_1^2 = T_1^3 + T_1^2 - T_1$$

and Ξ^2 be the elliptic curve, isomorphic to $20B$, with equation

$$S_2^2 = T_2^3 - 2T_2^2 + 5T_2.$$

It is easy to check that

$$T_1 = X_1,\ S_1 = Y_1 + X_1 \tag{2}$$

and

$$T_2 = X_2,\ S_2 = Y_2 + X_2 + 1 \tag{3}$$

give isomorphisms

$$\Xi^1 \simeq E_1,\quad \Xi^2 \simeq E_2.$$

Also we have the 2-isogeny [6] given by

$$\begin{array}{llll} \Phi: & \Xi^1 & \longrightarrow \Xi^2 & \\ & (T_1, S_1) & \longmapsto \left(\frac{S_1^2}{T_1^2}, -\frac{S_1(T_1^2+1)}{T_1^2}\right). \end{array} \tag{4}$$

Using (2), (3) and (4) we can see Φ as

$$\begin{array}{lll}\Phi: & E_1 & \longrightarrow E_2 \\ & (X_1,Y_1) & \longmapsto (X^\Phi,Y^\Phi)=(\frac{(Y_1+X_1)^2}{X_1^2},-\frac{(Y_1+X_1+1)(Y_1+X_1^2+X_1)}{X_1^2}).\end{array}$$

The elliptic curve E_1 can be considered as a double cover of $\mathbf{P}^1$ by

$$\pi_{X_1}: E_1 \longrightarrow \mathbf{P}^1$$

ramified at the zeros of $X_1^3+X_1^2-X_1$, i.e. $0, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$. The closed curve $\sigma_1=\pi_{X_1}^{-1}([0,\frac{-1+\sqrt{5}}{2}])$ generates $H_1(E_1,\mathbb{Z})^-$.

The elliptic curve E_2 can be considered as a double cover of $\mathbf{P}^1$ by

$$\pi_{X_2}: E_2 \longrightarrow \mathbf{P}^1$$

ramified at the zeros of $X_2^3-2X_2^2+5X_2$, i.e. $0, 1+2i, 1-2i$. The closed curve $\sigma_2=\pi_{X_2}^{-1}([1-2i,1+2i])$ generates $H_1(E_2,\mathbb{Z})^-$.

Using the 2-isogeny we get

$$\begin{array}{ccc} 2P_1,\ \ 5P_1 & \xrightarrow{\Phi} & 2P_2 \\ P,\ \ Q & \xrightarrow{\Phi} & 3P_2 \\ P_1,\ \ 4P_1 & \xrightarrow{\Phi} & 4P_2 \\ 3P_1,\ \ O_1 & \xrightarrow{\Phi} & O_2 \end{array}$$

where

$$P=(-\varphi,\varphi),\quad Q=(-\frac{1}{\varphi},\frac{1}{\varphi}),\quad Q=P+3P_1,\ \varphi=\frac{1+\sqrt{5}}{2}.$$

Also, when (X_2,Y_2) describes σ_2, (X^Φ,Y^Φ) describes twice the closed curve

$$\sigma=\{(X_1,Y_1)\in E_1(\mathbb{C})/|X_1|=1\},$$

which generates $H_1(E_1,\mathbb{Z})^-$, because it's in the same homology class as σ_1. Hence,

$$r(\{X_2,Y_2\})=\pm\frac{1}{2}r\left(\left\{X^\Phi,Y^\Phi\right\}\right). \tag{5}$$

We have

$$\begin{array}{rcl}(1+X_1+Y_1) & = & [2P_1]+2\,[5P_1]-3\,[O_1] \\ (X_1+Y_1) & = & [3P_1]+[P]+[Q]-3\,[O_1] \\ (Y_1+X_1+X_1^2) & = & [3P_1]+[5P_1]+2[2P_1]-4\,[O_1]\,.\end{array}$$

Performing the necessary computation, we obtain

$$\begin{aligned}(X^{\Phi})\Diamond(Y^{\Phi}) &= -12cl([P_1]) + 12cl([2P_1]) + 12cl([P-2P_1]) + 12cl([P-2Q]) \\ &\quad +12cl([Q-2P_1]) + 12cl([Q-2P])\end{aligned}$$

and

$$\begin{aligned}(1+X_1+Y_1)\Diamond(X_1+Y_1) &= 5cl([P_1]) - cl([2P_1]) - 3cl([P-2P_1]) - 3cl([P-2Q]) \\ &\quad -3cl([Q-2P_1]) - 3cl([Q-2P]).\end{aligned}$$

Using the fact that

$$D^E((1+X_1+Y_1)\Diamond(X_1+Y_1)) = 0$$

we get

$$D^{E_1}(X^{\Phi}\Diamond Y^{\Phi}) = 8D^{E_1}(P_1) + 8D^{E_1}(2P_1)$$

so by Theorem 1 and Proposition 1

$$r\left(\left\{X^{\Phi}, Y^{\Phi}\right\}\right) = -2r\left(\left\{X_1, Y_1\right\}\right). \tag{6}$$

By (5), (6) and remark 2 we get

$$r\left(\left\{X_2, Y_2\right\}\right) = r\left(\left\{X_1, Y_1\right\}\right).$$

■

Let E_1, E_2 be as above. We have the following theorem [8].

Theorem 3 *We have the following equalities*

$$\begin{aligned} 1) \quad D^{E_1}(P_1) &= -2D^{E_2}(P_2) + 3D^{E_2}(2P_2) \\ 2) \quad D^{E_1}(2P_1) &= -2D^{E_2}(P_2) + 2D^{E_2}(2P_2). \end{aligned}$$

Proof. The proof follow the same way of the proof of Theorem 3.2 in [7]

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Now, we are able to give a new exotic relation for the curve $20A$.

Corollary 1 *We have the linear relation*

$$16D^{E_1}(P_1) - 11D^{E_1}(2P_1) = 0.$$

Proof. It results from Proposition 1 and Theorem 2 that

$$-4D^{E_1}(P_1) - 4D^{E_1}(2P_1) = 6D^{E_2}(P_2) + 6D^{E_2}(2P_2);$$

so by theorem 3, we get

$$16D^{E_1}(P_1) - 11D^{E_1}(2P_1) = 0.$$

■

Remark 3 *1. In turn, by Theorem 3, the relation*

$$16D^{E_1}(P_1) - 11D^{E_1}(2P_1) = 0$$

becomes

$$5D^{E_2}(P_2) - 13D^{E_2}(2P_2) = 0.$$

This achieves the proof of the (conjectured) exotic relation for the curve $20B$ *given by Bloch and Grayson in [3].*

2. In [3] only elliptic curves with negative discriminant are considered, so our new exotic relation does not appear in the list of Bloch and Grayson because the curve $20A$ *have a positive discriminant.*

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