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Bifurcation and numerical study in an EHD convection problem

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Abstract

A bifurcation study for the eigenvalue problem governing the stability of an oil insulated layer [7] in an EHD convection problem is performed. The analytical and numerical study is then completed using a Galerkin type spectral method based on Legendre and Chebyshev polynomials, leading to good numerical results.

1 The physical problem

The investigated physical model is one of the two EHD models of Roberts [10], based on the Gross'experiments [7] which are concerned with a layer of insulating oil confined between two horizontal conducting planes and heated from above and cooled from below. The experimental investigations showed that the presence of a vertical electric field of sufficient strength across the layer, lead to a tesselated pattern of motions, in a manner similar to that of the classical Bénard convection [13]. Gross [7] suggested that this phenomenon may be due to the variation of the dielectric constant of the fluid with the temperature.

In the first model investigated by Roberts [10] the dielectric constant is allowed to vary with the temperature. The homogeneous insulating fluid is assumed to be situated in a layer of depth d (the fluid occupies the region between the planes $z = \pm 0.5d$, which are maintained at uniform but different temperatures), with vertical, parallel applied gradients of temperature and electrostatic potential. The uniform electric field is applied in the z direction.

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For a constant density, ρ_0, Newtonian fluid (apart from the thermal buoyancy term) the momentum, the continuity and the energy equations governing the temperature field are

$$\begin{cases} u_{i,t} + u_j u_{i,j} = g_i + \nu \Delta u_i, \\ u_{i,i} = 0, \\ T_{,t} + u_i T_{,i} = \kappa \Delta T, \end{cases} \tag{1}$$

where $u_{i,t} \equiv \dfrac{\partial u_i}{\partial t}$, $u_j u_{i,j} \equiv u_j \dfrac{\partial u_i}{\partial x_j}$, $T_{,t} \equiv \dfrac{\partial T}{\partial t}$.

The physical parameters have the following meanings: κ is the thermal diffusivity coefficient, ν is the viscosity,

$$g_i = -w_{,i} - \frac{E^2}{8\pi\rho_0}(\frac{\partial \epsilon}{\partial T})_\rho T_{,i} - g[1 - \alpha(T - T_0)]k_i,$$

with g representing the gravity, α the thermal expansion coefficient, $k = (0,0,1)$, $w = \dfrac{p}{\rho_0} - \dfrac{E^2}{8\pi}(\dfrac{\partial \epsilon}{\partial T})_\rho$, ρ is the density, ϵ given for an isotropic material by $D = \epsilon E$, where D the electric displacement and E the electric field. The unknown fields are the velocity field u, the pressure p and the temperature T.

For an analysis based upon normal mode representations, the equation governing the EHD convection is [13]

$$(D^2 - a^2 - \sigma)(D^2 - a^2 - Pr\sigma)(D^2 - a^2)^2 F = La^4 F - R_a a^2 (D^2 - a^2) F \tag{2}$$

with the boundary conditions on F

$$\left.\begin{array}{l} F = D^2 F = D(D^2 - a^2 - Pr\sigma)F = 0 \\ ((D^2 - a^2)(D^2 - a^2 - \sigma)(D^2 - a^2 - Pr\sigma) + R_a a^2)(DF \pm kaF) = 0 \end{array}\right. \text{at } z = \pm 0.5. \tag{3}$$

Here, the unknown function F is the amplitude of the temperature perturbation Θ, i. e. $\Theta = F(z)e^{(i(lx+my)+st)}$, Pr is the Prandtl number, $k = \dfrac{\epsilon_m}{\widehat{\epsilon}}$, with ϵ_m the value of the dielectric field at the temperature $T_m = 0$ and $\widehat{\epsilon}$ the electric constant of the solid in $z > \dfrac{1}{2}$, a is the wavenumber, $a^2 = l^2 + m^2$, R_a is the Rayleigh number, L is a parameter measuring the potential difference between the planes. Roberts [10] investigated only the stationary case, i.e. $\sigma = 0$, so the eigenvalue problem consists from an eight-order differential equation

$$(D^2 - a^2)^4 F - La^4 F + R_a a^2 (D^2 - a^2) F = 0 \tag{4}$$

and the boundary conditions

$$F = D^2F = D(D^2-a^2)F = ((D^2-a^2)^3+R_a a^2)(DF \pm kaF) = 0 \text{ at } z = \pm 0.5. \tag{5}$$

He found that when the smallest Rayleigh number, $R_{a_{\min}} = \min_a R_a(a)$, varies from -1000 to 1707.762, L decreases from 3370.077 to 0.

The second model [10] was also been investigated by Turnbull [14], [15]. In this case, the variation of the dielectric constant is not important but the fluid is weakly conducting and its conductivity varies with temperature.

The eigenvalue equation has the form [13]

$$(D^2 - a^2)^3F + R_a a^2F + Ma^2DF = 0 \tag{6}$$

with M a dimensionless parameter measuring the variation of the electrical conductivity with temperature. The boundary conditions, written for the case of rigid boundaries at constant temperatures, read

$$F = D^2F = D(D^2 - a^2)F = 0 \text{ at } z = \pm 0.5 \tag{7}$$

P.H. Roberts [10] found that when M is increasing from 0 to 1000, $R_{a_{\min}}$ is increasing from 1707.062 to 2065.034.

B. Straughan [13] also investigated these EHD convection problems, developing a fully nonlinear energy stability analysis for non-isothermal convection problems in a dielectric fluid.

2 The bifurcation analysis

The linear stability of the motion or the equilibrium of a fluid in many problem from hydrodynamic, electrohydrodynamic or hydromagnetic stability theory is governed by a linear higher-order ordinary differential equation with constant coefficients and homogeneous boundary conditions. The exact solution of such equations or, for the case of eigenvalue problems, the exact eigenvalues are most of the times impossible to find. That is why, numerical methods, usually implying an infinite number of terms, leading however to an approximative solution by some specific truncations to a finite number of terms, are used. However, the theoretical methods can impose restrictions regarding to the numerical results.

For the considered problem let us introduce the direct method [4] based on the roots of the characteristic equation.

The characteristic equation associated to (6) is

$$(\lambda^2 - a^2)^3 + Ma^2\lambda + R_a a^2 = 0 \tag{8}$$

The general form of the solution of the two-point problem (6) -(7) is written in terms of the roots of the characteristic equation (8) associated with the differential equation (6). In addition, this form depends on the multiplicity of the characteristic roots. Introducing the general solution into the boundary conditions, the secular equation is obtained and it depends on the multiplicity of the characteristic roots. As a consequence, the secular equation has different forms in different regions of the parameter space. Each eigenvalue is a solution of the obtained secular equation, so the eigenvalue depends on all other physical parameters. The neutral manifolds (the most convenient manifolds from the physical point of view), generated by the secular equation separate the domain of stability from the domain of instability.

Let us consider the general case when the roots of the characteristic equation λ_1, λ_2, ..., λ_6 are distinct. Then the general solution of (6) has the form $F(z) = \sum\limits_{i=1}^{6} A_i e^{\lambda_i z}$. Introducing it into the boundary conditions (7) we obtain the secular equation [2]

$$\Delta(a, M, R_a) = 0, \tag{9}$$

where Δ is a determinant. Its i-th column has the same form in λ_i as any other j-th column in λ_j. If $\lambda_i = \lambda_j$, then the i-th and the j-th columns in Δ are identical. Therefore $\Delta \equiv 0$. In fact, in this situation, (9) is not entitled to serve as a secular equation and the direct numerical computations is invalid. In fact, when the characteristic equation has multiple roots the straightforward application of numerical method can lead in some cases to false secular points [4]. When $M \neq 0$, some particular cases interesting from the bifurcation point of view, arise due to the existence of bifurcation sets of the characteristic manifold.That is why, these cases have been investigated separately.

Proposition 1 *For $M = 0$, the only secular points are those situated on*

$$NS_n : R_a = \frac{((2n-1)^2\pi^2 + a^2)^3}{a^2}, \ \forall n \in \mathbb{N}.$$

Proof.

For $M = 0$, the characteristic equation (8) reduces to

$$(\lambda^2 - a^2)^3 + a_2 = 0, \ \text{with } a_2 = R_a a^2. \tag{10}$$

In this classical case [1], the roots of (10) have the form

$$\lambda_{1,2} = \sqrt{a^2 + \sqrt[3]{-a_2}\epsilon_{1,2}}, \ \lambda_3 = \sqrt{a^2 + \sqrt[3]{-a_2}},$$

$$\lambda_4 = -\lambda_1, \lambda_5 = -\lambda_2, \lambda_6 = -\lambda_3$$

so the general solution of (6) has the form

$$F = \sum_{i=1}^{3} A_i \cosh(\lambda_i z) + B_i \sinh(\lambda_i z).$$

Replacing the solution F into the boundary conditions (7), we get the secular equation

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & m_1 & m_2 & m_3 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 m_1 & \lambda_2^2 m_2 & \lambda_3^2 m_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & 0 & 0 & 0 \\ -\lambda_1\mu_1 m_1 & -\lambda_2\mu_2 m_2 & -\lambda_3\mu_3 m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \end{vmatrix} = 0, \tag{11}$$

with $m_i = \tanh(\lambda_i/2)$, $\mu_i = \lambda_i^2 - a^2$, $i = 1, 2, 3$.

When $\cosh(\lambda_i/2) \neq 0$, $i = 1, 2, 3$, we can rewrite the secular equation as $\Delta = \Delta_1 \cdot \Delta_2$ with $\Delta_1 = \lambda_1\mu_1 m_1(\lambda_2^2 - \lambda_3^2) + \lambda_2\mu_2 m_2(\lambda_3^2 - \lambda_1^2) + \lambda_3\mu_3 m_3(\lambda_1^2 - \lambda_2^2)$ and $\Delta_2 = \lambda_1^2 m_1(\lambda_3\mu_3 m_2 - \lambda_2\mu_2 m_2) + \lambda_2^2 m_2(\lambda_1\mu_1 m_3 - \lambda_3\mu_3 m_1) + \lambda_3^2 m_3(\lambda_2\mu_2 m_1 - \lambda_1\mu_1 m_2)$. For $a > 0$, the equations $\Delta_1 = 0$ and $\Delta_2 = 0$ have only null solutions $R = 0$, so no secular points exists on these surfaces.

The condition $\cosh(\lambda_i/2) \neq 0$, $i = 1, 2, 3$ is not fulfilled only for $i = 3$, i.e. $\cosh(\lambda_3/2) = 0 \Leftrightarrow \cos(\lambda_3/2) = 0 \Leftrightarrow \lambda_3^2 = -(2n-1)^2\pi^2$, which implies that the secular curve is NS_n. And, indeed, the critical values of the Rayleigh number R_a belong to NS_1 only, identical to the classical one from Chandrasekhar [1].

Let us consider the surface S_0 defined by the points $(a, M, R_a) = (a, M, a^4)$. In this case we have the following result.

Proposition 2 *Let us define the surfaces*

$$S_i : M = \frac{(33 \mp 3\sqrt{21})\sqrt{90 \pm 10\sqrt{21}}a^3}{250}, \ i = 1, 2.$$

The surface $S_0 \cap S_i$, $i = 1, 2$, is a bifurcation set of the characteristic manifold defined by (8). The points on $S_0 \cap S_i$, $i = 1, 2$, are not secular.

Proof. If $(a, M, R_a) \in S_0$ then $R_a = a^4$ and one of the roots of the characteristic equation is, for instance, $\lambda_1 = 0$. Assuming that $M \neq 0$ and that the wavenumber a is also a positive number, $a > 0$, λ_1 is not a double root of (8). The search of multiple roots reduces then to the equation

$$\lambda^5 - 3\lambda^4 a^2 + 3\lambda^2 a^4 + M a^2 = 0. \tag{12}$$

No multiple roots of algebraic multiplicity order greater than 2 exists. The double roots of (12) must also be roots of its derivative. In these conditions the possible double roots are $\lambda_{2,3} = \lambda = -\dfrac{1}{10}\sqrt{90 \pm 10\sqrt{21}a}$ only for $(a, M, R_a) \in S_i$, $i = 1, 2$, i.e. the surfaces $S_0 \cap S_i$, $i = 1, 2$, are bifurcation sets for the characteristic equation (8). In the case of multiple roots, the general form of the solution of (6) is $F(z) = \sum\limits_{i=1}^{n} P_i(z)e^{\lambda_i z}$, where P_i is an algebraic polynomial of $m_i - 1$ degree, m_i being the algebraic multiplicity of λ_i, in our case $F(z) = A + (B + Cz)e^{\lambda}z + \sum\limits_{i=4}^{6} A_i e^{\lambda_i z}$.

Formally, the secular equation is deduced from (9), by writing the column i for λ_i while the columns $i+1$, $i+2$,..., $i+m_i-1$ are obtained by differentiating l, $l = 1, 2, ..., m_i - 1$ times the $(i+l)$-th column of (9) with respect to λ_{i+l} and then replacing λ_{i+l} by λ_i.

However, the numerical evaluations show that secular points exists on these surfaces.

3 Study based on spectral methods

The second part of our study regards the numerical treatment of the two-point problem (6) - (7) using spectral methods.

Large classes of eigenvalue problems can be solved numerically using spectral methods, where, typically, the various unknown fields are expanded upon sets of orthogonal polynomials or other orthogonal functions. The convergence of such methods is in most cases easy to assure and they are efficient, accurate and fast. Our numerical study is performed using a weighted residual (Galerkin type) spectral method.

Introducing the new function $U = (D^2 - a^2)F$, the generalized eigenvalue problem

$$\begin{cases} (D^2 - a^2)^2 U = -R_a a^2 F - M a^2 DF, \\ (D^2 - a^2)F = U, \\ F = U = DU = 0 \text{ at } z = \pm 0.5 \end{cases} \tag{13}$$

is obtained. Following [8], we consider the orthogonal sets of functions

$$\{\phi_i\}_{i=1,2,...,N} : \phi_i(z) = \int_{-0.5}^{z} L_i^*(t)dt, \text{ verifying } \phi_i(\pm 0.5) = 0,$$

$$\{\beta_i\}_{i=1,2,...,N} : \beta_i(z) = \int_{-0.5}^{z}\int_{-0.5}^{s} L_i^*(t)dtds,$$

verifying

$$\beta_i(\pm 0.5) = D\beta_i(\pm 0.5) = 0,$$

with $L_i^* = L_i(2x)$ the shifted Legendre polynomials on $(-0.5, 0.5)$ and L_i the Legendre polynomials on $(-1, 1)$.

The unknown functions from (13), U, F, are written as truncated series of functions β_i, and, respectively ϕ_i,

$$U = \sum_{i=1}^{N} U_i \beta_i(z), \;\; F = \sum_{i=1}^{N} F_i \phi_i(z).$$

The boundary conditions on U and F are automatically satisfied. Replacing these expressions in (13), imposing the condition of orthogonality on the vectors $(\beta_k, \phi_k)^T$, $k = 1, 2, ..., N$, we get an algebraic system in the unknown, but not all null, coefficients U_i, F_i. The secular equation, written as the determinant of the obtained algebraic system, gives us the values of the Rayleigh number as a function of the other physical parameters. The smallest values of the Rayleigh number for various values of the parameters a and M form the neutral surface that separates the domain of stability from the instability domain. All the expression of the scalar products resulting in the algebraic system are given in [3] for the case of shifted Legendre polynomials on $(0, 1)$, but they are easy to adjust to the interval $(-0.5, 0.5)$. The specific choice of basis functions led to sparse matrices, with banded sub-matrices of dimension $N \times N$.

The numerical evaluations of the critical Rayleigh numbers were obtained for a small number of terms N ($N = 6$) in the truncated series confirming the well-known accuracy of spectral methods. We obtained that critical values of R_a are increasing from 1734.120 to 2082.808 when M is increasing from 0 to 1000, similar to the ones of Roberts [10].

The unknown vector fields from (13) can also be expanded upon complete sequences of functions in $L^2(-0.5, 0.5)$ defined by using Chebyshev polynomials that satisfy the boundary conditions of the problem. Keeping the above notations, the functions ϕ_i, $i = 1, 2, ..., N$, are defined by $\phi_i(z) = T_i^*(z) - T_{i+2}^*(z)$ and β_i, $i = 1, 2, ..., N$, by $\beta_i(z) = T_i^*(z) - \dfrac{2(i+2)}{i+3} T_{i+2}^*(z) + \dfrac{i+1}{i+3} T_{i+4}^*(z)$ [12] with T_i^*, $i = 1, 2, ..., N$, the shifted Chebyshev polynomials on $(-0.5, 0.5)$ defined in a similar manner as the shifted Legendre polynomials. All the evaluations of the scalar products were based on the orthogonality relations

$$\int_{-0.5}^{0.5} T_n^*(z) T_m^*(z) w^*(z) dz = \begin{cases} \dfrac{\pi}{2} c_n \delta_{nm}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{14}$$

with respect to the weight function $w^*(z) = \dfrac{1}{\sqrt{1/4 - z^2}}$.

a	M	$R_a - SLP$	$R_a - SCP$
3.117	0	1734.120	1775.955
3.117	10	1734.154	1775.987
3.117	1000	2082.802	2100.935
1.5	0	3116.286	31199.286
1.5	5	3116.381	3199.289
10	0	11409.157	14909.559
10	100	14414.05	14419.963
10	500	14531.694	14994.747
20	0	166779.036	182878.881

Table 1: Numerical values for the Rayleigh number for various values of the parameters a, M obtained by spectral methods based on shifted Legendre (SLP) and shifted Chebyshev (SCP) polynomials.

The numerical results were obtained for a larger number of terms in the expansion sets ($N = 11$) and they show that the shifted Legendre based method is more effective in this case. We can mention that the Chebyshev polynomials are considered suitable more likely for the tau method or the collocation type methods. Some numerical evaluations of R_a as a function of a and M are given in Table 1.

Other sets of complete orthogonal functions based on Chebyshev polynomials and satisfying various boundary conditions can be found in [6], [9].

4 Conclusions

In this paper we performed a bifurcation analysis and a numerical treatment for an electrohydrodynamic convection problem. When eigenvalue problems from linear stability theory are investigated only numerically spurious eigenvalue can be encountered, especially when bifurcation sets of the characteristic manifold occur. In order to detect the false secular points a bifurcation study of the problem becomes necessary. An example of this type of problems was investigated in [5], e.g. for an electrohydrodynamic convection problem in the case of free-free boundaries the numerical methods led to the existence of false secular points.

The numerical study was performed here by using a Galerkin type spectral method which implied that the boundary conditions are satisfied by the orthogonal sets of expansion functions. However, when this condition is not fulfilled, the tau method or the collocation method can also be applied. All these methods are widely used in the numerical investigation of eigenvalue problems governing the linear stability of motions or equilibrium of fluid in convection problems. From the physical point of view, the evaluations of the Rayleigh number R_a showed an enlargement of the stability domain when the parameter M is increasing, the dependence of R_a of M is not however exponential. These evaluations were easy to compute and proved to be similar to the ones existing in the literature.

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