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Weight and metrizability of inverses under hereditarily irreducible mappings

Ivan LONČAR

Abstract

The main purpose of this paper is to study the weight under hereditarily irreducible mappings between continua. The main result states that if $f : X \to Y$ is an hereditarily irreducible and surjective mapping of a D-continuum X, then $w(X) = w(Y)$.

1 Introduction

A topological space X is called a *compact space* [5, p. 165] if X is a Hausdorff space and every open cover of X has a finite subcover.

Definition 1.1 *Let X be a compact space. The weight of a space X is the least cardinal of a basis for X and is denoted by* $\mathrm{w}(X)$.

Let X, Y be compact spaces. A mapping $f : X \to Y$ is *light* (*zero-dimensional*) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [5, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger that one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

The problem of estimating the weight of inverses under light mappings has been investigated by Mardešić [17, Theorem 1, p. 162]. His result reads as follows.

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Theorem 1.1 *Let X and Y be two compact spaces, and $f : X \to Y$ a continuous light mapping onto Y. If X is locally connected, then* $\mathrm{w}(X) = \mathrm{w}(Y)$ *whenever* $\mathrm{w}(Y)$ *is infinite; if* $\mathrm{w}(Y)$ *is finite, then* $\mathrm{w}(X)$ *is finite too, and* $\mathrm{w}(X) \geq \mathrm{w}(Y)$.

An analogue of this theorem for non-compact spaces has been given by Proizvolov [21].

In the paper [6], the assumption that X is a locally connected space is replaced by the assumption that Y is locally connected and that f satisfies some additional conditions, i.e., that f is locally confluent.

The following very interesting result has been obtained by Tuncali [22, Theorem 1.4, p. 465].

Theorem 1.2 *Let $f : X \to Y$ be a light mapping of a non-degenerate continuum X onto a space Y. If X admits a basis of open sets whose boundaries have wight $\leq w(Y)$, then $w(X) = w(Y)$.*

The notion of an irreducible mapping was introduced by Whyburn [23, p. 162]. If X is a continuum, a surjection $f : X \to Y$ is *irreducible* provided no proper subcontinuum of X maps onto all of Y under f. Some theorems for the case when X is semi-locally-connected are given in [23, p. 163].

Definition 1.2 *A mapping $f : X \to Y$ is said to be hereditarily irreducible* [19, p. 204, (1.212.3)] *provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.*

Every hereditarily irreducible mapping is light.

Let X be a topological space. We define its hyperspaces as the following sets:

$$\begin{aligned}
&2^X = \{F \subseteq X : F \text{ is closed and nonempty}\},\\
&C(X) = \{F \in 2^X : F \text{ is connected}\},\\
&C^2(X) = C(C(X)),\\
&X(n) = \{F \in 2^X : F \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}.
\end{aligned}$$

For any finitely many subsets S_1, ..., S_n, let

$$\langle S_1, ..., S_n \rangle = \left\{ F \in 2^X : F \subset \bigcup_{i=1}^{n} S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i \right\}.$$

The topology on 2^X is the Vietoris topology, i.e., the topology with a base $\{< U_1, ..., U_n >: U_i$ is an open subset of X for each i and each $n < \infty$ $\}$, and $C(X), X(n)$ are subspaces of 2^X. Moreover, $X(1)$ is homeomorphic to X.

Let X and Y be topological spaces and let $f : X \to Y$ be a mapping. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [18, p. 170, Theorem 5.10], 2^f is continuous and $2^f(C(X)) \subset C(Y)$, $2^f(X(n)) \subset Y(n))$. The restriction of 2^f to $C(X)$ is denoted by $C(f)$.

Proposition 1 [19, p. 204, (1.212.3)] *If $f : X \to Y$ is a mapping between continua, then $C(f) : C(X) \to C(Y)$ is light if and only if f is hereditarily irreducible.*

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [19, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

a) if $\{A\}, \{B\} \in \Lambda$ such that $A \subset B, A \neq B$, then $g(\{A\}) < g(\{B\})$ and

b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([19, pp. 24-26], [9, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [2]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [2]. Moreover, if X is a non-metrizable locally connected or a rim-metrizable continuum, then X admits no Whitney map for $C(X)$ [11, Theorem 8, Theorem 11].

The following external characterization of non-metric continua which admit a Whitney map for $C(X)$ is given in [12, Theorem 2.3] and uses hereditarily irreducible mappings.

Theorem 1.3 *Let X be a non-metric continuum. Then X admits a Whitney map for $C(X)$ if and only if for each σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of continua which admit Whitney maps for $C(X_a)$ and $X = \lim \mathbf{X}$ there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible.*

Hereditarily irreducible mappings play an important role in the dissertation [7] and in the paper [8].

Definition 1.3 [7, Definition 3.1., p. 22] *Let $f : X \to Y$ be a continuous function between continua. Then f is said to be Whitney preserving if there are Whitney maps $\mu : C(X) \to \mathbb{R}$ and $\upsilon : C(Y) \to \mathbb{R}$ such that for every real number $s \in [0, \mu(X)]$, $C(f)(\mu^{-1}(s)) = \upsilon^{-1}(t)$ for some $t \in [0, \upsilon(Y)]$.*

Definition 1.4 [7, Definition 3.14.] *A Whitney preserving mapping, $f : X \to Y$ between continua, is said to be strictly Whitney preserving if for any two different Whitney levels $\mu^{-1}(s)$ and $\mu^{-1}(r)$ of $C(X)$ we have that $C(f)(\mu^{-1}(s)) \cap C(f)(\mu^{-1}(r)) = \emptyset$. In other words, the images of two different Whitney levels under $C(f)$ are different Whitney levels.*

Strictly Whitney preserving mappings are related to hereditarily irreducible mappings.

Theorem 1.4 [7, Theorem 3.16.]. *If $f : X \to Y$ is strictly Whitney preserving, then f is hereditarily irreducible.*

Proposition 2 [7, Proposition 3.18.] *Let $f : X \to Y$ be hereditarily irreducible. If f is Whitney preserving, then f is strictly Whitney preserving.*

Lemma 1.5 *If $f : X \to Y$ is a hereditarily irreducible and monotone mapping between continua, then f is one-to-one.*

It is clear that the lightness of $2^f : 2^X \to 2^Y$ implies the lightness of $C(f) : C(X) \to C(Y)$, but not conversely. The following result is known.

Theorem 1.6 [1, Theorem 5.4] *Let continua X and Y and a mapping $f : X \to Y$ be given. Consider the following conditions:*

(3.11) $C(f) : C(X) \to C(Y)$ *is light.*

(5.3) *For every two continua $P, Q \in C(X) \setminus X(1)$ with $P \cap Q = \emptyset$ the inequality $f(P) \setminus f(Q) \neq \emptyset$ holds.*

(3.12) $2^f : 2^X \to 2^Y$ *is light.*

Then (3.12) *implies* (5.3)*, and* (5.3) *implies* (3.11). *Consequently,* (3.12) *implies* (3.11). *The other implications do not hold.*

A family $\mathcal{N} = \{M_s : s \in S\}$ of subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighbourhood U of x there exists an $s \in S$ such that $x \in M_s \subset U$ [5, p. 170]. The *network weight* of a space X is defined as the smallest cardinal number of the form $\mathrm{card}(\mathcal{N})$, where $\mathcal{N}$ is a network for X; this cardinal number is denoted by $nw(X)$.

Theorem 1.7 [5, p. 171, Theorem 3.1.19] *For every compact space X we have $nw(X) = w(X)$.*

In the sequel we shall use the following result [20, p.226, Exercise 11.52].

Lemma 1.8 *If X is a continuum and if A and B are mutually disjoint subcontinua of X, then there is a component K of $X \setminus (A \cup B)$ such that $\mathrm{Cl}\, K \cap A \neq \emptyset$ and $\mathrm{Cl}\, K \cap B \neq \emptyset$.*

2 Hereditarily irreducible mappings onto arboroids

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points (end points) x, y. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

A well-known theorem of G. T. Whyburn [23, Theorem 2.4, p. 188] says that, if $f : X \to Y$ is a light open mapping from a compact space X onto Y, and a dendrite D is contained in Y , then for each point $x_0 \in f^{-1}(D)$ there is a dendrite $D' \subset X$ with $x_0 \in D'$ such that f maps D' homeomorphically onto D. This result has been extended in several ways (see e.g. [15] and [16]). It is shown in [3] that the property considered in Whyburn's theorem characterizes dendrites among all continua. In the paper [4] the characterization is further generalized.

In this section we will consider hereditarily irreducible mappings onto arboroids and we will show result similar in some sense to Whyburn's theorem.

An *arboroid* is a hereditarily unicoherent arcwise connected continuum. A metrizable arboroid is a *dendroid*. If X is an arboroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in X with endpoints x and y.

A point t of an arboroid X is said to be a *ramification point* of X if t is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.

If an arboroid X has only one ramification point t, it is called a *generalized fan* with the top t. A metrizable generalized fan is called a *fan*.

Lemma 2.1 *If X is an arcwise connected continuum and if Y is an arboroid which contains finitely many ramification points, then every hereditarily irreducible and surjective mapping $f : X \to Y$ is a homeomorphism.*

Proof. Suppose that f is not a homeomorphism. Then there exists a point $y \in Y$ such that $f^{-1}(y)$ is not a single point. This means that there exist points $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = y$. Since X is an arboroid there exists a generalized arc Z in X such that x_1, x_2 are end points of Z.

Claim 1. *There exists a segment $[a, b]$ of Z such that $f^{-1}(y) \cap (a, b) = \emptyset$ and $f^{-1}(y) \cap [a, b] = \{a, b\}$.* It is clear that $f^{-1}(y)$ is not dense in Z. In the opposite case we have that Z is a proper subcontinuum of $f^{-1}(y)$. This is impossible since $f^{-1}(y)$ contains no continuum. It follows that there exists a segment $[c, d] \subset Z$ such that $f^{-1}(y) \cap Z \subset [c, d]$ and $\{c, d\} \subset f^{-1}(y) \cap Z$. It is again clear that there exists a subinterval (a_1, b_1) of $[c, d]$ such that $f^{-1}(y)$ $\cap(a_1, b_1) = \emptyset$. Let $\mathcal{A}$ be a family of all segments (a_α, b_α) which contains (a_1, b_1) and $f^{-1}(y) \cap(a_\alpha, b_\alpha) = \emptyset$. It is clear that the union of all elements of $\mathcal{A}$ is a subsegment (a, b) of $[c, d]$. Let us prove that $a, b \in f^{-1}(y)$. Suppose that $a \notin f^{-1}(y)$. Then $f(a) \neq y$. There exists an open set U containing a such that $f(U)$ does not contain the point y. It is clear that there exists a segment (e, h) contained in U. Then $(a, b) \cup (e, h)$ is a segment which contains (a_1, b_1). It is clear that $(a, b) \cup (e, h)$ is not in $\mathcal{A}$, a contradiction. Hence, $a \in f^{-1}(y)$. Similarly, one can prove that $b \in f^{-1}(y)$.

In the remaining part of the proof we shall consider the restriction $g = f|[a, b]$. Let us recall that g is hereditarily irreducible and that $W = f([a, b])$, as a subcontinuum of Y, is an arboroid. Thus we have a hereditarily irreducible surjection g of the arc $[a, b]$ onto a dendroid W such that $g^{-1}(y) = \{a, b\}$.

Claim 2. *There exist subarcs* $[a, x]$ *and* $[z, b]$ *such that* $g([a, x]) \subset g([z, b])$ or $g([a, x]) \supseteq g([z, b])$.

Let U_y be a neighborhood of y such that $U_y \setminus \{y\}$ does not contain ramification points. There exist segments $[a, x]$ and $[z, b]$ such that $g([a, x]) \subset U_y$ and $g([z, b]) \subset U_y$. It follows that $g([a, x])$ and $g([z, b])$ are arcs since $g((a, x])$ and $g([z, b))$ do not contain ramification points. Suppose that $g([a, x]) \cap g([z, b]) = \{y\}$. Then $C = g([a, x]) \cup g([z, b])$ is a continuum. Because of Claim 1, $g([x, z])$ is a continuum not containing the point y. It follows that $C \cap g([x, z])$ is not a continuum since $C \cap g([x, z])$ contains $\{y\}$ and two disjoint subsets $g([a, x]) \cap g([x, z] \supseteq \{g(x)\}$ and $g([x, z]) \cap g([z, b] \supseteq \{g(z)\}$ not containing $\{y\}$. This is impossible since is W is hereditarily unicoherent. Hence, $D = g([a, x]) \cap g([z, b])$ is a non-degenerate continuum containing the point $\{y\}$. It is clear that D does not contain ramification points. It follows that $g([a, x]) \subset g([z, b])$ or $g([a, x]) \supseteq g([z, b])$ since in the opposite case we obtain a triod in U_y.

Claim 3. *We may assume that* $g([a, x]) \supseteq g([z, b])$. Now, $g([a, z]) = g([a, b])$ since $g([a, x]) \supseteq g([z, b])$. This is impossible since g is hereditarily irreducible. Hence, f is one-to-one and, consequently, a homeomorphism.

Corollary 2.2 *If X is an arcwise connected continuum and if Y is a generalized fan, then every hereditarily irreducible and surjective mapping $f : X \to Y$ is a homeomorphism.*

We say that a surjection $f : X \to Y$ is *weakly confluent* if for every subcontinuum C of Y there exists a subcontinuum D of X such that $f(D) = C$.

Theorem 2.3 *Let X be an arcwise connected continuum and let Y be a hereditarily unicoherent continuum. If $f : X \to Y$ is a hereditarily irreducible and weakly confluent mapping, then f is a homeomorphism.*

Proof. The proof is broken into several steps. Let us note that f is light. It suffices to prove that f is one-to-one.

Step 1. *Suppose that f is not one-to-one.* There exists a point $y \in Y$ such that $f^{-1}(y)$ contains two different points x_1 and x_2. There exists an arc L with endpoints x_1 and x_2 since X is an arboroid.

Step 2. *There exists an subarc $[x_3, x_4]$ of L such that $f(x_3) = f(x_4) = y$ and $([x_3, x_4] \setminus \{x_3, x_4\}) \cap f^{-1}(y) = \emptyset$.* The set $[x_1, x_2] \cap f^{-1}(y)$ is closed and not dense on L since then $[x_1, x_2] \cap f^{-1}(y) = L$. We infer that $L \subset f^{-1}(y)$. This is impossible since f is light. Thus, there exists open interval $(a, b) \subset L$ such that $f^{-1}(y) \cap (a, b) = \emptyset$. Let $\mathcal{F}$ be a family of such intervals. It is easy to see that the union of every chain of intervals in $\mathcal{F}$ is an interval in $\mathcal{F}$. This means that there exists a maximal interval (x_3, x_4) with property $f^{-1}(y) \cap (x_3, x_4) = \emptyset$. Let us prove that $f(x_3) = y$ and $f(x_4) = y$. Suppose that $f(x_3) \neq y$. There exists an open interval U containing x_3 such that $y \notin f(U)$. This means that $(x_3, x_4) \cup U$ is an open interval disjoint with $f^{-1}(y)$. This is impossible since (x_3, x_4) is a maximal such interval. Similarly, it follows that $f(x_4) = y$.

Step 3. Let x_5 be any point of $[x_3, x_4] \setminus \{x_3, x_4\}$. It follows that $f(x_5) \neq y$. The intersection $f([x_3, x_5]) \cap f([x_5, x_4])$ contains y and $f(x_5)$. Hence, the intersection $f([x_3, x_5]) \cap f([x_5, x_4])$ is a non-degenerate continuum since Y is hereditarily unicoherent. There exists a continuum $K \subset X$ such that $f(K) = f([x_3, x_5]) \cap f([x_5, x_4])$. If K intersects $[x_3, x_4]$, then $L = K \cup [x_3, x_4]$ is a continuum. It follows that $f(L) = f(K)$, a contradiction if $L \supset K$. If $K \subset [x_3, x_4]$ is a segment which must contain either x_3 or x_4 or both, since y is in $f([x_3, x_5]) \cap f([x_5, x_4])$ and $([x_3, x_4] \setminus \{x_3, x_4\}) \cap f^{-1}(y) = \emptyset$. Suppose that $x_3 \in K$. Hence K is an arc $[x_3, a]$, where $x_3 < a < x_4$. If we suppose that $a \geq x_5$, then $f([x_3, a]) \subseteq f([x_3, x_5)$. This means that $f([x_3, a]) = f([x_3, x_5)$, a contradiction since f is hereditarily irreducible. The proof is similar if $x_4 \in K$. It remains to consider the case when $K \cap [x_3, x_4] = \emptyset$. Let $a \in K$ and $b \in [x_3, x_4]$ and let $L = [a, b]$. If $[x_3, x_4] \cup L$ is a proper subcontinuum of $[x_3, x_4] \cup L \cup K$, then $f([x_3, x_4] \cup L) = f([x_3, x_4] \cup L \cup K)$ contradicts the assumption that f is hereditarily irreducible. If $[x_3, x_4] \cup L = [x_3, x_4] \cup L \cup K$, then $K = L$. Hence K is a subarc $[a, c]$ of L. Let $L_1 = [b, c]$. Now, $f([x_3, x_4] \cup L_1) = f([x_3, x_4] \cup L_1 \cup K)$. This is impossible since $[x_3, x_4] \cup L_1$ is a proper subcontinuum of $[x_3, x_4] \cup L_1 \cup K$ and f is hereditarily irreducible.

Corollary 2.4 *Let X be an arcwise connected continuum and let Y be a hereditarily unicoherent continuum. If $f : X \to Y$ is a hereditarily irreducible, then f is a homeomorphism if and only if f is a weakly confluent mapping.*

We say that a surjective mapping $f : X \to Y$ is *arc-preserving* provided for each arc $L \subset X$ the image $f(L)$ is an arc or a point.

Theorem 2.5 *Let $f : X \to Y$ be an arc-preserving mapping of an arcwise connected continuum X onto a dendroid Y. Then f is hereditarily irreducible if and only if f is a homeomorphism.*

Proof. Suppose that f is not one-to-one. There exists a point $y \in Y$ such that $f^{-1}(y)$ is not a single point. This means that there exist points $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = y$. Since X is an arboroid there exists a generalized arc Z in X such that x_1, x_2 are end points of Z. By Step

2 of the proof of Theorem 2.3 there exists an arc $[x_3, x_4]$ of X such that $f(x_3) = f(x_4) = y$ and $([x_3, x_4] \setminus \{x_3, x_4\}) \cap f^{-1}(y) = \emptyset$. It follows that $f([x_3, x_4])$ is not a point. Hence $f([x_3, x_4])$ is an arc L_1. From the continuity of f, it follows that there exist two subsegments $[x_3, a]$ and $[b, x_4]$ of $[x_3, x_4]$ such that $f([x_3, a])$ and $f([b, x_4])$ are contained in L_1. We may assume that $x_3 < a < b < x_4$. Now we have the following cases: a) $f([x_3, a]) \subset f([b, x_4])$, b) $f([x_3, a]) = f([b, x_4])$ and c) $f([x_3, a]) \supset f([b, x_4])$. If a) then we have $f([x_4, a]) = f([x_3, x_4])$. Hence, f is not hereditarily irreducible. For the case b) we have that $f([x_4, a]) = f([x_3, x_4])$. Hence f is not hereditarily irreducible. If c) then $f([x_3, b]) = f([x_3, x_4])$. This is impossible since f is hereditarily irreducible. Finally, we conclude that f is one-to-one, i.e., f is a homeomorphism.

3 D-continua

A continuum X is called a *D-continuum* if for every pair C, D of its disjoint non-degenerate subcontinua there exists a subcontinuum $E \subset X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.

Lemma 3.1 *[13, Lemma 2.3]. If X is an arcwise connected continuum, then X is a D-continuum.*

Lemma 3.2 *[13, Lemma 2.4]. If X is a locally connected continuum, then X is D-continuum.*

Theorem 3.3 *Let X be a continuum. Then $Con(X))$ is a D-continuum.*

Proof. $Con(X)$ is arcwise connected and, consequently, D-continuum.

A continuum X is said to be *colocally connected* provided that for each point $x \in X$ and each open se $U \ni x$ there exists an open set V containing x such that $V \subset U$ and $X \setminus U$ is connected.

Lemma 3.4 *Each colocally connected continuum X is a D-continuum.*

Proof. Let C, D be a pair of non-degenerate disjoint subcontinua of X. Let x be a point in C. There exists an open set U such that $x \in U, C \setminus U \neq \emptyset$ and $U \cap D = \emptyset$. From the colocal connectedness of X, it follows that there exists an open set V such that $x \in V \subset U$ and $X \setminus V$ is connected. Setting $E = X \setminus V$ we see that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$. Hence, X is a D-continuum.

Lemma 3.5 *The cartesian product of two non-degenerate continua is a colocally connected continuum.*

Proof. Let (x, y) be a point of $X \times Y$. We have to prove that there exists a neighbourhood $U = U_x \times U_y$ of (x, y) such that $E = X \times Y \setminus U$ is connected. We may assume that $U_x \neq X$ and $U_y \neq Y$. Let (x_1, y_1), (x_2, y_2) be a pair of different points in E. For each point $(z, w) \in X \times Y$, we consider a continuum

$$E_{zw} = \{(z, y) : y \in Y\} \cup \{(x, w) : x \in X\}.$$

Claim 1. *For each point $(x', y') \in E$, there exists a point $(z, w) \in E$ such that $(x', y') \in E_{zw}$ and $E_{zw} \cap U = \emptyset$*. If $E_{x'y'} \cap U = \emptyset$ the proof is completed. In the opposite case we have either $\{(x', y) : y \in Y\} \cap U \neq \emptyset$ or $\{(x, y') : x \in X\} \cap U \neq \emptyset$. Suppose that $\{(x', y) : y \in Y\} \cap U \neq \emptyset$. Then $\{(x, y') : x \in X\} \cap U = \emptyset$. There exists a point $z \in X$ such that $z \notin U$. Setting $y' = w$, we obtain a point $(z, w) \in E$ such that $(x', y') \in E_{zw}$ and $E_{zw} \cap U = \emptyset$. The proof in the case $\{(x, y') : x \in X\} \cap U \neq \emptyset$ is similar.

Now, by Claim 1, for (x_1, y_1) there exists a continuum E_{z_1, w_1} such that $E_{z_1 w_1} \cap U = \emptyset$ and $(x_1, y_1) \in E_{z_1, w_1}$. Similarly, there exist a continuum E_{z_2, w_2} such that $E_{z_2 w_2} \cap U = \emptyset$ and $(x_2, y_2) \in E_{z_2, w_2}$.

Claim 2. *The union $E_{z_1, w_1} \cup E_{z_2, w_2}$ is a continuum which contains the points (x_1, y_1), (x_2, y_2) and is contained in $E = X \times Y \setminus U$*. Obvious.

Finally, we infer that $E = X \times Y \setminus U$ is connected. The proof is completed.

Theorem 3.6 *The cartesian product of two non-degenerate continua is a D-continuum.*

Proof. Apply Lemmas 3.5 and 3.4.

A surjective mapping $f : X \to Y$ is said to be *confluent* provided for every subcontinuum K of Y each component L of $f^{-1}(K)$ maps under f onto K, i.e., $f(L) = K$.

Now we shall prove the following result.

Theorem 3.7 *Let X be a D-continuum. Each confluent hereditarily irreducible mapping $f : X \to Y$ is a homeomorphism.*

Proof. Let K be a subcontinuum of Y. Let us prove that $f^{-1}(K)$ has only one component. Suppose that C and D are two different component of $f^{-1}(K)$. This means that $C \cap D = \emptyset$. There exists a subcontinuum E such that $C \subset E, D \neq D \cap E \neq \emptyset$ since X is a D-continuum. Now $f(E \cup D) = f(E)$ which is impossible since f is hereditarily irreducible. Hence $f^{-1}(K)$ has only one component. This means that $f^{-1}(K)$ is connected for every non-degenerate continuum $K \subset Y$. We shall prove that f is monotone, i.e., for $y \in Y$ the fiber $f^{-1}(y)$ is connected. Suppose that $f^{-1}(y)$ is not connected. Then there exists a pair U, V of disjoint open subsets of X such that $f^{-1}(y) \subset U \cup V$. There exists an open se W of Y such that $y \in W$ and $f^{-1}(W) \subset U \cup V$. Let Z

be an open set such that $y \in Z \subset \ \mathrm{Cl}\, Z \subset W$. There exists a component C of $\mathrm{Cl}\, Z$ such that $y \in C$ and $C \cap \ \mathrm{Bd} Z \neq \emptyset$. This means that C is non-degenerate and $f^{-1}(y) \subset f^{-1}(C) \subset U \cup V$. This is impossible since $f^{-1}(C)$ is connected. Finally, Lemma 1.5 completes the proof.

If $f : X \to Y$ is not confluent, the we have the following result.

Theorem 3.8 *Let $f : X \to Y$ be an hereditarily irreducible and surjective mapping of a D-continuum X. Then $w(X) = w(Y)$.*

Proof. It is obvious that $w(Y) \leq w(X)$ [5, p. 171, Theorem 3.1.22]. Let us prove that $w(Y) \geq w(X)$. The proof is broken into several steps.

Step 1. *$C(f) : C(X) \to C(Y)$ is one-to-one on $C(X) \setminus X(1)$. Moreover, $C(f)$ is a homeomorphism of $C(X) \setminus X(1)$ onto $C(f)(C(X) \setminus X(1))$.* Suppose that $C(f)$ is not one-to-one. Then there exist a continuum F in Y and two continua C, D in X such that $f(C) = f(D) = F$. It is impossible that $C \subset D$ or $D \subset C$ since f is hereditarily irreducible. Otherwise, If $C \cap D \neq \emptyset$, then for a continuum $Z = C \cup D$ we have that C and D are subcontinua of Z and $f(Z) = f(C) = f(D) = F$ which is impossible since f is hereditarily irreducible. We infer that $C \cap D = \emptyset$. There exists a subcontinuum E such that $C \subset E, D \neq D \cap E \neq \emptyset$ since X is a D-continuum. Now $f(E \cup D) = f(E)$ which is impossible since f is hereditarily irreducible. Furthermore, $C(f)^{-1}(Y(1)) = X(1)$ since from the hereditarily irreducibility of f it follows that no non-degenerate subcontinuum of X maps under f onto a point.We infer that $C(f)^{-1}[Y \setminus Y(1)] = C(X) \setminus X(1)$. It follows that the restriction $P = C(f)|(C(X) \setminus X(1))$ is one-to-one and closed [5, p. 95, Proposition 2.1.4]. From $C(f)^{-1}[Y \setminus Y(1)] = C(X) \setminus X(1)$ it follows that P is surjective. Hence, P is a homeomorphism.

Step 2. $w(C(X) \setminus X(1)) \leq w(Y)$. Now we have $w(C(X) \setminus X(1)) = w(C(f)|(C(X) \setminus X(1))) \leq w(C(Y) \setminus Y(1)) \leq w(2^X) = w(Y)$ since $w(2^X) = w(Y)$ [5, p. 306, Problem 3.12.26 (a)].

Step 3. $w(X) \leqq w(Y)$. Let $\mathcal{B} = \{B_\alpha : \alpha \in A\}$ be a base of $C(X) \setminus X(1)$. For each B_α let $C_\alpha = \{x \in X : x \in B,\ B \in B_\alpha\}$, i.e., the union of all continua B contained in B_α.

Claim 1. *The family $\{C_\alpha : \alpha \in A\}$ is a network of X.* Let X be a point of X and let U be an open subset of X such that $x \in U$. There exists an open set V such that $x \in V \subset \mathrm{Cl} V \subset U$. Let K be a component of $\mathrm{Cl} V$ containing x. By Boundary Bumping Theorem [20, p. 73, Theorem 5.4] K is non-degenerate and, consequently, $K \in C(X) \setminus X(1)$. Now, $\langle U \rangle \cap (C(X) \setminus X(1))$ is a neighbourhood of K in $C(X) \setminus X(1)$. It follows that there exists a $B_\alpha \in \mathcal{B}$ such that $K \in B_\alpha \subset \langle U \rangle \cap (C(X) \setminus X(1))$. It is clear that $C_\alpha \subset U$ and $x \in C_\alpha$ since $x \in K$. Hence the family $\{C_\alpha : \alpha \in A\}$ is a network of X.

Claim 2. $nw(X) = w(C(X) \setminus X(1))$. Apply Claim 1.

Claim 3. $w(X) = w(C(X)\diagdown X(1))$. By Claim 2 we have $nw(X) = w(C(X)\diagdown X(1))$. Moreover, by Theorem 1.7 $w(X) = w(C(X)\diagdown X(1))$.

Claim 4.*Finally,* $w(X) \leq w(Y)$. Apply Step 2 and Claim 3.

The proof is complete since we have $w(X) \leq w(Y)$ and $w(Y) \leq w(X)$.

The proof above can be modified to prove the following theorem.

Theorem 3.9 *Let $f : X \to Y$ be an hereditarily irreducible mapping of a continuum X. If for every two continua $P, Q \in C(X)\diagdown F_1(X)$ with $P \cap Q = \emptyset$ the inequality $f(P)\diagdown f(Q) \neq \emptyset$ holds, then $w(X) = w(Y)$.*

Proof. Modify the proof of Theorem 3.8 in such a way that Step1 is replaced by the following.

Step 1*. *$C(f) : C(X) \to C(Y)$ is one-to-one on $C(X)\diagdown X(1)$. Moreover, $C(f)$ is a homeomorphism of $C(X)\diagdown X(1)$ onto $C(f)(C(X)\diagdown X(1))$.* Suppose that $C(f)$ is not one-to-one. Then there exist a continuum F in Y and two continua C, D in X such that $f(C) = f(D) = F$. It is impossible that $C \subset D$ or $D \subset C$, since f is hereditarily irreducible. Otherwise, if $C \cap D \neq \emptyset$, then for a continuum $Z = C \cup D$ we have that C and D are subcontinua of Z and $f(Z) = f(C) = f(D) = F$ which is impossible since f is hereditarily irreducible. We infer that $C \cap D = \emptyset$. Now, $C \cap D = \emptyset$ and $f(C) = f(D) = F$, i.e., $f(C)\diagdown f(D) = \emptyset$. This contradicts the assumption of the Theorem.

Theorem 3.9 can be reformulated as follows.

Theorem 3.10 *Let $f : X \to Y$ be an hereditarily irreducible mapping of a continuum X onto Y. If $2^f : 2^X \to 2^Y$ is light, then $w(X) = w(Y)$.*

Proof. Apply Theorems 1.6 and 3.9.

The following two results are consequences of Theorem 3.8.

Corollary 3.11 *Let $X \times Y$ be a product of two non-degenerate continua. If there exists a hereditarily irreducible mapping $f : X \times Y \to Z$, then $w(X \times Y) = w(Z)$.*

Proof. Apply Theorems 3.6 and 3.8.

Corollary 3.12 *Let X be a continuum. If $f : Con(X) \to Y$ is a hereditarily irreducible mapping, then $w(Con(X)) = w(Y)$.*

Proof. Apply Theorems 3.3 and 3.8.

A continuum X is said to be *τ-rim-d-continuum* if X admits a basis of open sets whose boundaries are the union of $\leq \tau$ D-continua.

Now we shall prove the following generalization of Theorem 1.2.

Theorem 3.13 *If $f : X \to Y$ is a hereditarily irreducible mapping of τ-rim-d-continuum X onto a continuum Y such that $w(Y) = \tau$, then $w(X) = w(Y)$.*

Proof. Let $\mathcal{B} = \{B_\alpha : \alpha \in A\}$ be a basis of open sets of X such that every $\mathrm{Bd}(B_\alpha)$ is the union of $\leq \tau$ D-continua $C_{\alpha\mu}$. Consider the restriction $f_{\alpha\mu}$ of f onto $C_{\alpha\mu}$, i.e., $f_{\alpha\mu} : C_{\alpha\mu} \to f_{\alpha\mu}(C_{\alpha\mu})$. From Theorem 3.8 it follows that $w(C_{\alpha\mu}) = w(f_{\alpha\mu}(C_{\alpha\mu})) \leq w(Y) = \tau$ since $f_{\alpha\mu}$ is hereditarily irreducible. By [5, p. 171, Theorem 3.1.20] we have $w(\mathrm{Bd}(B_\alpha)) \leq \tau = w(Y)$. Using Theorem 1.2 we complete the proof since each hereditarily irreducible mapping is light.

4 Near locally connected continua

A continuum X is said to be *near locally connected at a point* $x \in X$ provided for every open set U containing x there is a continuum C such that $x \in C \subset U$ and $\mathrm{Int}(C) \neq \emptyset$. A continuum is said to be a *NLC-continuum* provided it is near locally connected at every of its point. Each locally connected continuum is NLC-continuum.

The concept of aposyndesis was introduced by Jones in [10]. A continuum is said to be *semi-aposyndetic* [9, p. 238, Definition 29.1], if for every $p \neq q$ in X, there exists a subcontinuum M of X such that $\mathrm{Int}_X(M)$ contains one of the points p, q and $X \setminus M$ contains the other one. Each locally connected continuum is semi-aposyndetic.

Example. *There exists a non-locally connected non-semi-aposyndetic NLC-continuum X.* Let $\mathbb{R}^2$ be the Euclidean plane endowed with the ordinary rectangular coordinate system Oxy. We define the continuum X as a subcontinuum of $\mathbb{R}^2$ which is the union of the following sets:

a) $[-1, 0] \times [-1, 1]$,

b) $\left\{(x, \sin \frac{1}{x}) : 0 < x \leq 1\right\}$,

c) $\left\{(x, \sin \frac{1}{2-x}) : 1 \leq x < 2\right\}$,

d) $[2, 3] \times [-1, 1]$.

It is clear that X is not locally connected. It is not semi-aposyndetic. Namely, if $(0, \frac{1}{3})$ and $(0, \frac{1}{2})$ are two points of X, then each continuum with non-empty interior which contains $(0, \frac{1}{3})$ contains also $(0, \frac{1}{2})$. It is clear that X is locally connected at each point of $X \setminus (\{0\} \times [-1, 1] \cup \{2\} \times [-1, 1])$. Hence, X is NLC-continuum at each point of $X \setminus (\{0\} \times [-1, 1] \cup \{2\} \times [-1, 1])$. On the other hand, at every point A of $\{0\} \times [-1, 1] \cup \{2\} \times [-1, 1]$ and for every open set containing A, there exists a continuum K containing A such that $\mathrm{Int}(K) \neq \emptyset$. Hence X is a NLC-continuum.

Now we shall consider subspace $C_{int}(X)$ of $C(X)$ containing all subcontinua of X with nonempty interior. It is clear that $C_{int}(X)$ is non-empty since $X \in C_{int}(X)$.

Lemma 4.1 *The hyperspace $C_{int}(X)$ is arcwise connected.*

Proof. Let $K \in C_{int}(X)$. There exists an order arc α from K to $X \in C(X)$ [14, p. 1209, Theorem]. It is clear that each $L \in \alpha$ has a non-empty interior (in X) since $K \subset L$ and K has a non-empty interior in X. Thus, $\alpha \subset C_{int}(X)$.

It is a question when $C_{int}(X) = C(X)$. We say that a continuum X is *completely regular* if each non-degenerate subcontinuum of X has a non-empty interior in X . Each completely regular continuum is hereditarily locally connected.

Lemma 4.2 *If X is a continuum, then $C_{int}(X) = C(X)$ if and only if X is completely regular.*

Theorem 4.3 *If X is a NLC-continuum and $f : X \to Y$ is hereditarily irreducible mapping, then $w(X) = w(Y)$.*

Proof. It is obvious that $w(Y) \leq w(X)$ [5, p. 171, Theorem 3.1.22]. Let us prove that $w(Y) \geq w(X)$. The proof is broken into several steps.

Step 1. *For every pair C, D of disjoint non-degenerate subcontinua of X with non-empty interiors, there exists a non-degenerate subcontinuum $E \subset X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.* It suffices to apply Lemma 1.8 to the union $C \cup D$ and we obtain a component K of $X \setminus (C \cup D)$ such that $\mathrm{Cl}K \cap C \neq \emptyset$ and $\mathrm{Cl}K \cap D \neq \emptyset$. Then $E = \mathrm{Cl}K$ is a continuum with properties $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$ since $\mathrm{Int}_X(C) \cap E = \emptyset$ or $\mathrm{Int}_X(D) \cap E = \emptyset$.

Step 2. *Every restriction*

$$C(f)|C_{int}(X) : C_{int}(X) \to C(f)(C_{int}(X)) \subset C(Y)$$

is one-to-one and closed. Hence, it is a homeomorphism. See the proof of Step 1 of the proof of Theorem 3.8.

Step 3. $w(C_{int}(X)) \leq w(Y)$. Now we have

$$w(C_{int}(X)) = w(C(f)|(C_{int}(X)) \leq w(C(Y)) \leq w(2^X) = w(Y),$$

since $w(2^X) = w(Y)$ [5, p. 306, Problem 3.12.26 (a)].

Step 4. Let $\mathcal{B} = \{B_\mu : \mu \in M\}$ be a base of $C_{int}(X)$. For each B_μ let $C_\mu = \cup\{x \in X : x \in B,\ B \in B_\mu\}$, i.e., the union of all continua B contained in B_i.

Claim 1. *The family $\{C_\mu : \mu \in M\}$ is a network of X.* Let X be a point of X and let U be an open subsets of X such that $x \in U$. There exists an open set V such that $x \in V \subset \mathrm{Cl}V \subset U$. Let K be a component of

ClV containing x. By Boundary Bumping Theorem [20, p. 73, Theorem 5.4] K is non-degenerate and, consequently, $K \in C_{int}(X)$ since X is an NLC-continuum. Now, $\langle U \rangle \cap (C_{int}(X))$ is a neighbourhood of K in $C_{int}(X)$. It follows that there exists a $B_\mu \in \mathcal{B}$ such that $K \in B_\mu \subset \langle U \rangle \cap (C_{int}(X))$. It is clear that $C_\mu \subset U$ and $x \in C_\mu$ since $x \in K \subset U$. Hence the family $\{C_\mu : \mu \in M\}$ is a network of X.

Claim 2. $nw(X) = w(C_{int}(X)) \leq w(Y)$. Apply Claim 1.

Claim 3. $w(X) \leq w(Y)$. Apply Claim 2 and Step 1.

Finally, from Claim 3 and $w(Y) \leq w(X)$, it follows that $w(X) = w(Y)$.

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Ivan Lončar, Trenkova 51,
42000 Varaždin, Croatia
e-mail: ivan.loncar@foi.hr or
ivan.loncar1@vz.htnet.hr