An. Şt. Univ. Ovidius Constanţa Vol. 15(2), 2007, 19–28

A Metric Induced by the Geometric Interpretation of Rolle's Theorem

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Abstract

In this note we discuss a geometric viewpoint on Rolle's Theorem and we show that a particular setting of the form of Rolle's Theorem yields a metric that is the hyperbolic metric on the disk. Our result is related to recent developments in the study of Barbilian's metrization procedure.

1 Introduction

Barbilian's metrization procedure was first introduced in 1934 [2], in a work cited many times in the last decade. This metrization procedure allows us to construct a distance which naturally leads to the construction of a metric. That metric is closely related to the geometric configuration studied in the classical Rolle's Theorem.

Rolle's Theorem and the Mean Value Theorems have been, over the years, in the center of attention of many researchers. Some of the results have brought up interesting geometric interpretations. For example, an interesting view on the geometric content of the Mean Value Theorem is in Barrett and Jacobson's work [7]. Other works have explored the Mean Value Theorem in the complex plane (see, e.g., [12, 13]). A multidimensional version of Rolle's theorem has been proven by Furi and Martelli in [14]. The topic has beed studied by other authors (see, for example, [1, 8, 19, 20, 21, 22]).

Key Words: Hyperbolic metric, Apollonian metric, Barbilian's metrization procedure, Rolle's Theorem.
Mathematics Subject Classification: 51M10, 53B40, 53A35, 53C60.
Received: March, 2007
Accepted: June, 2007

We found that there is a natural connection between Barbilian's metrization procedure and Rolle's Theorem. In the remaining part of the introduction we present several facts related to Barbilian's metrization procedure. Barbilian's metrization procedure was originally introduced in 1934 [2] with the aim of generalizing a construction inspired from the study of Klein-Beltrami model of hyperbolic geometry. Later contributions include P. J. Kelly's work [15] and major developments are due to D. Barbilian himself [3, 4, 5, 6]. Recently, in [11] it has been shown that Barbilian's metrization procedure in the plane generates Riemannian or Lagrange generalized metrics irreducible to Finslerian or Langrangian metrics.

The following construction describes Barbilian's metrization procedure in its most general setting. It originates in [3] and it develops the idea from [2]. For new developments on this problem, see [9, 11], and for the history of the theory see [10]. For important developments of the theory see, for example, [17, 18].

Consider two arbitrary sets K and J. The function $f : K \times J \to \mathbb{R}_+^*$ is called an influence of the set K over J if for any $A, B \in J$ the ratio $g_{AB}(P) = \frac{f(P,A)}{f(P,B)}$ has a maximum $M_{AB} \in \mathbb{R}$ when $P \in K$. Note that $g_{AB} : K \to \mathbb{R}_+^*$. In [3] it is pointed out that if we assume the existence of $\max g_{AB}(P)$, when $P \in K$, then there also exists $m_{AB} = \min_{P\in K} g_{AB}(P) = \frac{1}{M_{BA}}$. It is known since [3] that $d : J \times J \to \mathbb{R}_+$ given by

$$d(A,B) = \ln \frac{\max_{P\in K} g_{AB}(P)}{\min_{P\in K} g_{AB}(P)} \tag{1}$$

is a semidistance, i.e.: (a) if $A = B$ then $d(A,B) = 0$; (b) d is symmetric; (c) d satisfies triangle inequality. In [3] it is specified in what conditions this semidistance is a distance.

On the other hand, we would like to remind here a particular form of the result from [5], part 2, paragraph 7. This result represents also a version of the argument used by P. A. Hästö in [16], in the proof of his Lemma 3.5.

Lemma 1 *Let K and J be two subsets of the Euclidean plane $\mathbb{R}^2$, and $K = \partial J$. Consider the influence $f(M,A) = ||MA||$, where by $||MA||$ we denote the Euclidean distance. Consider*

$$g_{AB}(M) = \frac{f(M,A)}{f(M,B)} = \frac{||MA||}{||MB||}$$

and consider the distance induced on J by the Barbilian's metrization procedure, $d^B(A,B)$. Suppose furthermore that for $M \in K$ the extrema $\max g_{AB}(M)$ and $\min g_{AB}(M)$ for any A and B in J are attained each in an unique point in K. Then:

(a) For any $A \in J$ and any line d passing through A there exist exactly two circles tangent to K and to d in A.

(b) The metric induced by the Barbilian distance has the form

$$ds^2 = \frac{1}{4}\left(\frac{1}{R} + \frac{1}{r}\right)^2 (dx_1^2 + dx_2^2), \tag{2}$$

where R and r are the radii of the circles described in (a).

We do not use this Lemma in the contruction presented in our paper. However, this Lemma suggests the geometric idea that arise the natural construction we consider below in (3).

In fact, Barbilian's metrization procedure has been discovered by studying an extremum problem in the Beltrami-Klein model of hyperbolic geometry [2]. It is natural to connect its study to the study of mean value theorems.

2 A Metric Induced by Mean Value Theorem Yields the Hyperbolic Metric on the Disk

In this section we show how we can naturally construct a metric by pursuing an idea similar to a result obtained in [5].

Consider the simple closed planar curve K of class C^1 with the property that its interior is convex. Let $A \in Int\ \ K$, and Δ a given line passing through A. This line intersects K in exactly two points A_1 and A_2. On curve K, there are two arcs joining A_1 and A_2. On each of these arcs we can apply the Mean Value Theorem. It follows that there exist precisely two points T_1 and T_2 such that the two tangents to the curve t_1 (in T_1) and t_2 (in T_2), are parallel to the line A_1A_2.

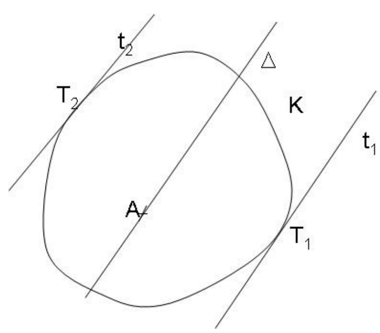

Figure 1.

Apply an inversion $\rho_{A,k}$ of pole A and arbitrary power k. The curve K is mapped into the curve K', which is also simple and closed. The tangents t_1 and t_2 are mapped by $\rho_{A,k}$ into the circles t_1' and t_2', which pass through A, are tangent in A, and their common tangent is the line Δ, which is invariated by $\rho_{A,k}$. (See Figure 2.) Denote by R_1 and R_2 the radii of the two circles $t_i', i = 1, 2$.

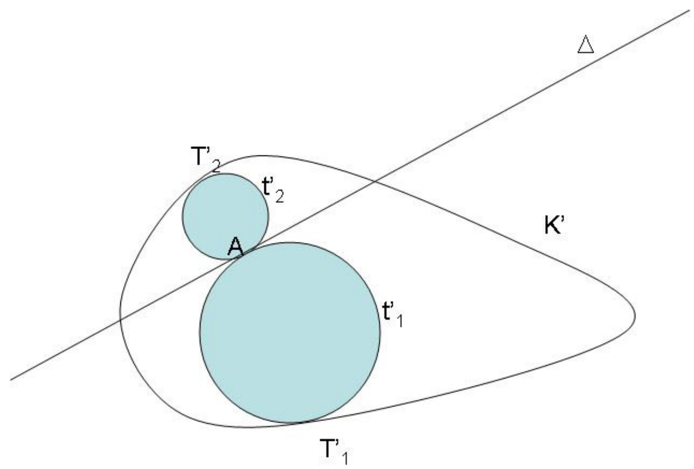

Figure 2.

Obviously, when the line Δ rotates about A, the radii of the two circles modify.

Among the many metrics whose coefficients depends on R_1 and R_2, let us define in the interior of the curve K' the metric

$$ds^2 = \frac{1}{4}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^2 (dx_1^2 + dx_2^2), \tag{3}$$

where R_1 and R_2 are the radii of the circles t_1' and t_2', respectively. This metric is inspired by the configuration obtained by applying the inversion $\rho_{A,k}$ to the figure suggested by the Mean Value Theorem. Also, the arc element of the metric (3) is inspired by a similar construction studied by Barbilian in [5]. The natural question is to study what kind of metric we obtain by this construction in the present setting.

We should point out here the geometric meaning of this construction. The metric (3) depends on the circles t_1' and t_2', respectively. These circles are tracing the contour of the closed curve K', when the line Δ rotates about A. That's why we can say that the contour induces on its interior the metric (3). This construction is in the spirit of Barbilian's construction from [5].

In the remaining part of the paper, we will discuss the case when the initial curve K is a circle centered in the origin and of radius R. We will show that, in the case of the disk centered in the origin and of arbitrary radius R, the metric (3) is the hyperbolic metric.

Theorem 1 *Consider the circle K centered in the origin and of radius R. Then the metric given by (3) has the form*

$$ds^2 = \frac{4R^2}{[R^2 - (x^2 + y^2)]^2} \cdot (dx^2 + dy^2).$$

Furthermore, the metric obtained by this procedure has the Gaussian curvature -1.

Proof. Let A of coordinates (x_0, y_0) in the interior of K. The two parallel tangents t_1 and t_2 correspond to the antipodal points $T_1 \in K$, and $T_2 \in K$. Apply an inversion $\rho_{A,k}$ of pole A and ratio equal to the power of the point A with respect to the circle K, i.e. $k = |OA|^2 - R^2$. (See Figures 3 and 4. By inversion, Figure 3 transforms into Figure 4.)

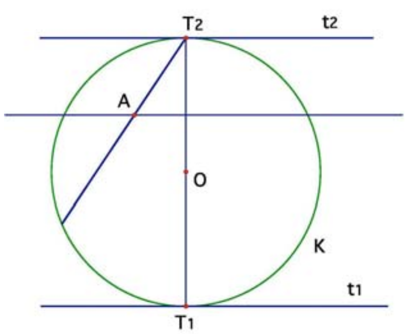

Figure 3.

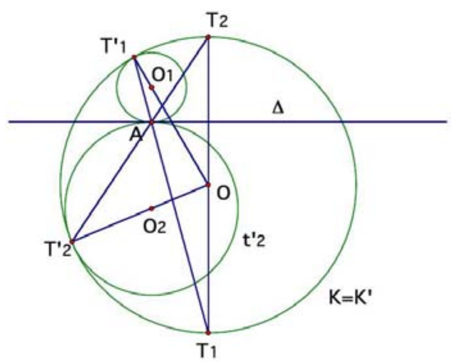

Figure 4.

First, remark that the circle K is fixed by $\rho_{A,k}$. The images of T_1 and T_2 through $\rho_{A,k}$ are $\{T_1'\} = K \cap [T_1 A$, and $\{T_2'\} = K \cap [T_2 A$. Remark also that pole A separates the points from their images when the negative ratio of the inversion is used. The circles t_1' and t_2' are tangent with a common tangent Δ, and, as above, are tangent in T_1' and T_2' to the circle $K = K'$. As an aside remark, the arc of circle K given by $T_1'AT_2'$ is twice orthogonal to the circle K, since it comes from the transformation by inversion of the diameter T_1T_2, which is twice orthogonal to the circle K. Denote by $O_1(x_1, y_1)$ and $O_2(x_2, y_2)$ the centers of the two circles and by m the slope of the line Δ. Line O_1O_2 has the equation

$$y - y_0 = -\frac{1}{m}(x - x_0).$$

Therefore, the points O_1 and O_2 have the coordinates $(x_i, y_0 - \frac{1}{m}(x_i - x_0))$, for $i = 1, 2$. Furthermore,

$$R_i^2 = |O_iA|^2 = \frac{m^2+1}{m^2}(x_i - x_0)^2, \qquad i = 1, 2.$$

Without loss of generality, we assume that $x_1 - x_0 \leq 0$ and $x_2 - x_0 \geq 0$, with the equality case reached when $\Delta \| Ox$. Remark that $x_1 - x_0 < 0$, if $m > 0$. Thus

$$|O_1A| = \frac{\sqrt{m^2+1}}{m}(x_0 - x_1),$$

and

$$|O_2A| = \frac{\sqrt{m^2+1}}{m}(x_2 - x_0).$$

Therefore, the circle t_1' has center $(x_1, y_0 - \frac{1}{m}(x_1 - x_0))$ and radius $R_1 = \frac{\sqrt{m^2+1}}{m}(x_0 - x_1)$, and circle t_2' is the circle of center $(x_2, y_0 - \frac{1}{m}(x_2 - x_0))$ and radius $R_2 = \frac{\sqrt{m^2+1}}{m}(x_2 - x_0)$.

To obtain the coordinates of the point T_1', we recall that it lies at the intersection between the circle $x^2 + y^2 = R^2$ and the line

$$y = \frac{1}{x_1}[y_0 - \frac{1}{m}(x_1 - x_0)]x,$$

which passes through the collinear points O, O_1 and T_1'. Solving the system, we get the coordinates of T_1' as follows

$$\left(\frac{Rx_1}{\sqrt{x_1^2 + (y_0 - \frac{1}{m}(x_1 - x_0))^2}}, \frac{R(y_0 - \frac{1}{m}(x_1 - x_0))}{\sqrt{x_1^2 + (y_0 - \frac{1}{m}(x_1 - x_0))^2}}\right).$$

By direct computation, we get

$$|O_1T_1'| = R - \sqrt{x_1^2 + (y_0 - \frac{1}{m}(x_1 - x_0))^2}.$$

Since the segments O_1T_1' and O_1A are radii of the circle of center O_1 and radius R_1 we set up the equalities

$$\frac{\sqrt{m^2+1}}{m}(x_0-x_1)=R_1=R-\sqrt{x_1^2+(y_0-\frac{1}{m}(x_1-x_0))^2}.$$

It follows that

$$x_0-x_1=\frac{R_1m}{\sqrt{m^2+1}}.$$

Thus

$$(R-R_1)^2=x_1^2+\left(y_0+\frac{R_1}{\sqrt{m^2+1}}\right)^2.$$

Since

$$x_1=x_0-\frac{R_1m}{\sqrt{m^2+1}},$$

we obtain

$$(m^2+1)(R-R_1)^2-(y_0\sqrt{m^2+1}+R_1)^2=(x_0\sqrt{m^2+1}-R_1m)^2$$

which gives

$$R_1=\frac{\sqrt{m^2+1}}{2}\cdot\frac{R^2-x_0^2-y_0^2}{R\sqrt{m^2+1}-x_0m+y_0}.$$

In a similar way we obtain,

$$R_2=\frac{\sqrt{m^2+1}}{2}\cdot\frac{R^2-x_0^2-y_0^2}{R\sqrt{m^2+1}+x_0m-y_0}.$$

Hence, we proved the metric relation

$$\frac{1}{4}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2=\frac{4R^2}{(R^2-x_0^2-y_0^2)^2}.$$

By straightforward computation, we can easily see that the Gaussian curvature of this metric is $\kappa_g=-1$. Therefore this metric generates the hyperbolic geometry on the disk $D(O,R)$.

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