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MILNOR ALGEBRAS VERSUS MODULAR GERMS FOR UNIMODAL HYPERSURFACE SINGULARITIES

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Abstract

We find and describe unexpected isomorphisms between two different objects associated to hypersurface singularities. One object is the Milnor algebra of a function, while the other object associated to a singularity is the local ring of the flatness stratum of the singular locus in a miniversal deformation, an invariant of the contact class of a defining function. Such isomorphisms exist for unimodal hypersurface singularities. However, for the moment it is badly understood, which principle causes these isomorphisms and how far this observation generalizes.

0 Introduction

Let $X_0 \subseteq \mathbb{C}^n$ be a germ of an isolated hypersurface singularity defined by an analytic function $f(x) = 0$, $f \in \mathbb{C}\{x\}$. An imported topological invariant of the germ is the Milnor number, which can be computed as the $\mathbb{C}$-dimension of the so-called Milnor algebra $Q(f) = \mathbb{C}\{x\}/(\partial f/\partial x)$, [Mil68]. The Milnor algebra carries a canonical structure of a $\mathbb{C}[T]$-algebra defined by the multiplication with f. A special version of the Mather-Yau theorem states that the $\mathcal{R}$-class (right-equivalence class) of the function $f(x)$ with isolated critical point is fully determined by the isomorphism class of $Q(f)$ as $\mathbb{C}[T]$-algebra [Mar85]. However, there is richer structure on the Milnor algebra connected with the relative Milnor algebra associated to a universal unfolding $F(x, s)$ of the function $f(x)$ and the associated Frobenius manifold. A moduli space

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functions with respect to $\mathbb{R}$-equivalence can be constructed from it [Her02]. This will be not discussed here.

By computational experiments, we have found another occurence of the Milnor algebra – this time connected with the $\mathcal{K}$-class (the contact equivalence class) of $f(x)$, i.e. with the isomorphism-class of the germ X_0. Our observation concerns unimodal functions that are not quasihomogeneous. Here we consider a miniversal deformation $F : X \to S$ of the singularity X_0. It has a smooth base space of dimension τ, with τ being the Tjurina number, i.e. the $\mathbb{C}$-dimension of the Tjurina algebra $T(f) := Q(f)/fQ(f)$. We consider the relative singular locus $Sing(X/S)$ of X over S and its flatness stratum $\mathbb{F} := \mathbb{F}_S(Sing(X/S)) \subset S$, which depends only on X_0, up to isomorphism. The flatness stratum is computable for sufficiently simple functions using a special algorithm [Mar02]. Surprisingly, the local ring of the flatness stratum of a unimodal singularity is isomorphic in all computed cases, either to the Milnor algebra of the defining function (in case $dim(\mathbb{F}) = 0$), or to the Milnor algebra of a 'nearby' function with non-isolated critical point, otherwise.

The notion of a modular stratum was developed by Palamodov, [Pal78], in order to find a moduli space for singularities. It coincides with the flatness stratum $\mathbb{F} = \mathbb{F}_S(Sing(X/S))$ [Mar03], which has been described for unimodal functions in [Mar06]. Only for some singularities from the T-series the modular stratum has expected dimension 1 with smooth curves and embedded fat points as primary components. The combinatorial pattern of its occurrence was found and the phenomenon of a splitting singular locus along a τ-constant stratum was discovered. Here we extend our observation that the modular stratum is the spectrum of the Milnor algebra of an associated non-isolated limiting singularity.

The modular stratum is a fat point of multiplicity μ isomorphic to $\mathrm{S}pec\, Q(f)$ in all other (computed) cases of T-series singularities. The same holds for all 14 exceptional and non-quasihomogeneous unimodal singularities. In the case of a quasihomogenous exceptional singularity, the modular stratum is a smooth germ, hence corresponding to a trivial Milnor algebra.

For completeness, we will first recall the basic results on modular strata and prove that they are algebraic. Second, we collect and complete results on the modular strata of unimodal functions, which are already found in [Mar06]. Subsequently, some of the non-trivial unexpected isomorphisms are presented. A further example of higher modality is discussed in section 4. Hypotheses toward a possible generalization of these experimental results are formulated. All computations were executed in the computer algebra system Singular [GPS02].

1 Characterizations of a modular germ

The definition of modularity was introduced by Palamodov, cf. for instance [Pal78], and was simultaneously discussed by Laudal for the case of formal power series under the name 'prorepresentable substratum'. While this notion can be considered for any isolated singularity with respect to several deformation functors or to deformations other objects, cf. [HM05], for simplicity we restrict ourselves mostly to the following case: a germ of an isolated complex hypersurface singularity $X_0 = \{f(x) = 0\} \subseteq \mathbb{C}^n$, or an isolated complete intersection singularity (ICIS).
A *deformation* of X_0 is a flat morphism of germs $F : X \longrightarrow S$ with its special fibre isomorphic to X_0. It is called *versal*, if any other deformation of X_0 can be induced via a morphism of the base spaces up to isomorphism. It is called *miniversal*, if the dimension of the base space is minimal. Miniversal deformations exists for isolated singularities and are unique up to a noncanonical isomorphism. In case of a hypersurface, a miniversal deformation has a smooth base space, i.e. the deformations are unobstructed. It can be represented as an 'embedded' deformation $F : X \subset \mathbb{C}^n \times S \to S$, $S = \mathbb{C}^\tau$, $F(x, s) = f(x) + \sum_{i=1}^{\tau} s_\alpha m_\alpha$, where $\{m_1, \ldots, m_\tau\} \subset \mathbb{C}\{x\}$ induces a $\mathbb{C}$-basis of the Tjurina algebra $T(f)$.
Obviously, a miniversal deformation has not the properties of a moduli space, because there are always isomorphic fibres or even locally trivial subfamilies. Hence the inducing morphism of another deformation is not unique. One can, however, look for subfamilies of a miniversal deformation with this universal property.

Definition 1.1. *Let $F : X \to S$ be a miniversal deformation of a complex germ X_0. A subgerm $M \subseteq S$ of the base space germ is called* modular *if the following universal property holds: If $\varphi : T \to M$ and $\psi : T \to S$ are morphisms such that the induced deformations $\varphi^*(F_{|M})$ and $\psi^*(F)$ over T are isomorphic, then $\varphi = \psi$.*

The union of two modular sub germs inside a miniversal family is again modular. Hence, a unique maximal modular subgerm exists. Its is called *modular stratum* of the singularity. Note, that any two modular strata of a singularity are isomorphic by definition.

Example 1.2. If X_0 is an isolated complete intersection singularity with a good $\mathbb{C}^*$-action, i.e. defined by quasihomogeneous polynomials, then its modular stratum coincides with the τ-constant stratum and is smooth, cf. [Ale85].

Palamodov's definition of modularity is difficult to handle. It made it challenging to find non-trivial explicit examples. Even the knowledge of the basic characterizations of modularity in terms of cotangent cohomology, which were already discussed by Palamodov and Laudal, did lead to identify more examples.

Proposition 1.3. *Given a miniversal deformation $F : X \to S$ of an isolated singularity X_0, the following conditions are equivalent for a subgerm of the base space $M \subseteq S$:*

i) M is modular.

ii) M is infinitesimally modular, i.e. injectivity of the relative Kodaira-Spencer map $T^0(S, \mathcal{O}_M) \longrightarrow T^1(X/S, \mathcal{O}_S)_{|M}$ holds.

iii) M has the lifting property of vector fields of the special fiber, i.e.

$$T^0(X/S, \mathcal{O}_S)_{|M} \longrightarrow T^0(X_0, \mathbb{C})_{|M}$$

is surjective.

Note that T^0 corresponds to the module of associated vector fields, while T^1 describes all infinitesimal deformations. It is given here by the (relative) Tjurina algebra $T^1(X/S) = T(F) = \mathbb{C}\{x, s\}/(F, \partial_x F)$.
As a corollary the tangent space of the modular stratum inside the tangent space of S can be identified in terms of the cotangent cohomology. The infinitesimal deformations are identified with the tangent vectors to the base space by construction of a miniversal deformation, i.e. $T^1(X_0) \cong \mathcal{T}_0(S)$.

Lemma 1.4. *Take the Lie bracket in degree $(0, 1)$ of the tangent cohomology*

$$[-, -] : T^0(X_0) \times T^1(X_0) \to T^1(X_0).$$

Then an element $t \in T^1(X_0)$ is tangent to $M \subseteq S$, iff the Lie bracket map $[-, t]$ vanishes.

Example 1.5. For a quasihomogeneous singularity, the only non-trivial derivation in $T^0(X_0)$ is the Euler derivation $\delta_E = \sum w_i x_i \partial/\partial x_i$ induced from the weights w_i of the coordinates. $\delta_E(f) = f$ holds. Take a tangent vector $t \in \mathcal{T}_0(S)$ corresponding to a quasihomogenous $g(x)$, then $[\delta_E, t] = class((deg_w(g) - 1)g(x)) \in T(f)$ is zero iff $deg_w(g) = 1$. Hence, the tangent space to the modular stratum corresponds to the zero graded subspace with respect to the associated grading of $\mathcal{T}_0(S) \cong T^1(X_0)$.

All objects are belonging to the category of analytic germs. But an isolated singularity is always algebraic, i.e. its defining equations can be chosen as polynomials. It is not ad hoc clear whether the modular stratum is algebraic, too, and to our knowledge it has not been investigated yet. Here, we add the proof for an isolated complete intersection singularity.

Lemma 1.6. *Let X_0 be a germ of an isolated complete intersection singularity. Then its modular stratum $M(X_0) \subset \mathbb{C}^\tau$ is an algebraic subgerm.*

The proof uses the characterization of modularity as flatness stratum of the Tjurina-module. A more general result holds under weaker assumptions than ICIS, too, cf. [Mar03].

Proposition 1.7.
Let $X_0 \subseteq \mathbb{C}^n$ be an isolated complete intersection singularity defined by p equations $f \in \mathbb{C}\{x\}^p$ with miniversal deformation $F : X \to S$. Then the modular space coincides with the flatness stratum of the relative Tjurina module $T^1(X/S) = \mathcal{O}_X^p/(\partial F/\partial x)\mathcal{O}_X^p$ as $\mathcal{O}_S$-module.

Proof of the lemma: We may choose the defining equations $f = (f_1, \ldots, f_p)$ of the germ X_0 as polynomials by finite determination of isolated singularities. The affine variety defined by these polynomials $V(f) \subset \mathbb{C}^n$ has in general other singularities than the zero point. But, we can choose the embedding (not necessary minimal) such that $Sing(V(f))$ is concentrated at zero. This holds if and only if global and local Tjurina numbers are equal

$$\dim_{\mathbb{C}}(\mathbb{C}[x]^p/(f\mathbb{C}[x]^p, \partial f/\partial x)) = \dim_{\mathbb{C}}(\mathbb{C}\{x\}^p/(f\mathbb{C}\{x\}^p, \partial f/\partial x)) = \tau.$$

Consider the $\mathbb{C}[s,x]$-module $B := \mathbb{C}[s,x]^p/(F\mathbb{C}[s,x]^p, \partial F/\partial x)$. The module B is finite as a $\mathbb{C}[s]$-module. Its flatness stratum over S at zero $\mathbb{F}_{S,0}(B) \subset S$, $S := \mathbb{C}^\tau = Spec(\mathbb{C}[s])$, is a well-defined by the fitting ideal of a representation of B as a $\mathbb{C}[s]$-module. The $\mathbb{C}\{s,x\}$-module $T^1(X/S, \mathcal{O}_S)$ is finite as $\mathbb{C}\{s\}$-module. Consider the modules $B_0 := B/sB$ and $T^1(X_0) = T^1(X/S, \mathcal{O}_S)_{|s=0}$, then the localization at $x = 0$ of B_0 and $T^1(X_0)$ have identical module-structures which are both already given as $\mathbb{C}[x]/(x)^k$-modules: $B_{0\ (x)} = T^1(X_0)$, hence the germ at zero $\mathbb{F}(B)_{(s,x)}$ coincides with the flattening stratum of $T^1(X/S, \mathcal{O}_S)$.
At this place we add some remarks concerning the flatness criterion:

- The support of $T^1(X/S, \mathcal{O}_S)$ is exactly the relative singular locus of the mapping germ $F : \mathbb{C}^n \times S \longrightarrow \mathbb{C}^p \times S$ over S. In case of a hypersurface, i.e. $p = 1$, $T^1(X/S, \mathcal{O}_S)$ coincides with the $\mathcal{O}_S$-algebra of the relative singular locus, that is the relative Tjurina-algebra $T(F) = \mathcal{O}_{Sing(X/S)}$.
- The support of the flatness-stratum $\mathbb{F}_{\mathcal{O}_S}(Sing(X/S))$ is the τ-constant stratum, because $T^1(X/S, \mathcal{O}_S)$ is a finite $\mathcal{O}_S$-module.
- It follows from a non-trivial result, cf. [LR76], that the germ of the μ-constant stratum is irreducible. But the analogous statement for the τ-constant stratum does not hold, see below. This phenomenon we have called *splitting singular locus* inside the τ-constant stratum.

- The possible reducibility of $\mathbb{F}_{\mathcal{O}_S}(Sing(X/S))$ causes that a 'correct' τ-constant stratum of a deformation has to be considered in the category of deformations of multi-germs, or one has to be aware that under τ-deformations a singular germ may split into a multi-germ.

2 Computing the modular germs of unimodal singularities

Applying the algorithm for computing the flatness stratum, cf. [Mar02], we can compute the modular stratum of not too complicated singularities. More precisely, the output of the algorithm is the k-jet of the germ of the flatness stratum for some positive integer k. If the modular stratum is a fat point we are done with some big number k. We cannot prove or even expect in general to end up with an algebraic representation. But, it does occur, as visible in the examples given below.
The classification of singularities starts with the simple singularities, the ADE-singularities. These are all quasihomogeneous, their modular strata are all trivial, i.e. simple points. Following the classification of functions by Arnol'd [AGZV85], the next more complicated singularities are the unimodal ones. They are characterized by the fact that in a neighborhood of the function only $\mathcal{R}$-orbit families occur, which are depending at most on one parameter.
Recall their classification: We have the T-series singularities and 14 so called exceptional unimodal singularities. We may restrict their representation to three variables up to stable equivalence (i.e. adding squares of new variables). Any type is representing an one-parameter μ-constant family of $\mathcal{R}$-equivalence classes. The exceptional ones are all semi-quasihomogeneous. Thus, the μ-constant family can be written as

$$f_\lambda = f_0(x) + \lambda h_f(x), \quad \lambda \in \mathbb{C},$$

where f_0 is quasihomogeneous and $h_f(x) := det(\frac{\partial^2 f_0}{\partial x_i \partial x_j})$ is the Hesse form of f_0. Such a family splits into exactly two $\mathcal{K}$-classes, one quasihomogeneous ($\lambda = 0$) and one semi-quasihomogeneous ($\lambda \neq 0$, we call it of *Hesse-type*), and $\tau(f_1) = \mu(f_1) - 1$ holds. The modular strata of the quasihomogeneous singularities are trivial (simple point), while the modular strata of the semi-quasihomogeneous ones are fat points of multiplicity μ.
The singularities of the T-series are defined by the equations

$$T_{p,q,r}: \ x^p + y^q + z^r + \lambda xyz, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1.$$

For $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} < 1$, $\lambda \neq 0$, the singularity $T_{p,q,r}$ is called *hyperbolic* and its

$\mathcal{K}$-class is independent of λ. Its Newton boundary has three maximal faces and the singularity is neither quasihomogeneous nor semi-quasihomogeneous. We have $\tau(T_{p,q,r}) = \mu(T_{p,q,r}) - 1 = p + q + r - 2$.
In exactly three cases we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. These singularities are quasihomogeneous. They are called the *parabolic* singularities P_8, X_9 and J_{10} in Arnold's notation or *elliptic* hypersurface singularities $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$, in Saito's paper [Sai74]:

$$\tilde{E}_6 = P_8 = T_{3,3,3} : \ x^3 + y^3 + z^3 + \lambda xyz, \ \lambda^3 \neq -3^3, \ \tau = \mu = 8;$$

$$\tilde{E}_7 = X_9 = T_{4,4,2} : \ x^4 + y^4 + z^2 + \lambda xyz, \ \lambda^4 \neq 2^6, \ \tau = \mu = 9;$$

$$\tilde{E}_8 = J_{10} = T_{6,3,2} : \ x^6 + y^3 + z^2 + \lambda xyz, \ \lambda^6 \neq 2^4 3^3, \ \tau = \mu = 10.$$

Note, that the families $T_{4,4,2}(\lambda)$ and $T_{6,3,2}(\lambda)$ are not contained in a miniversal family. They form a double covering of the τ-constant line in a miniversal deformation, which can be demonstrated by substituting $z \mapsto z - (1/2)\lambda xy$:

$$x^4 + y^4 + z^2 + \lambda xyz \ \mapsto \ x^4 + y^4 + z^2 - \frac{1}{4}\lambda^2 x^2 y^2,$$

$$x^6 + y^3 + z^2 + \lambda xyz \mapsto \ x^6 + y^3 + z^2 - \frac{1}{4}\lambda^2 x^2 y^2,$$

i.e. these types are stable equivalent to functions of two variables. In all three cases the $\mathcal{K}$-equivalence relations on the λ-lines are induced by an action of a finite group.
Some modular strata for the T-series are discussed as far as computed in [Mar06]. We recall these results and will study their properties in more detail. Obviously, the modular strata of the three parabolic (quasihomogeneous) function are smooth curves. The modular strata of the hyperbolic singularities are more complicated. Some of them are 1-dimensional, others are just fat points. The 1-dimensional modular strata are all reducible. The regularity of the appearance of 1-dimensional components was already treated in [Mar06].
A hyperbolic singularity of type $T_{p,q,r}$ is adjacent to another T-series singularity if and only if all its three parameters (p, q, r) are greater or equal to the parameters of the second. Hence any hyperbolic singularity is adjacent to at least one parabolic singularity. Inspecting the list we find exactly six types of hyperbolic singularities, which have the same Tjurina numbers as an adjacent parabolic singularity: If two of the numbers (p, q, r) are the same as of the parabolic one, the third had to differ by one. Such types are candidates for possessing a 1-dimensional modular stratum.

Proposition 2.1. *The following six 'exceptional' hyperbolic singularities are adjacent to a parabolic singularity of same Tjurina number and have a τ-constant line in their miniversal deformation:*

$$\begin{array}{llll} T_{4,3,3} & & & \Longrightarrow \quad P_8, \\ T_{4,4,3}, & T_{5,4,2} & & \Longrightarrow \quad X_9, \\ T_{6,3,3}, & T_{6,4,2}, & T_{7,3,2} & \Longrightarrow \quad J_{10}. \end{array}$$

Indeed a τ-constant line is given by $f_t = f_0 + tg$, where g stands for the missing monomial of the associated parabolic singularity.

Example 2.2. $f_t := x^4+y^3+z^3+xyz+tx^3$ is a τ-constant deformation of $T_{4,3,3}$ with generic fiber type P_8. The modular deformation f_t fits into the λ-line of $P_8(\lambda)$ at infinity: $f_t \sim_{\mathcal{K}} P_8(t^{-1/3})$ for $t \neq 0$, i.e. we get a threefold covering of the λ-line, $\lambda \neq 0$, by the t-line, $t \neq 0$. We may think of a compactification of the modular λ-line of P_8 at infinity with a point corresponding to the $T_{4,3,3}$-singularity.

Example 2.3. The same holds for $T_{4,4,3}$ and $T_{5,4,2}$ with respect to X_9. But, this causes two different compactifications of the same modular family over the punctured disc $X_9(1/\lambda)$, $\lambda > N$, at the special point zero to a modular family over the disc. We obtain a first example of the failure of separation property for a 'hypothetical' moduli space of function with respect to the $\mathcal{K}$-equivalence that could be constructed by gluing representatives of modular germs. In all three cases, the support of the modular stratum is the indicated τ-constant line, but it has a non-reduced structure generated by an embedded fat point at zero. Equations are given below.

The situation is more complicated for the three types associated with J_{10}. While the above observation holds similar for $T_{7,3,2}$, we find new phenomena for the types $T_{6,4,2}$ and $T_{6,3,3}$. The modular stratum of the first has another line component and the second even has three line components.

Example 2.4. $T_{6,4,2}$ is adjacent to J_{10} as well as to X_9. While one line component has simple type J_{10}, the other line is a modular family defined by the equation $x^6 + y^4 + z^2 + xyz + 2tx5 + t^2x^4$. Here, the fiber at $t \neq 0$ has a singularity of type X_9. Hence it is not τ-constant as deformation of germs (with zero-section). Why is it modular? The affine hypersurface $V(f_t)$ has another singularity at point $(-t, 0, 0)$ of type A_1. The Tjurina numbers of both singularities add to 10 and the A_1-point approaches zero as t goes to zero, i.e. f_t is τ-constant as deformation of multi-germs. We call such a modular deformation a τ-constant *splitting line*. The singular point of the special fiber splits into two singularities under the deformation. The two line components form the reduced modular stratum, which is completed again by a fat point at zero.

Example 2.5. $T_{6,3,3}$ is adjacent to J_{10} as well as to P_8. First we find two τ-constant lines of identical simple type similar to example 2.2 and caused by the additional symmetry of two equal parameters:

$$x^6 + y^3 + z^3 + xyz + ty^2 \quad \text{and, resp.} \quad x^6 + y^3 + z^3 + xyz + tz^2.$$

The third line is a splitting line with generic singularity P_8 at zero and A_2 at $(-t, 0, 0)$

$$x^6 + y^3 + z^3 + xyz + 3tx^5 + 3t^2x^4 + t^3x^3.$$

The question arises, which of the T-series singularities have one splitting line and which have more than one line component in their modular stratum. What shape has the modular stratum of the other T-series singularities? The following was shown in [Mar06].

Proposition 2.6. *Any of the six 'exceptional' hyperbolic singularities given above is heading an exceptional sub-series of $T_{p,q,r}$, whose modular strata contain a splitting line.*

Comments:

- The six exceptional sub-series are

$$T_{k,3,3},\ k > l = 4, \quad T_{k,4,2},\ k > l = 5, \quad T_{4,4,k},\ k > l = 3,$$

$$T_{k,3,2},\ k > l = 7, \quad T_{6,k,2},\ k > l = 4, \quad T_{6,3,k},\ k > l = 3.$$

- The families over the splitting lines with index k are given by (up to the obvious permutation of variables)

$$f_t := x^p + y^q + xyz + z^l(x+t)^{k-l}.$$

- The fiber singularities over $t \neq 0$ are a singularity of the associated parabolic type and one singularity of type A_{k-d-1}.

- We find three cases with two splitting lines of same type due to the symmetry of parameters, all associated to J_{10}:

$$T_{6,6,2}, \quad T_{6,6,3}, \quad T_{6,3,3},$$

  of splitting types $J_{10} + A_2$, $J_0 + A_3$ and $P_8 + A_2$ respectively, and $T_{4,4,4}$ has three lines of identical type $X_9 + A_1$.

- We have two types that have splitting lines to different parabolic types:

$$T_{6,4,2} \quad \text{has two lines of types} \quad X_9 + A_1 \quad \text{and, resp.} \quad J_{10},$$

$$T_{6,3,3} \quad \text{has one lines of type} \quad P_8 + A_2 \quad \text{and two lines of type} \quad J_{10}.$$

  These are cases of multi-component modular strata of the exceptional sub-series.

- The modular strata of these singularities have besides the lines another embedded primary component (a fat point). The only exception is the highly symmetric singularity $T_{4,4,4}$, whose modular stratum is the transversal crossing of three lines.

All other computed examples of modular strata of T-series singularities, not belonging to the above six exceptional sub-series, are fat points. We cannot prove this in general, but the clear combinatorial pattern of the occurrence of positive dimensional modular strata is a strong indication that the exceptional sub-series together with the parabolic singularities are the only unimodal singularities with a 1-dimensional modular stratum.
As in the case of the 14 exceptional semi-quasihomogeneous singularities, the fat points have multiplicity $\mu = \mu(f)$, the Milnor number of the singularity. It was already demonstrated that even the Hilbert function of the fat point coincides with the Hilbert function of the Milnor algebra of the singularity, cf. [Mar06].

3 New explicit results on modular strata

A careful inspection of the cases of many computation produced further results about the modular strata of unimodal functions. While the picture is complete for all 14 exceptional functions, the new propositions for the T-series singularities has been checked for all functions of Milnor number smaller than 45.
It is be seen from the examples that a general proof fails, because of the complexity of the occurring equations. We discuss only some examples in full detail.

Proposition 3.1. *All* 14 *exceptional semi-quasihomogeneous unimodal singularities fulfill: The local ring of their modular stratum is isomorphic to their Milnor algebra.*

Below we will discuss in detail the non-trivial isomorphisms for three singularities.

Proposition 3.2. *All modular strata of hyperbolic singularities belonging to an exceptional sub-series are isomorphic as long as they have one line component only. The local rings of their modular strata are isomorphic to the Milnor algebra of the non-isolated 'limiting singularity' given by the equation* $f_\infty := x^p + y^q + xyz$*, i.e. omitting the 'varying monomial'.*

This is a consequence of 3.4, provided the proposition can be proved for all values of the parameters.

Proposition 3.3. *The local algebra of a modular stratum is isomorphic to the Milnor algebra of a non-isolated singularity for the five T-series singularities of the exceptional subseries with 2 or 3 line-components:*

$$\begin{array}{rll} Q(xyz) & \textit{for } T_{4,4,4}, & \textit{and } T_{6,3,3} \\ Q(x^2+xyz) & \textit{for } T_{6,4,2} & \textit{and } T_{6,6,2}, \\ Q(x^3+xyz) & \textit{for } T_{6,3,3}. & \end{array}$$

Again this follows from the equations below, see example 3.5.

Proposition 3.4. *The equations of the modular stratum of a hyperbolic T-series singularity of corank 3 are given by the formulas in Example 3.5.*

We checked this result for all singularities of Milnor number smaller than 45 by computation. Common formulas have been derived in terms of the parameters (p, q, r). The cases $2 < p \leq q \leq r$ include three of the six exceptional subseries. The vanishing of one special coefficient results in some special cases the occurrence of a splitting line. Similar formulas exist for T-series singularities of corank 2, i.e. of type with an z^2-term. Here, we omit these equations.

Example 3.5 (T-series)**.** Let X_0 be the germ of a hypersurface defined by $f = x^p + y^q + z^r + xyz$ with $p \geq 3$, $q \geq 3$, $r \geq 3$. Then

$$F = f + t_1 x^{p-1} \ldots t_{p-1} x + t_p + u_1 y^{q-1} \ldots u_{q-1} y + v_1 z^{r-1} \ldots v_{r-1} z$$

defines a miniversal deformation $X \to S$ of X_0, with $\mathcal{O}_S = \mathbb{C}\{t, u, v\}$.
We obtained the following polynomials generating the ideal $I_M \subset \mathcal{O}_S$ of the modular stratum $M \subset S$ in all computed cases ($p + q + r \leq 46$).

$$\begin{aligned} I_M \quad = \quad & (f_2, \ldots, f_p, g_2, \ldots, g_{q-1}, h_2, \ldots, h_{r-1}, \\ & u_1 v_1 - P_p(p,q,r) P(p,q,r)^2 t_1^{p-1}, \\ & t_1 v_1 - P_q(q,r,p) P(p,q,r)^2 u_1^{q-1}, \\ & t_1 u_1 - P_r(r,p,q) P(p,q,r)^2 v_1^{r-1}), \end{aligned}$$

where

$$f_i := t_i - P_i(p,q,r)t_1^i,\ g_i := u_i - P_i(q,r,p)u_1^i,\ h_i := v_i - P_i(r,p,q)v_1^i,$$

and with coefficients

$$P_i(p,q,r) := \frac{\prod_{k=1}^{i} P(p-k+1,q,r)}{i!P(p,q,r)^i}$$

and

$$P(p,q,r) := pqr(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}).$$

A coefficients $P_p(p,q,r)$ is zero if and only if $\frac{1}{k} + \frac{1}{q} + \frac{1}{r} = 1$ for some $1 \leq k \leq p$. This being the case exactly when $T_{p,q,r}$ belongs to an exceptional sub-series $T_{3,3,k}$, $T_{4,4,k}$, $T_{6,3,k}$. For $T_{4,4,4}$, $T_{6,6,3}$ and $T_{6,3,3}$ more than one of the coefficients $P_p(p,q,r), P_q(q,r,q), P_r(r,p,q)$ is zero and we obtain $\mathcal{O}_M$ to be isomorphic to $Q(xyz)$ for $T_{4,4,4}$, and to $Q(x^3+xyz)$ for $T_{6,3,3}$ and $T_{6,6,3}$. Only one coefficient $P_r(r,q,p)$ vanishes for all other singularities of an exceptional sub-series. In this case all local algebras $\mathcal{O}_M$ are isomorphic to $Q(x^3+y^3+xyz)$, or to $Q(x^4+y^4+xyz)$, or to $Q(x^6+y^3+xyz)$ respectively. If the singularity is not from an exceptional sub-series, none of the coefficients vanishes, and the local algebra of the modular stratum is isomorphic to the Milnor algebra $Q(f)$ of the function itself. The isomorphisms are induced by a diagonal change of variables

$$t_1 \mapsto \alpha t_1, u_1 \mapsto \beta u_1, v_1 \mapsto \gamma v_1.$$

In the next example we shall take a closer look at three singularities from the 14 exceptional semi-quasihomogeneous cases. The isomorphisms between the local rings of their modular strata and the Milnor algebras of the defining functions are listed, which turn out to be rather complicated. They are all computed with a special algorithm.

Example 3.6 (W_{12}, S_{11} and Z_{11})**.** We start with $f = x^4 + y^5 + x^2y^3$ and choose $(b_{11},\dots,b_1) := (1,x,x^2,y,xy,x^2y,y^2,xy^2,xy^3,y^4)$ as representatives of a $\mathbb{C}$-basis of the Tjurina algebra $T(f)$. Now, $F = f + s_1b_1 + \dots + s_{11}b_{11} \in \mathbb{C}\{x,y\} \otimes \mathcal{O}_S$ defines a miniversal deformation $X \to S$ of X_0.

The ideal $I_M \subset \mathcal{O}_S$ of the maximal modular subgerm $M \subset S$, computed with Singular is gives by the following completely interreduced generators:

$$
\begin{array}{lll}
\mathbf{s_1^4} & - & \mathbf{\frac{30445}{7392} s_1^2 s_2^2 + \frac{4240139}{1897280} s_1^3 s_2^2}, \\
\mathbf{s_2^3} & - & \mathbf{\frac{2696}{48125} s_1^3 s_2}, \\
s_{11} & + & \frac{11699}{144375} s_1^3 s_2^2, \\
s_{10} & - & \frac{3904}{48125} s_1^3 s_2, \\
s_9 & + & \frac{52}{625} s_1^3 - \frac{951}{7000} s_1 s_2^2 + \frac{592717}{8421875} s_1^2 s_2^2 - \frac{119567878949}{5187875000000} s_1^3 s_2^2, \\
s_8 & + & \frac{1304}{5775} s_1^2 s_2^2 - \frac{1411481}{18528125} s_1^3 s_2^2, \\
s_7 & - & \frac{618}{1925} s_1^2 s_2 + \frac{1024869}{37056250} s_1^3 s_2, \\
s_6 & + & \frac{6}{25} s_1^2 + \frac{3}{80} s_2^2 - \frac{21}{3125} s_1^3 + \frac{531}{20000} s_1 s_2^2 - \frac{31001023}{5390000000} s_1^2 s_2^2, \\
 & + & \frac{25063327841}{207515000000000} s_1^3 s_2^2, \\
s_5 & - & \frac{2}{25} s_1^3 + \frac{9}{16} s_1 s_2^2 - \frac{114057}{539000} s_1^2 s_2^2 + \frac{6306416817}{83006000000} s_1^3 s_2^2, \\
s_4 & - & \frac{6}{7} s_1 s_2 + \frac{1227}{67375} s_1^2 s_2 - \frac{16557777}{2593937500} s_1^3 s_2, \\
s_3 & - & \frac{2}{5} s_1^2 + \frac{9}{16} s_2^2 - \frac{9}{625} s_1^3 - \frac{621}{4000} s_1 s_2^2 + \frac{49325643}{1078000000} s_1^2 s_2^2, \\
 & - & \frac{644553838881}{41503000000000} s_1^3 s_2^2.
\end{array}
$$

$\mathcal{O}_M$ is a zero-dimensional local algebra of embedding dimension 2. A minimal embedding is defined by the two polynomials printed in bold. The mapping

$$
\begin{array}{rcl}
\varphi : \mathcal{O}_M & \to & \mathbb{C}\{x,y\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \\
\overline{s_1} & \mapsto & \frac{2668050}{2051993}\overline{y} - \frac{11759762521878525}{25638801731506361}\overline{y}^2 \\
\overline{s_2} & \mapsto & \frac{2134440}{2051993}\sqrt{-\frac{1386}{6089}} \cdot \overline{x}
\end{array}
$$

defines an isomorphism between this local algebra and the Milnor algebra of f.

Note that this mapping induces an isomorphism $\mathcal{O}_M/(\overline{s_{11}}) \to T(f)$, too. Hence, the maximal modular subgerm in the truncated miniversal deformation (omitting the deformation of the monomial 1) is isomorphic to the singular

locus of X_f. This holds for all our examples – the isomorphism to the Milnor algebra of f, or of f_∞ respectively induces an isomorphism $\mathcal{O}_M/(\overline{s}_\tau) \cong T(f)$, or $\mathcal{O}_M/(\overline{s}_\tau) \cong T(f_\infty)$ repectively.
We give the isomorphism for Z_{11} and S_{11}:

name:	S_{11}	
equation:	$f =$	$y^2z + xz^2 + x^4 + x^3z$
deformation:	$F =$	$f + s_1x^2z + s_2x^2y + s_3x^3 + s_4xz + s_5z + s_6xy+$
		$+s_7y + s_8x^2 + s_9x + s_{10}$
isomorphism:	$\overline{s_1} \mapsto$	$-\frac{3^6 5^2 7^2 11^2}{2^6 23^2 67^2}\overline{x} + \frac{3^6 5^2 7^3 11^2 \cdot 19 \cdot 163}{2^{12} 23^3 67^3}\overline{z}$
	$\overline{s_2} \mapsto$	$-\sqrt{-\frac{3^{13} 5^5 7^5 11^5}{2^{15} 23^5 67^5}\overline{y}}$
	$\overline{s_3} \mapsto$	$\frac{3^9 5^3 7^3 11^3}{2^{10} 23^3 67^3}\overline{z} + \frac{3^{11} 5^4 7^3 11^3 13}{2^{10} 23^4 67^4}\overline{x}^2 -$
		$-\frac{3^9 5^3 7^3 11^3 41 \cdot 307 \cdot 587 \cdot 32677569187}{2^{28} 23^7 67^7}\overline{y}^2$
		$-\frac{3^9 5^4 7^3 11^3 71 \cdot 1759 \cdot 516147191239}{2^{27} 23^7 67^7}\overline{xz} -$
		$-\frac{3^9 5^4 7^3 11^3 31 \cdot 2280560407042079}{2^{30} 23^7 67^7}\overline{z}^2$

name:	Z_{11}	
equation:	$f =$	$x^3y + xy^4 + y^5$
deformation:	$F =$	$f + s_1y^4 + s_2xy^3 + s_3y^3 + +s_4xy^2 + s_5y^2 + s_6xy+$
		$+s_7ys_8x^2 + s_9x + s_{10}$
isomorphism:	$\overline{s_1} \mapsto$	$-\frac{2^{28} 3^3 7^4 11^4}{2399^4}\overline{x} - \frac{2^{29} 3^2 7^4 11^4 23 \cdot 53 \cdot 4405133}{5^3 2399^6}\overline{y}^2$
	$\overline{s_2} \mapsto$	$-\frac{2^{20} 3^2 7^3 11^3 173 \cdot 5879}{5^2 2399^4}\overline{x} + \frac{2^{20} 3^3 7^3 11^3}{2399^3}\overline{y}$
		$+\frac{2^{18} 7^3 11^3 59 \cdot 569 \cdot 49081 \cdot 52566671 \cdot 113887106221771273}{3^2 5^7 2399^7 271}\overline{xy}$
		$+\frac{2^{18} 7^3 11^3 41 \cdot 13677187 \cdot 10991949493076828837 9}{3 \cdot 5^5 2399^6 271}\overline{y}^2$

4 Further examples and questions

We have calculated modular strata for singularities of higher modality, too. The results raise hope that our observation generalizes. We give one example of a singularity of modality greater two.

Example 4.1. We consider the hypersurface singularity given by the semi-quasihomogeneous singularity of Hesse type $f = x^{10} + y^3 + x^4y^2$. A miniversal deformation is defined by

$$\begin{aligned} f \quad = \quad & s_1 + s_2x + s_3x^2 + s_4x^3 + s_5x^4 + s_6x^5 + s_7x^6 + s_8x^7 + s_9x^8 + s_{10}y \\ & +s_{11}xy + s_{12}x^2y + s_{13}x^3y + s_{14}x^4y + s_{15}x^5y + s_{16}x^6y + s_{17}x^7y \end{aligned}$$

The maximal modular subgerm M in the base this deformation is given by the ideal

$$
\begin{array}{rcl}
J_M = (s_1 & + & \mathcal{O}(s^2), \\
 & \vdots & \\
s_8 & + & \mathcal{O}(s^2), \\
s_9^2 & - & \frac{9}{256} s_{17}^4 s_9 - \frac{29342801}{335104000} s_{17}^6 s_9 - \frac{9963}{343146496} s_{17}^8 - \frac{831341932017399}{3283872972800000} s_{17}^{10}, \\
s_{10} & + & \mathcal{O}(s^2), \\
 & \vdots & \\
s_{16} & + & \mathcal{O}(s^2), \\
s_{17}^9 & - & \frac{67372}{106029} s_{17}^7 s_9).
\end{array}
$$

The local ring $\mathcal{O}_M = \mathcal{O}_{17}/J_M$ is again isomorphic to $Q(f)$ via

$$
\begin{array}{rcl}
\varphi : \mathcal{O}_M & \to & \mathbb{C}\{x,y\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}), \\
\overline{s_{17}} & \mapsto & a_1 x, \\
\overline{s_9} & \mapsto & a_2 by + a_3 x^4 + a_4 x^2 y + a_5 x^6 + a_6 x^4 y + a_7 x^8,
\end{array}
$$

with coefficients

$$
\begin{array}{rcl}
a_1 & = & 8 \sqrt[4]{\frac{17943573032}{1269497754275}}, \\
a_2 & = & \frac{2261952}{84215} \sqrt{\frac{17943573032}{1269497754275}}, \\
a_3 & = & \frac{753984}{84215} \sqrt{\frac{17943573032}{1269497754275}} + \frac{1291937258304}{1269497754275}, \\
a_4 & = & \frac{922023862124292878566319898163\,2}{1007265342568292675484765625}, \\
a_5 & = & \frac{25742505984143872}{158687219284375} \sqrt{\frac{17943573032}{1269497754275}} + \frac{30734128737476429285543996605 44}{1007265342568292675484765625)}, \\
a_6 & = & \frac{547510092328050056695293440974819328}{37772450346310975330678710937 5} \sqrt{\frac{17943573032}{1269497754275}}, \\
a_7 & = & \frac{547510092328050056695293440974819328}{113317351038932925992036132812 5} \sqrt{\frac{17943573032}{1269497754275}}.
\end{array}
$$

In all examples, we have considered a function f defining an isolated hypersurface singularity X_0, and relate its modular stratum to the Milnor algebra of f. If we take another $\mathcal{K}$-equivalent function f', the isomorphism-class of the modular stratum does not change by definition. While $\mu(f)$ is an invariant of $\mathcal{K}$-class, this is in general not true for the isomorphism-class of the Milnor algebra [BY90].
Nevertheless, for singularities with $\tau = \mu - 1$, we have the following lemma.

Lemma 4.2. *Let be f an analytic function with isolated critical point with $\tau(f) = \mu(f) - 1$, then its Milnor algebra is $\mathcal{K}$-invariant.*

Proof. We have a decomposition of $Q(f)$ as a vector space $Q(f) \cong T(f) \oplus \mathbb{C} \cdot \overline{f}$. Look at the exact sequence

$$0 \to \mathrm{Ann}(f) \to Q(f) \xrightarrow{\cdot \overline{f}} Q(f) \to T(f) \to 0.$$

Then $\mathrm{Ann}(f)$ has $\mathbb{C}$-dimension $\mu - 1$ and equals the maximal ideal $\mathfrak{m}_{Q(f)} = \mathfrak{m}_{T(f)} \oplus \mathbb{C} \cdot \overline{f}$ of $Q(f)$.
The multiplication induces

$$(g + c \cdot \overline{f}) \cdot (g' + c' \cdot \overline{f}) = gg' + ((c'g(0) + cg'(0)) \cdot \overline{f}) \tag{1}$$

with $c, c' \in \mathbb{C}$ and $g, g' \in T(f)$. Assume $f' \sim_{\mathcal{K}} f$, then $\mu(f') = \mu(f)$ and $\tau(f') = \tau(f)$ hold. Moreover, we have an isomorphism $\varphi : T(f) \cong T(f')$. Thus $Q(f)$ and $Q(f')$ are isomorphic as vector spaces via

$$\begin{array}{rcl} T(f) \oplus \mathbb{C} \cdot \overline{f} & \longrightarrow & T(f') \oplus \mathbb{C} \cdot \overline{f'}, \\ g + c \cdot \overline{f} & \mapsto & \varphi(g) + c \cdot \overline{f'}. \end{array}$$

Because of (1) this linear isomorhism is indeed an algebra homomorphism. □

Due to the last lemma we can speak of the Milnor algebra of a hypersurface singularity in the case $\tau = \mu - 1$. Hence we can state the following conjecture, motivated by our examples.

Hypothesis 4.3. *Consider a hypersurface singularity f with $\tau = \mu - 1$. Then the local ring of the modular stratum $\mathcal{O}_{M(f)}$ is of Milnor type, i.e. there exists a germ of an analytic function f' such that $Q(f') \cong \mathcal{O}_M$. If f has an Artinian modular stratum, then the local ring of the modular stratum is isomorphic to the Milnor algebra of f itself.*

We found the modular strata to be of Milnor type in all computed examples. So one could ask more generally: *For which singularities is the modular stratum of Milnor type?*

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